

HUSIMI Q-FUNCTIONS ATTACHED TO HYPERBOLIC LANDAU LEVELS

*Z. MOUAYN, ^bH. CHHAIBA, [‡]H. KASSOGUE, [‡]P. K. KIKODIO

* *Department of Mathematics, Faculty of Sciences and Technics (M'Ghila), Sultan Moulay Slimane University, P.O. Box. 523, Béni Mellal, Morocco*

^b *Residence num 408 Hassan 2 avenue appt 6, Diour Jamaa, Rabat, Morocco*

[‡] *High Institute of Applied Technologies (TechnoLAB-ISTA), Bamako, Mali*

[‡] *13, Market avenue, Mongt-Ngafula, Kinshasa, DR Congo*

ABSTRACT. We are concerned with a phase-space probability distribution which is known as Husimi Q -function of a density operator with respect to a set of coherent states $|\tilde{\kappa}_{z,B,R,m}\rangle$ attached to an m th hyperbolic Landau level and labeled by points z of an open disk of radius R , where $B > 0$ is proportional to a magnetic field strength. For a density operator representing a projector on a Fock state $|j\rangle$ we obtain the Q_j distribution and discuss some of its basic properties such as its characteristic function and its main statistical parameters. We achieve the same program for the thermal density operator (mixed states) of the isotonic oscillator for which we establish a lower bound for the associated thermodynamical potential. We recover most of the results of the Euclidean setting (flat case) as the parameter R goes to infinity by making appeal to asymptotic formulas involving orthogonal polynomials and special functions. As a tool, we establish a summation formula for the special Kampé de Fériet function $F_{2;0;0}^{1;2;2}$.

1. INTRODUCTION

Coherent states (CS) are an overcomplete family of normalized *ket* vectors $|\zeta\rangle$, $\zeta \in X$, in a Hilbert space \mathcal{H} corresponding to a specific quantum model and provide \mathcal{H} with the following resolution of the identity

$$1_{\mathcal{H}} = \int_X |\zeta\rangle \langle \zeta| d\mu(\zeta). \quad (1.1)$$

Here, X represents the phase-space domain and $d\mu(\zeta)$ the associated integration measure on X . These CS have long been known for the harmonic oscillator, whose properties have been used as a model for more quantum systems [1]. In a such system of states the density operator $\hat{\rho}$ for an arbitrary (pure or mixed) state can be represented by the classical Husimi Q -function [2]:

$$Q_{\hat{\rho}}(\zeta) := \langle \zeta | \hat{\rho} | \zeta \rangle. \quad (1.2)$$

In particular, for a pure state of the form $\hat{\rho} = |\phi\rangle \langle \phi|$ where $|\phi\rangle \in \mathcal{H}$ is an arbitrary vector, the Q -function can also be expressed as

$$Q_{|\phi\rangle \langle \phi|}(\zeta) = (\mathcal{N}_{\zeta})^{-1} |W[\phi](\zeta)|^2 \quad (1.3)$$

in terms of the associated *coherent state transform* W defined from \mathcal{H} into a specific L^2 subspace of X . Here, \mathcal{N}_{ζ} is a factor ensuring the normalization condition $\langle \zeta | \zeta \rangle = 1$. For a general density operator $\hat{\rho}$, the function $Q_{\hat{\rho}}(\zeta)$ is normalized, since

$$1 = Tr[\hat{\rho}] = \int_X Q_{\hat{\rho}}(\zeta) d\mu(\zeta). \quad (1.4)$$

As $Q_{\hat{\rho}}$ is introduced in a way that guarantees it to be non negative, then it allows the representation of quantum states by a *probability distribution* in the phase space X . It provides, equivalently to the Glauber-Surdashan or Wigner representations, a basis for a formal equivalence between the quantum and classical descriptions of optical coherence [3].

In this paper, we are concerned with the Husimi function of a density operator with respect to a set of generalized CS (GCS) $|\tilde{\kappa}_{z,B,R,m}\rangle$ belonging to the Hilbert space $L^2(\mathbb{R}_+)$, labeled by points $z \in \mathbb{D}_R = \{\zeta \in \mathbb{C}, |\zeta| < R\}$, $R > 0$ and attached to a higher m th hyperbolic Landau level in the presence of a magnetic field whose strength is proportional to $B > 0$. We first introduce GCS via a group theoretical method by displacing an m th Laguerre function by mean of a square integrable irreducible unitary representation of the affine group. These GCS are then used to obtain the Husimi distribution, $Q_j^{(B,R,m)}$, for a projector operator $\hat{\rho}_j = |j\rangle\langle j|$ on a Fock state $|j\rangle$, which is known as a pure state. For this distribution we write down the characteristic function from which we derive the main value and the variance. As in the Euclidean setting [4] this distribution may be useful for tackling the hyperbolic version of Ginibre-type processes [5]. Next, from $Q_j^{(B,R,m)}$ we deduce the Husimi function $Q_\beta^{(B,R,m)}$ for the thermal density operator (heat semi-group) $\hat{\rho}_\beta$ of the Hamiltonian of the isotonic oscillator. Since $Q_j^{(B,R,m)}$ may be also viewed as a lower symbol for $\hat{\rho}_\beta$ we apply a Berezin-Lieb inequality [6] (with a specific choice of a convex function) to obtain a lower bound for the thermodynamical potential associated with this Hamiltonian. We also establish a formula for the characteristic function of $Q_\beta^{(B,R,m)}$, from which we derive the mean and the variance parameters. Finally, we recover most of the results [4] of the flat case as the radius parameter R goes to infinity. As a tool we establish a summation formula for the special Kampé de Fériet function $F_{2:0:0}^{1:2:2}$.

The paper is organized as follows. In Section 2, we recall some notations and facts about orthogonal polynomials and special functions we will be using. In section 3, we discuss the construction of GCS labeled by elements of the affine group as well as their relationship to hyperbolic Landau levels. Section 4 deals with the Q-representation for a pure state as well as the corresponding characteristic function and the main statistical parameters. The same task is achieved in Section 5 for the thermal density operator of the isotonic oscillator. For the latter one, we also give a lower bound for the thermodynamical potential. In Section 6, we establish a summation formula for the Kampé de Fériet function $F_{2:0:0}^{1:2:2}$. Section 7 is devoted to some concluding remarks. Proofs of our results are detailed in appendices.

2. SOME NOTATIONS AND DEFINITIONS

This section collects the basic notations and definitions of special functions and orthogonal polynomials used in the rest of the paper. The reader may proceed to Section 3 and refer back here as necessary. For more details on the theory of these functions we refer to [7–11].

1. For $a \in \mathbb{C}$, the shifted factorial or Pochhammer symbol is defined by

$$(a)_k = a(a+1)\dots(a+k-1), \quad k \in \mathbb{N} \quad (2.1)$$

where by convention $(a)_0 = 1$. When $a = -n$ with $n \in \mathbb{N}^* = \mathbb{N} - \{0\}$,

$$(-n)_k = \begin{cases} \frac{(-1)^k n!}{(n-k)!}, & 0 \leq k \leq n \\ 0, & k > n \end{cases} \quad (2.2)$$

2. For $a \in \mathbb{C}$ and $k \in \mathbb{N}$, the binomial coefficient is defined by

$$\binom{a}{k} = \frac{a(a-1)\dots(a-k+1)}{k!} = \frac{(-1)^k (-a)_k}{k!} \quad (2.3)$$

3. For $z \in \mathbb{C}$, the gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \operatorname{Re} z > 0. \quad (2.4)$$

Note that $\Gamma(n+1) = n!$ if $n \in \mathbb{N}$, and

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} \quad (2.5)$$

if $a \in \mathbb{C} \setminus \mathbb{Z}_-$.

4. For $a, b \in \mathbb{C}$ such that $\operatorname{Re} a, \operatorname{Re} b > 0$, the beta function is defined by

$$\mathcal{B}(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \quad (2.6)$$

5. For $a_1, \dots, a_p \in \mathbb{C}$ and $c_1, \dots, c_q \in \mathbb{C} \setminus \mathbb{Z}_-$, the generalized hypergeometric function is defined by the series

$${}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ c_1, \dots, c_q \end{matrix} \middle| z \right) = \sum_{k=0}^{+\infty} \frac{(a_1)_k \cdots (a_p)_k}{(c_1)_k \cdots (c_q)_k} \frac{z^k}{k!}, \quad (2.7)$$

which terminates whenever at least one of the p parameters a_i equals $-1, -2, -3, \dots$. It converges for $|z| < \infty$ if $p \leq q$ or for $|z| < 1$ if $p = q + 1$ and it diverges for all $z \neq 0$ if $p > q + 1$. Special cases of this function are the Gauss hypergeometric function ${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| z \right)$, the confluent hypergeometric function ${}_1F_1 \left(\begin{matrix} a \\ c \end{matrix} \middle| z \right)$ and the binomial series ${}_1F_0 \left(\begin{matrix} a \\ - \end{matrix} \middle| z \right)$. The later one reduces to $(1-z)^{-a}$ if $|z| < 1$.

6. For $a \in \mathbb{R}$, the series

$$I_a(z) = \sum_{k=0}^{+\infty} \frac{\left(\frac{1}{2}z\right)^{a+2k}}{k! \Gamma(a+k+1)}, \quad z \in \mathbb{C} \quad (2.8)$$

is called the modified Bessel function of the first kind and order a .

7. The Jacobi polynomial of parameters a and b is defined by

$$P_n^{(a,b)}(x) = 2^{-n} \sum_{k=0}^n \binom{n+a}{k} \binom{n+b}{n-k} (x+1)^k (x-1)^{n-k}, \quad a, b > -1 \quad (2.9)$$

and can be expressed in terms of the terminating ${}_2F_1$ -sum as

$$P_n^{(a,b)}(x) = \frac{\Gamma(b+n+1)}{n! \Gamma(b+1)} \left(\frac{x-1}{2} \right)^n {}_2F_1 \left(\begin{matrix} -n, -(a+n) \\ b+1 \end{matrix} \middle| \frac{x+1}{x-1} \right). \quad (2.10)$$

8. The Laguerre polynomial of parameter a is defined by

$$L_n^{(a)}(x) = \sum_{k=0}^n (-1)^k \binom{n+a}{n-k} \frac{x^k}{k!}, \quad a > -1 \quad (2.11)$$

and can be expressed in terms of the terminating ${}_1F_1$ -sum as

$${}_1F_1 \left(\begin{matrix} -n \\ a+1 \end{matrix} \middle| x \right) = \frac{n!}{(a+1)_n} L_n^{(a)}(x). \quad (2.12)$$

9. For $a_1, \dots, a_p, b_1, \dots, b_q, c_1, \dots, c_k \in \mathbb{C}$ and $\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_m, \gamma_1, \dots, \gamma_n \in \mathbb{C} \setminus \mathbb{Z}_-$, the Kampé de Fériet function in two variables is defined by the series

$$F_{l:m:n}^{p:q:k} \left(\begin{matrix} a_1 \dots a_p; b_1 \dots b_q; c_1 \dots c_k \\ \alpha_1 \dots \alpha_l; \beta_1 \dots \beta_m; \gamma_1 \dots \gamma_n \end{matrix} \middle| x, y \right) = \sum_{r,s=0}^{+\infty} \frac{\prod_{i=1}^p (a_i)_{r+s} \prod_{i=1}^q (b_i)_r \prod_{i=1}^k (c_i)_s}{\prod_{i=1}^l (\alpha_i)_{r+s} \prod_{i=1}^m (\beta_i)_r \prod_{i=1}^n (\gamma_i)_s} \frac{x^r y^s}{r! s!} \quad (2.13)$$

where for convergence, $p+q < l+m+1$, $p+k < l+n+1$, $|x| < \infty$ and $|y| < \infty$. A special case is the Humbert series

$$\Phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| w, z \right) = F_{1:1:0}^{1:1:0} \left(\begin{matrix} a; b; - \\ c; -; - \end{matrix} \middle| w, z \right) = \sum_{k,l=0}^{+\infty} \frac{(a)_{k+l} (b)_k}{(c)_{k+l}} \frac{w^k z^l}{k! l!}; \quad |w| < 1, |z| < \infty \quad (2.14)$$

wich reduces to

$$\Phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| w, 0 \right) = {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| w \right) \quad (2.15)$$

for $z = 0$.

3. GENERALIZED COHERENT STATES (GCS)

3.1. GCS labeled by elements of the affine group. We recall that the affine group is the set $\mathbf{G} = \mathbb{R} \times \mathbb{R}^+$, endowed with group law $(x, y) \cdot (x', y') = (x + yx', yy')$. \mathbf{G} is a locally compact group with the left Haar measure $d\mu(x, y) = y^{-2} dx dy$. We shall consider one of the two inequivalent infinite dimensional irreducible unitary representations of the affine group \mathbf{G} , denoted π_+ , realized on the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^+, \xi^{-1} d\xi)$ as

$$\pi_+(x, y)[\varphi](\xi) := e^{\frac{i}{2}x\xi} \varphi(y\xi), \quad \varphi \in \mathcal{H}, \quad \xi \in \mathbb{R}^+. \quad (3.1)$$

This representation is square integrable since it is easy to find a vector $\phi_0 \in \mathcal{H}$ such that the function $(x, y) \mapsto \langle \pi_+(x, y)[\phi_0], \phi_0 \rangle_{\mathcal{H}}$ belongs to $L^2(\mathbf{G}, d\mu)$. This condition can also be expressed by saying that the self-adjoint operator $\delta : \mathcal{H} \rightarrow \mathcal{H}$ defined as $\delta[\varphi](\xi) = \xi^{-\frac{1}{2}} \varphi(\xi)$ gives

$$\int_{\mathbf{G}} \langle \varphi_1, \pi_+(x, y)[\psi_1] \rangle \langle \pi_+(x, y)[\varphi_2], \psi_2 \rangle d\mu(x, y) = \langle \varphi_1, \varphi_2 \rangle \left\langle \delta^{\frac{1}{2}}[\varphi_1], \delta^{\frac{1}{2}}[\varphi_2] \right\rangle \quad (3.2)$$

for all $\psi_1, \psi_2, \varphi_1, \varphi_2 \in \mathcal{H}$. The operator δ is unbounded because \mathbf{G} is not unimodular [12].

Now, as in [13], for $B > 0$ and $m = 0, 1, \dots, \lfloor B - 1/2 \rfloor$ where $\lfloor a \rfloor$ denotes the greatest integer not exceeding a , we consider a set of CS labeled by elements $(x, y) \in \mathbf{G}$, which were obtained by acting, via the representation operator $\pi_+(x, y)$, on the admissible vector

$$\phi_{B,m}(\xi) := \left(\frac{\Gamma(2B - m)}{m!} \right)^{-\frac{1}{2}} \xi^{B-m} e^{-\frac{1}{2}\xi} L_m^{(2(B-m)-1)}(\xi) \quad (3.3)$$

where $L_n^\alpha(\cdot)$ denotes the Laguerre polynomial defined in Eq.(2.11). Precisely,

$$|\tau_{(x,y),B,m}\rangle := \pi_+(x, y)[\phi_{B,m}] \quad (3.4)$$

and satisfy the resolution of the identity operator

$$\mathbf{1}_{\mathcal{H}} = c_{B,m} \int_{\mathbf{G}} d\mu(x, y) |\tau_{(x,y),B,m}\rangle \langle \tau_{(x,y),B,m}| \quad (3.5)$$

where $c_{B,m} := 2(B - m) - 1$ and the Dirac's bra-ket notation $|\Phi\rangle\langle\Phi|$ means the rank-one operator $\phi \mapsto \langle\Phi, \phi\rangle_{\mathcal{H}} \Phi$ with $\Phi, \phi \in \mathcal{H}$. In the ξ -coordinate, wavefunctions of the states in Eq.(3.4) read

$$\langle \xi | \tau_{(x,y),B,m} \rangle = \left(\frac{\Gamma(2B - m)}{m!} \right)^{-\frac{1}{2}} (\xi y)^{B-m} e^{-\frac{1}{2}\xi(y-ix)} L_m^{(c_{B,m})}(\xi y), \quad \xi > 0. \quad (3.6)$$

For $m = 0$, the states $\tau_{(x,y),B,0}$ coincide with the well known affine coherent states [14]. These affine states are closely related to analytic wavelets [15].

To describe the connection of these GCS with hyperbolic Landau levels we may first identify the affine group with the Poincaré upper half-plane $\mathbb{H}^2 \equiv \mathbf{G}$. Then, with vectors in Eq.(3.4) one can associate, as usual ([16, p.188]), the CS transform $\mathcal{B}_m : \mathcal{H} \rightarrow L^2(\mathbb{H}^2, d\mu_B)$ defined by [13]:

$$\mathcal{B}_m[\phi](x, y) = (c_{B,m})^{\frac{1}{2}} \int_0^{+\infty} \frac{\langle \xi | \tau_{(x,y),B,m} \rangle \phi(\xi) \xi^{-1} d\xi}{\sqrt{\xi}} \quad (3.7)$$

whose range is the eigenspace of the Schrödinger operator (in suitable units and up to an additive constant) describing the dynamics of a charged particle moving on \mathbb{H}^2 under the action of a magnetic field of strength proportional to B

$$\Delta_B = y^2 (\partial_x^2 + \partial_y^2) - 2iBy\partial_x, \quad (3.8)$$

associated with the eigenvalue $\epsilon_m^B = (B - m)(1 - B + m)$, $m = 0, 1, \dots, \lfloor B - \frac{1}{2} \rfloor$. That is, $\mathcal{B}_m[\mathcal{H}] \equiv \{\Phi \in L^2(\mathbb{H}^2, d\mu), \Delta_B \Phi = \epsilon_m^B \Phi\}$. The operator Δ_B is also known as B -weight Maass operator [17–20] which is an elliptic densely defined operator on the Hilbert space $L^2(\mathbb{H}^2, d\mu)$, with a unique self-adjoint realization also denoted by Δ_B . Its spectrum consists of two parts: a continuous part $[1/4, +\infty[$, corresponding to *scattering states* and the finite number of eigenvalues ϵ_m^B each one with infinite degeneracy called *hyperbolic Landau levels* [13]. Finally, the reproducing kernel of the Hilbert space $\mathcal{B}_m[\mathcal{H}]$ can be obtained from the overlapping function $\langle \tau_{w,B,m}, \tau_{\zeta,B,m} \rangle_{\mathcal{H}}$ between two CS as

$$K_m^B(w, \zeta) = \alpha_{B,m} \left(\frac{|w - \bar{\zeta}|^2}{4\Im w \Im \zeta} \right)^{-B+m} \left(\frac{\zeta - \bar{w}}{w - \bar{\zeta}} \right)^B {}_2F_1 \left(\begin{matrix} -m, -m - 2B \\ 2(B - m) \end{matrix} \middle| \frac{4\Im w \Im \zeta}{|w - \bar{\zeta}|^2} \right), \quad (3.9)$$

$w, \zeta \in \mathbb{H}^2$ where $\alpha_{B,m} := \frac{(-1)^m \Gamma(2B-m)}{m! \Gamma(2B-2m)}$ and ${}_2F_1$ being the Gauss hypergeometric function.

The finite number of the levels of infinite degeneracy increases with the strength of the magnetic field, a property. In [21] the first bounds on the size of the fundamental region, which correspond to a ‘cell’ (Pauli exclusion principle allows only one electron per cell), have been obtained. Note that [22] improves this estimate, so that, given a bounded region $\Delta \subset \mathbb{H}^2$ in a hyperbolic Landau level, assuming the Pauli exclusion principle, it is possible to use these discrete states to count the number of particles distributed on such Δ .

3.2. GCS labeled by points of the disk. We can also write a version of the GCS in Eq.(3.4) as states labeled by points z of the unit disk $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$ by using the inverse Cayley transform $\mathcal{C}^{-1} : \mathbb{D} \rightarrow \mathbf{G}$ given by

$$\mathcal{C}^{-1}(z) = \left(-2 \frac{\Im z}{|1 - z|^2}, \frac{1 - |z|^2}{|1 - z|^2} \right). \quad (3.10)$$

We precisely define GCS in $L^2(\mathbb{R}_+, \xi^{-1} d\xi)$ by setting

$$\kappa_{z,B,m} := \left(\frac{1 - \bar{z}}{1 - z} \right)^B \pi_+ (\mathcal{C}^{-1}(z)) [\phi_{B,m}]. \quad (3.11)$$

Direct calculations lead to their wave functions in ξ -coordinates as

$$\langle \xi | \kappa_{z,B,m} \rangle = \sqrt{\frac{m!}{\Gamma(2B - m)}} \frac{|1 - z|^{2m}}{(1 - z)^{2B}} ((1 - z\bar{z}) \xi)^{B-m} \exp\left(-\frac{\xi}{2} \frac{1+z}{1-z}\right) L_m^{(c_{B,m})} \left(\xi \frac{1 - z\bar{z}}{|1 - z|^2} \right). \quad (3.12)$$

By Eq.(3.12) we recover the CS in ([23, Eq.(4.4)]) (up to $(-1)^m$) where ν should be taken as our B and we may view Eq.(3.11) as the analog of “*displacing the vacuum*” ($D(z)|0\rangle$) way in the construction of canonical CS for the harmonic oscillator. When $m = 0$, the wavefunction of GCS in Eq.(3.12) reduces to

$$\langle \xi | \kappa_{z,B,0} \rangle = \frac{1}{\sqrt{\Gamma(2B)}} \left(\frac{(1 - z\bar{z}) \xi}{(1 - z)^2} \right)^B \exp\left(-\frac{\xi}{2} \frac{1+z}{1-z}\right), \quad \xi > 0 \quad (3.13)$$

and coincides with those constructed by Molanar *et al* ([24, Eq.(2.2)]) for the Morse potential by an algebraic way based on supersymmetry and shape invariance properties where the shape parameter may be taken as our $B > 0$. The GCS in Eq.(3.13) were first introduced by Nieto *et al* [25] as generalized minimal uncertainty states for the Hamiltonian of Morse potential [26].

Moreover, the GCS in Eq.(3.12) may slightly be modified in order to perform them as vectors of $L^2(\mathbb{R}_+, d\xi)$ labeled by points of the disk $\mathbb{D}_R := \{z \in \mathbb{C}, |z| < R\}$ as

$$\langle \xi | \tilde{\kappa}_{z,B,R,m} \rangle := \sqrt{\frac{2}{\xi}} \langle \xi^2 | \kappa_{\frac{z}{R}, BR^2, m} \rangle \quad (3.14)$$

for every fixed $m = 0, 1, 2, \dots, \lfloor BR^2 - \frac{1}{2} \rfloor$ provided that $2BR^2 > 1$. These new states obey the normalization condition $\langle \tilde{\kappa}_{z,B,R,m}, \tilde{\kappa}_{z,B,R,m} \rangle_{L^2(\mathbb{R}_+, d\xi)} = 1$ and satisfy the resolution of the identity operator as

$$1_{L^2(\mathbb{R}_+, d\xi)} = \int_{\mathbb{D}_R} |\tilde{\kappa}_{z,B,R,m}\rangle \langle \tilde{\kappa}_{z,B,R,m}| d\mu_{B,R,m}(z), \quad (3.15)$$

with respect to the measure

$$d\mu_{B,R,m}(z) := \frac{(2(BR^2 - m) - 1)}{\pi R^2 (1 - \frac{z\bar{z}}{R^2})^2} d\mu(z), \quad (3.16)$$

$d\mu(z)$ being the Lebesgue measure on the unit disk \mathbb{D} . Recalling that the GCS $|\tilde{\kappa}_{z,B,R,m}\rangle$ originate from Eq.(3.11) according to a group theoretical approach [27], it becomes natural to seek for a *number states expansion* [28] for them. Indeed, we can show that they decompose as

$$|\tilde{\kappa}_{z,B,R,m}\rangle = (\mathcal{N}_{B,R,m}(z))^{-\frac{1}{2}} \sum_{j=0}^{+\infty} \gamma_j^{B,R,m}(z) |j\rangle_{B,R,m} \quad (3.17)$$

where

$$\mathcal{N}_{B,R,m}(z) := \pi^{-1} (2(BR^2 - m) - 1) (1 - z\bar{z}R^{-2})^{-2BR^2}, \quad (3.18)$$

and the wavefunctions of the number states $|j\rangle_{B,R,m}$ are given by

$$\langle \xi | j \rangle_{B,R,m} := \left(\frac{2j!}{\Gamma(2(BR^2 - m) + j)} \right)^{\frac{1}{2}} \xi^{2(BR^2 - m) - \frac{1}{2}} e^{-\frac{\xi^2}{2}} L_j^{(2(BR^2 - m) - 1)}(\xi^2), \quad \xi \in \mathbb{R}_+, \quad (3.19)$$

form an orthonormal basis in $L^2(\mathbb{R}_+, d\xi)$. The coefficients

$$z \mapsto \gamma_j^{B,R,m}(z) := (-1)^m \left(\frac{(2(BR^2 - m) - 1)}{\pi} \frac{(m \wedge j)! \Gamma(2(BR^2 - m) + m \vee j)}{(m \vee j)! \Gamma(2(BR^2 - m) + m \wedge j)} \right)^{\frac{1}{2}} \frac{(-1)^{m \wedge j}}{(1 - \frac{z\bar{z}}{R^2})^m} \left(\frac{z\bar{z}}{R^2} \right)^{\frac{1}{2}|m-j|} e^{-i(m-j) \arg z} P_{m \wedge j}^{(|m-j|, 2(BR^2 - m) - 1)} \left(1 - \frac{2z\bar{z}}{R^2} \right) \quad (3.20)$$

are in fact orthonormalized functions of the Hilbert space $L^2\left(\mathbb{D}_R, R^{-2} (1 - \frac{z\bar{z}}{R^2})^{2BR^2 - 2} d\mu(z)\right)$. Here $m \wedge j := \min\{m, j\}$ and $m \vee j := \max\{m, j\}$. The key ingredient in proving Eq.(3.17) is the use of the generating formula ([10, p.137]):

$$\sum_{n=0}^{+\infty} \lambda^n {}_2F_1 \left(\begin{matrix} -n, b \\ 1+a \end{matrix} \middle| t \right) L_n^{(a)}(x) = \frac{(1-\lambda)^{b-1-a}}{(1-\lambda+t\lambda)^b} \exp\left(\frac{-x\lambda}{1-\lambda}\right) {}_1F_1 \left(\begin{matrix} b \\ 1+a \end{matrix} \middle| \frac{\lambda xt}{(1-\lambda)(1-\lambda+t\lambda)} \right) \quad (3.21)$$

where the parameters $a \geq 0$ and $|\lambda| < \min\{1, |1-t|^{-1}\}$.

We may note in passing that for $R = 1$, the GCS $|\tilde{\kappa}_{z,B,1,m}\rangle$ in Eq.(3.17) can be considered as a class of generalized negative binomial states [29] and the above functions $\gamma_j^{B,1,m}(z)$ constitute, in fact, an orthonormal basis for the eigenspace

$$\mathcal{E}_{B,m}(\mathbb{D}) = \left\{ f \in L^2\left(\mathbb{D}, (1 - z\bar{z})^{2B-2} d\mu(z)\right), \tilde{\Delta}_B f = \sigma_{B,m} f \right\} \quad (3.22)$$

of the B -weight Maass-Laplacian

$$\tilde{\Delta}_B = -4(1 - z\bar{z}) \left((1 - z\bar{z}) \frac{\partial^2}{\partial z \partial \bar{z}} - 2B\bar{z} \frac{\partial}{\partial \bar{z}} \right) \quad (3.23)$$

associated with eigenvalues $\sigma_{B,m} = 4m(2B - m - 1)$. Additionally, the mapping

$$W_{B,m}[\varphi](z) = (\mathcal{N}_{B,1,m}(z))^{\frac{1}{2}} \langle \varphi | \tilde{\kappa}_{z,B,1,m}\rangle_{L^2(\mathbb{R}_+, d\xi)} \quad (3.24)$$

is a unitary isomorphism from $L^2(\mathbb{R}_+, d\xi)$ onto the Hilbert space $\mathcal{E}_{B,m}(\mathbb{D})$ whose reproducing kernel can be obtained from the overlapping function $\langle \tilde{\kappa}_{z,B,1,m}, \tilde{\kappa}_{w,B,1,m} \rangle_{L^2(\mathbb{R}_+, d\xi)}$ between two GCS. Explicitly,

$$\begin{aligned} K_{B,m}(z, w) &= \pi^{-1} (2(B-m) - 1) (1 - z\bar{w})^{-2B} \left(\frac{|1 - z\bar{w}|^2}{(1 - z\bar{z})(1 - w\bar{w})} \right)^m \\ &\quad \times P_m^{(0, 2(B-m)-1)} \left(2 \frac{(1 - z\bar{z})(1 - w\bar{w})}{|1 - z\bar{w}|^2} - 1 \right) \end{aligned} \quad (3.25)$$

for $z, w \in \mathbb{D}$.

4. HUSIMI'S Q-FUNCTION FOR A PURE STATE

By using Eq.(3.17) to Eq.(3.20), one can write according to Eq.(1.2), the Husimi function of the pure state

$$\widehat{\rho}_j := |j\rangle_{B,R,m} \langle j|_{B,R,m} \quad (4.1)$$

as

$$Q_j^{B,R,m}(z) := \langle \tilde{\kappa}_{z,B,R,m} | \widehat{\rho}_j | \tilde{\kappa}_{z,B,R,m} \rangle, \quad z \in \mathbb{D}_R. \quad (4.2)$$

Explicitly,

$$Q_j^{B,R,m}(z) = \tau_j^{B,R,m} \left(1 - \frac{z\bar{z}}{R^2} \right)^{2(BR^2-m)} \left(\frac{z\bar{z}}{R^2} \right)^{|m-j|} \left(P_{m \wedge j}^{(|m-j|, 2(BR^2-m)-1)} \left(1 - \frac{2z\bar{z}}{R^2} \right) \right)^2 \quad (4.3)$$

where

$$\tau_j^{B,R,m} := \frac{(m \wedge j)! \Gamma(2(BR^2 - m) + m \vee j)}{(m \vee j)! \Gamma(2(BR^2 - m) + m \wedge j)} \quad (4.4)$$

for every fixed $m = 0, 1, \dots, \lfloor BR^2 - \frac{1}{2} \rfloor$ provided that $2BR^2 > 1$. Doing so enables us, by letting $R \rightarrow \infty$, to recover the (radial) Husimi's Q-function for a pure state associated with an m th Euclidean Landau level ([4, p.5]):

$$Q_j^{(m), \text{Euclid}}(\sqrt{2B}z) = \frac{(m \wedge j)!}{(m \vee j)!} e^{-2Bz\bar{z}} (2Bz\bar{z})^{|m-j|} \left(L_{m \wedge j}^{(|m-j|)}(2Bz\bar{z}) \right)^2, \quad m = 0, 1, 2, \dots, \quad z \in \mathbb{C}, \quad (4.5)$$

see Appendix A for the proof. We also note that the characteristic function for the random variable X having $Q_j^{(m), \text{Euclid}}$ as its density function reads ([4, p.6]):

$$\begin{aligned} \phi_X(u) &:= (1 - iu)^{-(m+j+1)} \sum_{k=0}^{m \wedge j} \binom{m}{k} \binom{j}{k} (iu)^{2k}, \quad u \in \mathbb{R} \\ &= \frac{(m+j)!}{m!j!} (1 - iu)^{-(m+j+1)} {}_2F_1 \left(\begin{matrix} -m, -j \\ -m-j \end{matrix} \middle| 1 + u^2 \right) \end{aligned} \quad (4.6)$$

and was involved by Shirai while discussing Ginibre-type processes [5]. Like the Euclidean setting, one can check that the Husimi function given by Eq.(4.3) is a probability distribution on the phase space \mathbb{D}_R . That is,

$$\int_{\mathbb{D}_R} Q_j^{B,R,m}(z) d\mu_{B,R,m}(z) = 1 \quad (4.7)$$

where $d\mu_{B,R,m}(z)$ is given by Eq.(3.16). Using polar coordinates $z = re^{i\theta}$, $r \in [0, R)$, $\theta \in [0, 2\pi)$ and setting

$$\Omega_j^{B,R,m}(\lambda) := R^{-2} (2(BR^2 - m) - 1) (1 - \lambda R^{-2})^{-2} Q_j^{B,R,m}(\lambda), \quad (4.8)$$

with $\lambda = r^2$, then Eq.(4.7) ensures

$$\int_0^{R^2} \Omega_j^{B,R,m}(\lambda) d\lambda = 1. \quad (4.9)$$

For $m = 0$, the density function in Eq.(4.8) reduces to

$$\mathfrak{Q}_j^{B,R,0}(\lambda) = R^{-2} \frac{\Gamma(2BR^2 - 1 + j + 1)}{\Gamma(2BR^2 - 1)\Gamma(j + 1)} \left(1 - \frac{\lambda}{R^2}\right)^{(2BR^2-1)-1} \left(\frac{\lambda}{R^2}\right)^{(j+1)-1} \quad (4.10)$$

which coincides (up to a scale factor $1/R^2$) with the Beta distribution $\mathcal{B}e(j + 1, 2BR^2 - 1)$ whose characteristic function is known to be given by the confluent hypergeometric series as

$$\phi_{\mathcal{B}e}(u) = {}_1F_1\left(\begin{matrix} j + 1 \\ 2BR^2 + j \end{matrix} \middle| iuR^2\right). \quad (4.11)$$

If for $R = 1$ we denote by $\mathfrak{X}_m^{(1)}$ the random variable having $\lambda \mapsto \mathfrak{Q}_j^{B,1,m}(\lambda)$ as its density function then $\mathfrak{X}_m^{(1)}$ would play the role of the hyperbolic analog of the continuous random variable denoted by $S_k^{n,b}$ in Shirai's paper [5] and

$$\Pr\left(\mathfrak{X}_m^{(1)} \leq r^2\right) = \int_0^{r^2} \mathfrak{Q}_j^{B,1,m}(\xi) d\xi \equiv \lambda_j^{m,B}(r^2) \quad (4.12)$$

would provide a probabilistic representation of eigenvalues $\lambda_j^{m,B}(r^2) := \|\gamma_j^{B,1,m} \mathbf{1}_{\mathbf{D}_r}\|^2$ of the restricted operator $(K_{B,m}(z, w))_{\mathbf{D}_r}$ to the disk $\mathbf{D}_r \subset \mathbb{D}$ of radius $r \leq 1$, where $K_{B,m}(z, w)$ defined in Eq.(3.25) is the projection operator onto the m th hyperbolic Landau eigenspace $\mathcal{E}_{B,m}(\mathbb{D})$ in Eq.(3.22). Furthermore, the operator $(K_{B,m}(z, w))_{\mathbf{D}_r}$ may be unitarily intertwined via the Bargmann-type transform $W_{B,m}$ in Eq.(3.24) to be acting on the quantum (or signals) Hilbert space $L^2(\mathbb{R}_+, d\xi)$ as a localization operator “à la Daubechies” [30] of the indicator function $\mathbf{1}_{\mathbf{D}_r}$ of the disk \mathbf{D}_r .

For these reasons, it would be useful to investigate properties of the random variable $\mathfrak{X}_m^{(R)}$ having $\mathfrak{Q}_j^{B,R,m}(\lambda)$ as its density function by first writing down its characteristic function which is defined, as usual, by the Fourier-type integral transform ([31, p.22]):

$$\phi_{\mathfrak{X}_m^{(R)}}(u) := \int_0^{R^2} e^{iu\lambda} \mathfrak{Q}_j^{B,R,m}(\lambda) d\lambda, \quad u \in \mathbb{R}. \quad (4.13)$$

Straightforward calculations (see Appendix B) lead to the expression

$$\begin{aligned} \phi_{\mathfrak{X}_m^{(R)}}(u) &= \tau_j^{B,R,m} (2(BR^2 - m) - 1) \sum_{k=0}^{2(m \wedge j)} C_{j,k}^{B,R,m} \Gamma(2(BR^2 - m) - 1 + k) \frac{(-iuR^2)^k}{k!} \\ &\times \frac{\Gamma(|m - j| + k + 1)}{\Gamma(2(BR^2 - m) + |m - j| + 2k)} {}_1F_1\left(\begin{matrix} |m - j| + k + 1 \\ 2(BR^2 - m) + |m - j| + 2k \end{matrix} \middle| iuR^2\right) \end{aligned} \quad (4.14)$$

where

$$\begin{aligned} C_{j,k}^{B,R,m} &= \frac{(2(m \wedge j))! (|m - j| + 1)_{2(m \wedge j)} ((2(BR^2 - m) + |m - j|)_{2(m \wedge j)})^2}{((m \wedge j)!)^2 ((2(BR^2 - m) + |m - j|)_{m \wedge j})^2} \\ &\times \frac{(-1)^k (2(BR^2 - m) + |m - j| - 1)_k (2k + 2(BR^2 - m) + |m - j| - 1)}{(2(m \wedge j) - k)! (|m - j| + 1)_k (2(BR^2 - m) + |m - j| - 1)_{2(m \wedge j) + k + 1}} \\ &\times F_{2:2:2}^{2:1:1}\left(\begin{matrix} -2(m \wedge j) + k, -2(BR^2 - m) - m - j + 1 - k, -m, -j, -m, -j \\ -2(m \wedge j), -m - j, -2(BR^2 - m) - m - j + 1, -2(BR^2 - m) - m - j + 1 \end{matrix} \middle| 1, 1\right). \end{aligned} \quad (4.15)$$

Here, $F_{2:2:2}^{2:1:1}$ denotes the Kampé de Fériet function of two variables defined by Eq.(2.13). For $m = 0$, Eq.(4.14) reduces to Eq.(4.11) as expected.

Now, by having $\phi_{\mathfrak{x}_m^{(R)}}(u)$, we immediately derive the main statistical parameters which are the mean value and the variance of $\mathfrak{x}_m^{(R)}$. Namely, we obtain the following formulas

$$\mathbb{E}(\mathfrak{x}_m^{(R)}) = \tau_j^{B,R,m} R^2 \frac{(|m-j|+1)! \Gamma(2(BR^2-m))}{\Gamma(2(BR^2-m)+|m-j|+1)} \left(C_{j,0}^{B,R,m} - \frac{(2(BR^2-m)-1)C_{j,1}^{B,R,m}}{(2(BR^2-m)+|m-j|+1)} \right) \quad (4.16)$$

and

$$\begin{aligned} \mathbb{V}ar(\mathfrak{x}_m^{(R)}) &= R^4 \tau_j^{B,R,m} \frac{(|m-j|+2)! \Gamma(2(BR^2-m))}{\Gamma(2(BR^2-m)+|m-j|+2)} \quad (4.17) \\ &\left(C_{j,0}^{B,R,m} - \frac{2((2(BR^2-m)-1)C_{j,1}^{B,R,m})}{(2(BR^2-m)+|m-j|+2)} + \frac{(2(BR^2-m)-1)(2(BR^2-m))C_{j,2}^{B,R,m}}{(2(BR^2-m)+|m-j|+2)(2(BR^2-m)+|m-j|+3)} \right) \\ &- \left[R^2 \tau_j^{B,R,m} \frac{(|m-j|+1)! \Gamma(2(BR^2-m))}{\Gamma(2(BR^2-m)+|m-j|+1)} \left(C_{j,0}^{B,R,m} - \frac{(2(BR^2-m)-1)C_{j,1}^{B,R,m}}{(2(BR^2-m)+|m-j|+1)} \right) \right]^2 \end{aligned}$$

For the proofs of Eq.(4.16) and Eq.(4.17) see Appendix D. In particular, for $m=0$ the above formulas reduce to

$$\mathbb{E}(\mathfrak{x}_0^{(R)}) = \frac{(j+1)R^2}{2BR^2+j}, \quad \mathbb{V}ar(\mathfrak{x}_0^{(R)}) = \frac{(j+1)(2BR^2-1)R^4}{(2BR^2+j)^2(2BR^2+j+1)}, \quad (4.18)$$

which are (up to a scale factor $1/R^2$) the mean value and variance of the Beta distribution $\mathcal{B}e(j+1, 2BR^2-1)$. Finally, we establish (see Appendix C) the following limit

$$\phi_{\mathfrak{x}_m^{(R)}}(u) \rightarrow \left(1 - \frac{iu}{2B}\right)^{-(m+j+1)} \sum_{k=0}^{m \wedge j} \binom{m}{k} \binom{j}{k} \left(\frac{iu}{2B}\right)^{2k}, \quad u < 2B \quad (4.19)$$

as $R \rightarrow +\infty$, which turn out to be (up to a scale factor $1/2B$) the characteristic function in the corresponding Euclidean setting in Eq.(4.6).

Remark 4.1. Note that $Q_j^{B,R,m}(z)$ vanishes at the origin $(0,0)$, the points of the circle of radius R and the points z such that that $R^2 - 2|z|^2$ are zeros of the Jacobi polynomials $P_{m \wedge j}^{(|m-j|, 2(B-m)-1)}(\cdot)$. Denoting by $x_i^{(m \wedge j)}$, $1 \leq i \leq m \wedge j$ the ordered consecutive zeros of this polynomial, the zeros of $Q_j^{B,R,m}(z)$ are then located on the concentric circles of radius $r_i = \sqrt{(R^2 - x_i^{(m \wedge j)})}/2$. A discussion on the number of zeros of the Husimi density and their location in connection with properties of harmonic and anaharmonic oscillators can be found in [32], particularly [4] in the Euclidean setting.

5. HUSIMI'S Q-FUNCTION FOR MIXED STATES

5.1. A Hamiltonian operator for the GCS. Recall that the functions $\langle \xi | j \rangle_{B,R,m}$ in the expansion (3.17) of the GCS $|\tilde{\kappa}_{z,B,R,m}\rangle$ were given by the normalized Laguerre functions (3.19) in $L^2(\mathbb{R}_+, d\xi)$. Moreover, one may use the Rodriguez formula for the Laguerre polynomial to show that these functions can also be obtained by a j -fold application of the first order differential operator

$$A_{\tau,m} = \frac{1}{\sqrt{2}} \left(-\frac{\tau - m - 1/2}{\xi} + \xi + \frac{d}{d\xi} \right), \quad \tau = BR^2, \quad (5.1)$$

on the ground state

$$\langle \xi | 0 \rangle_{B,R,m} := \left(\frac{2}{\Gamma(2(BR^2-m))} \right)^{\frac{1}{2}} \xi^{2(BR^2-m)-\frac{1}{2}} e^{-\frac{\xi^2}{2}}, \quad \xi \in \mathbb{R}_+, \quad (5.2)$$

as follows

$$|j\rangle_{B,R,m} = (A_{\tau,m})^j |0\rangle_{B,R,m}. \quad (5.3)$$

The formal adjoint operator of $A_{\tau,m}$ in $L^2(\mathbb{R}_+, d\xi)$ is given by

$$A_{\tau,m}^* = \frac{1}{\sqrt{2}} \left(-\frac{\tau - m - 1/2}{\xi} + \xi - \frac{d}{d\xi} \right). \quad (5.4)$$

Therefore we may associate to GCS $|\tilde{\kappa}_{z,B,R,m}\rangle$ the Hamiltonian operator

$$H_{\tau,m} = A_{\tau,m}^* A_{\tau,m} + 2(\tau - m). \quad (5.5)$$

Explicitly,

$$H_{\tau,m} := -\frac{1}{2} \frac{d^2}{d\xi^2} + \frac{(2(\tau - m) - 1)^2 - 1/4}{2\xi^2} + \frac{1}{2}\xi^2 \quad (5.6)$$

which appears in the literature under many names such as isotonic oscillator [33, 34], Gol'dman-Krivchenkov Hamiltonian [35], pseudoharmonic oscillator [36] or Laguerre operator [37], see also ([38, p.171]). The spectrum of $H_{\tau,m}$ in $L^2(\mathbb{R}_+, d\xi)$ is purely discrete and given by the eigenvalues

$$\eta_j = j + 4(\tau - m) - 1. \quad (5.7)$$

The latter ones, together with their associated eigenfunctions $|j\rangle_{B,R,m}$ can be used to define the heat semigroup associated with $H_{\tau,m}$ through the discret spectral resolution

$$e^{-\beta H_{\tau,m}} [f] := \sum_{j=0}^{+\infty} e^{-\beta \eta_j} \langle f | j \rangle_{B,R,m} \cdot |j\rangle_{B,R,m}, \quad f \in L^2(\mathbb{R}_+). \quad (5.8)$$

It is also well known that by using the Hille-Hardy formula ([39, p.242]), this semigroup has the integral representation

$$e^{-\beta H_{\tau,m}} [f] (r) = \int_0^{+\infty} W_{\beta}^{B,m}(r, \rho) f(\rho) d\rho \quad (5.9)$$

where

$$W_{\beta}^{B,m}(r, \rho) = \frac{2\sqrt{r\rho}e^{-\beta}}{(1 - e^{-2\beta})} \exp\left(-\frac{1}{2}(r^2 + \rho^2) \frac{1 + e^{-2\beta}}{1 - e^{-2\beta}}\right) I_{(2(BR^2 - m) - 1)}\left(\frac{2r\rho e^{-\beta}}{1 - e^{-2\beta}}\right). \quad (5.10)$$

Here $I_a(\cdot)$ denotes the modified Bessel function defined in Eq.(2.8).

5.2. Husimi's function. The standard statistical mechanics starts using the Gibb's canonical distribution, whose thermal density operator is represented by

$$\hat{\rho}_{\beta} = \frac{1}{Z} e^{-\beta \hat{H}} \quad (5.11)$$

where $Z = \text{Tr}(e^{-\beta \hat{H}})$ is the partition function, \hat{H} is the Hamiltonian of the system, $\beta = 1/(kT)$ the inverse temperature T and k the Boltzmann constant ($k = 1$). In our context, $\hat{H} \equiv H_{\tau,m}$ the isotonic oscillator and $\hat{\rho}_{\beta}$ is the associated normalized heat operator. The Husimi distribution is defined as the expectation value of the density operator $\hat{\rho}_{\beta}$ in the set of states $|\tilde{\kappa}_{z,B,R,m}\rangle$ as

$$Q_{\beta}^{B,R,m}(z) := \langle \tilde{\kappa}_{z,B,R,m} | \hat{\rho}_{\beta} | \tilde{\kappa}_{z,B,R,m} \rangle, \quad z \in \mathbb{D}_R. \quad (5.12)$$

By using Eq.(5.8), the R.H.S of Eq.(5.12) takes the form

$$Q_{\beta}^{B,R,m}(z) = \frac{1}{Z} \sum_{j=0}^{+\infty} e^{-\beta \eta_j} \left[Q_j^{B,R,m}(z) \right] \quad (5.13)$$

where $Q_j^{B,R,m}$ is the Husimi function for the pure state $|j\rangle\langle j|$ given by Eq.(4.3), where $|j\rangle \equiv |j\rangle_{B,R,m}$. Replacing the η_j by their expressions $j + 4(BR^2 - m) - 1$ and using direct calculations, we obtain the partition function as

$$Z = \frac{1}{1 - e^{-\beta}} e^{-(4(BR^2 - m) - 1)\beta} \quad (5.14)$$

such that

$$\int_{\mathbb{D}_R} Q_\beta^{B,R,m}(z) d\mu_{B,R,m}(z) = 1. \quad (5.15)$$

Therefore,

$$Q_\beta^{B,R,m}(z) = (1 - e^{-\beta}) \sum_{j=0}^{+\infty} (e^{-\beta})^j [Q_j^{B,R,m}(z)]. \quad (5.16)$$

Next, if we make the last sum look like the moment generating function of the generalized negative binomial distribution ([40, p.6]), then we get

$$\begin{aligned} Q_\beta^{B,R,m}(z) &= (1 - e^{-\beta}) \left(\frac{1 - \frac{z\bar{z}}{R^2}}{1 - \frac{z\bar{z}}{R^2} e^{-\beta}} \right)^{2BR^2} \left(\frac{(\frac{z\bar{z}}{R^2} - e^{-\beta})(1 - \frac{z\bar{z}}{R^2} e^{-\beta})}{(1 - \frac{z\bar{z}}{R^2})^2} \right)^m \\ &\times P_m^{(2(BR^2 - m) - 1, 0)} \left(1 + \frac{2e^{-\beta} (1 - \frac{z\bar{z}}{R^2})^2}{(\frac{z\bar{z}}{R^2} - e^{-\beta})(1 - \frac{z\bar{z}}{R^2} e^{-\beta})} \right). \end{aligned} \quad (5.17)$$

By letting $R \rightarrow \infty$ we prove (see Appendix E) that $Q_\beta^{B,R,m}(z) \rightarrow q_m(|\sqrt{2B}z|^2, \beta)$ where

$$q_m(|\sqrt{2B}z|^2, \beta) = (1 - e^{-\beta}) \exp\left(-2|\sqrt{2B}z|^2(1 - e^{-\beta})\right) e^{-\beta m} L_m^{(0)}\left(-4|\sqrt{2B}z|^2 \sinh^2 \frac{\beta}{2}\right) \quad (5.18)$$

is the Husimi function for mixed states of the flat case ([4, p.7]) as expected. Now, setting

$$\mathfrak{Q}_\beta^{B,R,m}(\lambda) := R^{-2} (2(BR^2 - m) - 1) (1 - \lambda R^{-2})^{-2} Q_\beta^{B,R,m}(\lambda), \quad (5.19)$$

with $\lambda = z\bar{z}$, Eq.(5.15) gives

$$\int_0^{R^2} \mathfrak{Q}_\beta^{B,R,m}(\lambda) d\lambda = 1. \quad (5.20)$$

Then, we denote by $\mathfrak{Y}_m^{(R)}$ the random variable having $\lambda \mapsto \mathfrak{Q}_\beta^{B,R,m}(\lambda)$ as its density function for which it would be useful to investigate some properties by starting by its characteristic function which is defined, as usual, by

$$\phi_{\mathfrak{Y}_m^{(R)}}(u) := \int_0^{R^2} e^{iu\lambda} \mathfrak{Q}_\beta^{B,R,m}(\lambda) d\lambda, \quad u \in \mathbb{R}. \quad (5.21)$$

Straightforward calculations (see Appendix F) lead to the expression

$$\begin{aligned} \phi_{\mathfrak{Y}_m^{(R)}}(u) &= (2(BR^2 - m) - 1) (1 - e^{-\beta}) e^{-m\beta} \sum_{k=0}^m \binom{2BR^2 - m - 1}{k} \binom{m}{k} \left(\frac{(1 - e^{-\beta})^2}{e^{-\beta}} \right)^k \\ &\mathcal{B}(k + 1, 2BR^2 - 2k - 1) \Phi_1 \left(\begin{matrix} k + 1, 2BR^2 \\ 2BR^2 - k \end{matrix} \middle| e^{-\beta}, -iuR^2 \right) \end{aligned} \quad (5.22)$$

where $\mathcal{B}(\cdot, \cdot)$ is the beta function and Φ_1 is the Humbert series defined in Eq.(2.14).

Now, by having $\phi_{\mathfrak{Y}_m^{(R)}}(u)$, we immediately derive (see Appendix G for the proof) the mean value

and the variance of $\mathfrak{Y}_m^{(R)}$ that are given respectively by

$$\begin{aligned} \mathbb{E} \left(\mathfrak{Y}_m^{(R)} \right) &= R^2 (2(BR^2 - m) - 1) (e^{-\beta} - 1) e^{-m\beta} \sum_{k=0}^m \binom{2BR^2 - m - 1}{k} \binom{m}{k} \left(\frac{(1 - e^{-\beta})^2}{e^{-\beta}} \right)^k \\ &\quad \mathcal{B}(k + 2, 2BR^2 - 2k - 1) {}_2F_1 \left(\begin{matrix} k + 2, 2BR^2 \\ 2BR^2 - k + 1 \end{matrix} \middle| e^{-\beta} \right) \end{aligned} \quad (5.23)$$

and

$$\begin{aligned} \text{Var}(\mathfrak{Y}_m^{(R)}) &= R^4 (2(BR^2 - m) - 1) (1 - e^{-\beta}) e^{-m\beta} \\ &\times \sum_{k=0}^m \binom{2BR^2 - m - 1}{k} \binom{m}{k} \left(\frac{(1 - e^{-\beta})^2}{e^{-\beta}} \right)^k \mathcal{B}(k + 3, 2BR^2 - 2k - 1) {}_2F_1 \left(\begin{matrix} k + 3, 2BR^2 \\ 2BR^2 - k + 2 \end{matrix} \middle| e^{-\beta} \right) \\ &- \left[R^2 (2(BR^2 - m) - 1) (1 - e^{-\beta}) e^{-m\beta} \sum_{k=0}^m \binom{2BR^2 - m - 1}{k} \binom{m}{k} \left(\frac{(1 - e^{-\beta})^2}{e^{-\beta}} \right)^k \right. \\ &\left. \times \mathcal{B}(k + 2, 2BR^2 - 2k - 1) {}_2F_1 \left(\begin{matrix} k + 2, 2BR^2 \\ 2BR^2 - k + 1 \end{matrix} \middle| e^{-\beta} \right) \right]^2. \end{aligned} \quad (5.24)$$

We also establish (see Appendix H) the following limit

$$\phi_{\mathfrak{Y}_m^{(R)}}(u) \rightarrow \frac{(1 - e^{-\beta})}{1 - e^{-\beta} - \frac{i u}{2B}} \left(\frac{1 - e^{-\beta} - \frac{i u}{2B} e^{-\beta}}{1 - e^{-\beta} - \frac{i u}{2B}} \right)^m, \quad u < 2B \quad (5.25)$$

as $R \rightarrow +\infty$, which turn out to be (up to a scale factor $\frac{1}{2B}$) the characteristic function in the Euclidean setting [4, p.8] as expected.

Remark 5.1. For $q_m(|\sqrt{2B}z|^2, \beta)$ in Eq.(5.18), we may set $\lambda = 2Bz\bar{z}$ and use a duality to define a non negative integer-valued random variable $\mathcal{Y}_{\lambda, \beta}$ by

$$\Pr(\mathcal{Y}_{\lambda, \beta}) = q_m(\lambda, \beta), \quad m = 0, 1, 2, \dots, \quad (5.26)$$

to recover the Laguerre probability distribution with parameters (λ, β) which can be presented as follows. Let $T = 1/k\beta$ represents in suitable units ($k = 1$), the absolute temperature and $N_T := (e^{\frac{1}{T}} - 1)^{-1}$ the average number of photons associated with the thermal noisy state, then the photon count probability distribution for a mixed light reads ([41]):

$$q_m(\lambda, \beta) = \frac{(N_T)^m}{(1 + N_T)^{m+1}} \exp\left(-\frac{\lambda}{1 + N_T}\right) L_m^{(0)}\left(-\frac{\lambda}{N_T(1 + N_T)}\right). \quad (5.27)$$

Finally, we can use the obtained Q-function $Q_\beta^{B, R, m}$ to get a lower bound for the thermodynamical potential associated with the Hamiltonian $H_{\tau, m}$, $\tau = BR^2$. This potential reads

$$\Theta_\beta^{\tau, \eta} := \frac{-1}{\beta} \text{Tr} \left(\log \left(1 + e^{-\beta(H_{\tau, m} - \eta)} \right) \right) \quad (5.28)$$

where η is the chemical potential. Putting $\epsilon = e^{\beta\eta}$, the form of the potential in Eq.(5.28) suggests us to consider the function

$$\zeta \mapsto \phi_\epsilon(\zeta) = -\log(1 + \epsilon\zeta). \quad (5.29)$$

So that we can rewrite Eq.(5.28) as

$$\beta\Theta_\beta^{\tau, \eta} = \text{Tr} \left(\phi_\epsilon \left(e^{-\beta H_{\tau, m}} \right) \right). \quad (5.30)$$

Applying the Berezin-Lieb inequality [6] for the lower symbol $Q_\beta^{B,R,m}$ of the operator $e^{-\beta H_{\tau,m}}$ as defined by Eq.(5.12), we obtain that

$$\int_{\mathbb{D}_R} [\phi_\epsilon \circ Q_\beta^{B,R,m}](z) d\mu_{B,R,m}(z) \leq \text{Tr}(\phi_\epsilon(e^{-\beta H_{\tau,m}})). \quad (5.31)$$

Making use of Eq.(5.29) and replacing the R.H.S of Eq.(5.31) by $\beta \Theta_\beta^{\tau,\eta}$ as in Eq.(5.30), we get an inequality that holds for every $m = 0, 1, \dots, \lfloor BR^2 - 1/2 \rfloor$. Therefore, we consider the maximum with respect to the integer m of the quantity in the L.H.S of Eq.(5.31) as

$$\max_{m \in \mathbb{Z}_+ \cap [0, BR^2 - 1/2]} \left[\frac{1}{\beta} \int_{\mathbb{D}_R} \log \left(\frac{1}{1 + \epsilon Q_\beta^{B,R,m}(z)} \right) d\mu_{B,R,m}(z) \right] \leq \Theta_\beta^{BR^2, \eta} \quad (5.32)$$

for every $\beta > 0$.

6. A FORMULA FOR A SPECIAL KAMPÉ DE FÉRIET FUNCTION

In this section we establish summation formula for the Kampé de Fériet function $F_{2:0:0}^{1:2:2}$, which will be limit in Eq.(4.5) (see appendix C).

Proposition 6.1. *Let $n \in \mathbb{N}^*$. For all real $x > 0$ or $x < -2\sqrt{2}$, the equality*

$$\sum_{k=0}^n \binom{n}{k} F_{2:0:0}^{1:2:2} \left(\begin{matrix} -n+k; a, b; c-a, c-b \\ -n, c; -; - \end{matrix} \middle| 1, 1 \right) x^k = (1+x)^n \left(\frac{x}{1+x} \right)^{a+b-c} {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| \frac{1+2x}{(1+x)^2} \right) \quad (6.1)$$

holds true. It also holds true for $x \in \mathbb{C}$ if $\max\{a, b\} \in \mathbb{Z}_-$ and $\max\{c-a, c-b\} \in \mathbb{Z}_-$.

Proof. We denote

$$S_n := \sum_{k=0}^n \binom{n}{k} F_{2:0:0}^{1:2:2} \left(\begin{matrix} -n+k; a, b; c-a, c-b \\ -n, c; -; - \end{matrix} \middle| 1, 1 \right) x^k. \quad (6.2)$$

for $n \in \mathbb{N}^*$. Using the definition of $F_{l:m:n}^{p:q:k}$ in Eq.(2.13) we obtain

$$S_n = \sum_{s,t \geq 0} \frac{(a)_s (b)_s (c-a)_t (c-b)_t}{(-n)_{s+t} (c)_{s+t}} \frac{1}{s!t!} \sum_{k=0}^n \binom{n}{k} (-n+k)_{s+t} x^k. \quad (6.3)$$

Since $(-n+k)_{s+t} = 0$ if $s+t > n-k$, the second sum in the R.H.S of Eq.(6.3) terminates as

$$\sum_{k=0}^n \binom{n}{k} (-n+k)_{s+t} x^k = (-n)_{s+t} \sum_{k=0}^{n-s-t} \binom{n-s-t}{k} x^k = (-n)_{s+t} (1+x)^{n-s-t} \quad (6.4)$$

when $(-n)_k = (-1)^k k! \binom{n}{k}$ for $0 \leq k \leq n$, $n \in \mathbb{N}^*$. Therefore, Eq.(6.3) takes the form

$$S_n = (1+x)^n \sum_{s,t \geq 0} \frac{(a)_s (b)_s (c-a)_t (c-b)_t}{(c)_{s+t}} \frac{1}{s!t!} \left(\frac{1}{1+x} \right)^{s+t} \quad (6.5)$$

which also reads

$$S_n = (1+x)^n \sum_{s \geq 0} \frac{(c-a)_s (c-b)_s}{(c)_s} {}_2F_1 \left(\begin{matrix} a, b \\ c+s \end{matrix} \middle| \frac{1}{1+x} \right) \frac{1}{s!} \left(\frac{1}{1+x} \right)^s. \quad (6.6)$$

Using the Euler transformation ([10, p.33]):

$${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| z \right) = (1-z)^{c-a-b} {}_2F_1 \left(\begin{matrix} c-a, c-b \\ c \end{matrix} \middle| z \right); \quad |\arg(1-z)| < \pi, \quad (6.7)$$

Eq.(6.6) becomes

$$S_n = (1+x)^n \left(\frac{x}{1+x} \right)^{c-a-b} \times \sum_{s \geq 0} \frac{(c-a)_s (c-b)_s}{s!(c)_s} \left(\frac{x}{(1+x)^2} \right)^s {}_2F_1 \left(\begin{matrix} c-a+s, c-b+s \\ c+s \end{matrix} \middle| \frac{1}{1+x} \right). \quad (6.8)$$

Next, by applying the generating formula ([42, p.348]):

$$\sum_{k \geq 0} \frac{(\bar{a})_k (\bar{b})_k}{k!(c)_k} t^k {}_2F_1 \left(\begin{matrix} \bar{a}+k, \bar{b}+k \\ c+k \end{matrix} \middle| y \right) = {}_2F_1 \left(\begin{matrix} \bar{a}, \bar{b} \\ c \end{matrix} \middle| t+y \right), \quad |t|, |y|, |t+y| < 1, \quad (6.9)$$

for $\bar{a} = c - a$, $\bar{b} = c - b$, $t = \frac{x}{(1+x)^2}$ and $y = \frac{1}{1+x}$, the sum S_n can be written in a closed form as

$$S_n = (1+x)^n \left(\frac{x}{1+x} \right)^{c-a-b} {}_2F_1 \left(\begin{matrix} c-a, c-b \\ c \end{matrix} \middle| \frac{1+2x}{(1+x)^2} \right). \quad (6.10)$$

where the conditions $|t|, |y|, |t+y| < 1$ translate into $x \in]-\infty, -2\sqrt{2}[\cup]0, +\infty[$. Note that if $\max\{a, b\} \in \mathbb{Z}_-$ and $\max\{c-a, c-b\} \in \mathbb{Z}_-$, the convergence conditions $|t|, |y|, |t+y| < 1$ are not necessary for applying the formula in Eq.(6.9) since both series in L.H.S and R.H.S of Eq.(6.8) will be finite sums. Then the resulting Eq.(6.10) holds true for all $x \in \mathbb{C}$ in this case. This completes the proof. \square

7. CONCLUDING REMARKS

We have been dealing with the Husimi Q-function of a density operator with respect to a set of coherent states attached to higher hyperbolic Landau levels and labeled by points of an open disk of radius R (phase space). For both a pure state representing a projector on a Fock state and the thermal density operator (mixed states) for the Hamiltonian of the isotonic oscillator, we explicitly have established formulas for the Q-function and we have written down the corresponding characteristic function from which the mean statistical parameters have been derived. We also have obtained a lower bound for a thermodynamical potential associated with the Hamiltonian under consideration. Most of the results of the Euclidean setting (flat case) as $R \rightarrow \infty$ have been recovered, by making use of asymptotic formulas involving special functions and orthogonal polynomials. In particular, and as tool, we obtained a summation formula for the Kampé de Fériet function $F_{2;0;0}^{-1;2;2}$. The obtained results may be useful for tackling the hyperbolic version of Ginibre-type processes [5]. They also can be exploited in the quantization procedure or localization operators "À la Daubechies" [30]. Finally, these Q-functions will be used to determine Wehrl entropies and discuss some related questions in our forthcoming paper.

APPENDIX A. PROOF OF EQ.(4.5)

By the asymptotic expansion for the ratio of the Gamma functions ([10, p.24]):

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta} \left(1 + \frac{(\alpha-\beta)(\alpha+\beta-1)}{2z} + \mathcal{O}(z^{-2}) \right) \quad (A.1)$$

$|z| \rightarrow \infty$, $|\arg(z)| \leq \pi - \varepsilon$, $0 < \varepsilon < \pi$, one obtains that

$$\frac{\Gamma(2(BR^2 - m) + m \vee j)}{\Gamma(2(BR^2 - m) + m \wedge j)} \simeq (2BR^2)^{|m-j|} \text{ as } R \rightarrow \infty. \quad (A.2)$$

Next, using the well known approximation ([43, p.196]):

$$\left(1 + \frac{x}{n} \right)^n \simeq e^x \text{ as } n \rightarrow \infty \quad (A.3)$$

we find that

$$\left(1 - \frac{|z|^2}{R^2}\right)^{2(BR^2-m)} = \left(1 - \frac{2B|z|^2}{2BR^2}\right)^{2BR^2} \left(1 - \frac{|z|^2}{R^2}\right)^{-2m} \simeq e^{-2B|z|^2} \text{ as } R \rightarrow \infty. \quad (\text{A.4})$$

By another side, we use the limiting formula ([39, p.247]):

$$L_n^{(a)}(x) = \lim_{b \rightarrow \infty} P_n^{(a,b)} \left(1 - \frac{2x}{b}\right) \quad (\text{A.5})$$

to show that

$$P_{m \wedge j}^{(|m-j|, 2(BR^2-m)-1)} \left(1 - \frac{2|z|^2}{R^2}\right) \simeq L_{m \wedge j}^{(|m-j|)}(2B|z|^2) \text{ as } R \rightarrow \infty. \quad (\text{A.6})$$

Finally by Eq.(A.2), Eq.(A.4) and Eq.(A.6) we arrive at

$$Q_j^{B,R,m}(z) \simeq \frac{(m \wedge j)!}{(m \vee j)!} e^{-2B|z|^2} (2B|z|^2)^{|m-j|} \left(L_{m \wedge j}^{(|m-j|)}(2B|z|^2)\right)^2 \text{ as } R \rightarrow \infty. \quad (\text{A.7})$$

The expression in the R.H.S of Eq.(A.7) is $Q_j^{(m), \text{Euclid}}(\sqrt{2B}z)$ as denoted in Eq.(4.5). \square

APPENDIX B. PROOF OF EQ.(4.14)

We start by recalling the characteristic function as defined in Eq.(4.13) by

$$\phi_{\mathfrak{x}_m^{(R)}}(u) = \int_0^{R^2} e^{iu\lambda} \mathfrak{Q}_j^{B,R,m}(\lambda) d\lambda. \quad (\text{B.1})$$

By Eq.(4.8) it can also be written as

$$\phi_{\mathfrak{x}_m^{(R)}}(u) = (2(BR^2 - m) - 1) \tau_j^{B,R,m} I_R \quad (\text{B.2})$$

where $\tau_j^{B,R,m}$ is given in Eq.(4.4) and

$$I_R := \int_0^1 e^{iuR^2x} (1-x)^{2(BR^2-m)-2} (x)^{|m-j|} \left(P_{m \wedge j}^{(|m-j|, 2(BR^2-m)-1)}(1-2x)\right)^2 dx. \quad (\text{B.3})$$

Next, we introduce the notations $b = 2(BR^2 - m) - 1$, $a = |m - j|$ and $l = m \wedge j$ with the variable change $v = 2x - 1$ in order to rewrite I_R as

$$I_R = \frac{1}{2^{a+b}} \int_{-1}^1 (v+1)^a (1-v)^{b-1} e^{\frac{iuR^2}{2}(v+1)} \left(P_l^{(a,b)}(-v)\right)^2 dv. \quad (\text{B.4})$$

For the square of the Jacobi polynomial in the last integral we may use the following Clebsh-Gordan type linearization formula ([44, p.611]):

$$P_i^{(\lambda,\delta)}(x) P_j^{(\mu,\gamma)}(x) = \sum_{k=0}^{i+j} C_k P_k^{(\alpha,\beta)}(x) \quad (\text{B.5})$$

where the coefficients

$$C_k = \frac{(i+j)!(\alpha+1)_{i+j}(\lambda+\delta+1)_{2i}(\mu+\gamma+1)_{2j}}{i!j!(\lambda+\delta+1)_i(\mu+\gamma+1)_j} \frac{(-1)^{i+j-k}(\alpha+\beta+1)_k(2k+\alpha+\beta+1)}{(i+j-k)!(\alpha+1)_k(\alpha+\beta+1)_{i+j+k+1}} \times F_{2:1:1}^{2:2:2} \left(\begin{matrix} -(i+j)+k, -\alpha-\beta-1-(i+j)-k; -i, -\lambda-i; -j, -\mu-j \\ -(i+j), -\alpha-(i+j); -2i-\lambda-\delta; -2j-\mu-\gamma \end{matrix} \middle| 1, 1 \right) \quad (\text{B.6})$$

are chosen with parameters $\lambda = \mu = a$, $\delta = \gamma = b$ and $i = j = l$. Therefore, I_R reads

$$I_R = \frac{1}{2^{a+b}} \sum_{k=0}^{2l} C_k (-1)^k \int_{-1}^{+1} (v+1)^a (1-v)^{b-1} e^{\frac{1}{2}iuR^2(v+1)} P_k^{(\beta,\alpha)}(v) dv \quad (\text{B.7})$$

with

$$C_k = \frac{(2l)!(\alpha+1)_{2l}((a+b+1)_{2l})^2 (-1)^{2l-k}(\alpha+\beta+1)_k(2k+\alpha+\beta+1)}{(l!)^2((a+b+1)_l)^2 (2l-k)!(\alpha+1)_k(\alpha+\beta+1)_{2l+k+1}} \times F_{2:2:2;2:1:1} \left(\begin{matrix} -2l+k, -\alpha-\beta-1-2l-k; -l, -a-l; -l, -a-l \\ -2l, -\alpha-2l; -2l-a-b; -2l-a-b \end{matrix} \middle| 1, 1 \right). \quad (\text{B.8})$$

Now, the integral in Eq.(B.7) can be computed by applying the formula (see [45, p.572]):

$$\int_{-\omega}^{\omega} (x+\omega)^{\gamma-1} (\omega-x)^{\rho} e^{-p(x+\omega)^r} P_n^{(\rho,\sigma)}\left(\frac{x}{\omega}\right) dx = \frac{(-1)^n}{n!} (2\omega)^{\gamma+\rho} \Gamma(\rho+n+1) \times \sum_{s \geq 0} \frac{1}{s!} (\sigma-\gamma-rs+1)_n \frac{\Gamma(\gamma+rs)}{\Gamma(\gamma+\rho+rs+n+1)} (-2^r \omega^r p)^s, \quad (\text{B.9})$$

$\omega, r, \Re\gamma > 0, \Re\rho > -1$ for parameters: $\omega = 1, \gamma = a+1 = |m-j|+1, \rho = \beta = b-1 = 2(BR^2-m)-2, p = -iuR^2/2, r = 1, \sigma = \alpha = \gamma-1 = |m-j|, n = k$. Summarizing the above calculations, Eq.(B.2) reads

$$\phi_{\mathfrak{x}_m^{(R)}}(u) = (2(BR^2-m)-1) \tau_j^{B,R,m} \sum_{k=0}^{2(m \wedge j)} C_{j,k}^{B,R,m} \frac{(-1)^k}{k!} \Gamma(2(BR^2-m)-1+k) \times \sum_{s \geq 0} \frac{1}{s!} \frac{\Gamma(|m-j|+s+k+1)}{\Gamma(2(BR^2-m)+|m-j|+s+2k)} (iuR^2)^{s+k} \quad (\text{B.10})$$

where

$$C_{j,k}^{B,R,m} = \frac{(2(m \wedge j))!(|m-j|+1)_{2(m \wedge j)} ((2(BR^2-m)+|m-j|)_{2(m \wedge j)})^2}{((m \wedge j)!)^2 ((2(BR^2-m)+|m-j|)_{m \wedge j})^2} \times \frac{(-1)^k (2(BR^2-m)+|m-j|-1)_k (2k+2(BR^2-m)+|m-j|-1)}{(2(m \wedge j)-k)!(|m-j|+1)_k (2(BR^2-m)+|m-j|-1)_{2(m \wedge j)+k+1}} \times F_{2:2:2;2:1:1} \left(\begin{matrix} -2(m \wedge j)+k, -2(BR^2-m)-m-j+1-k; -m, -j; -m, -j \\ -2(m \wedge j), -m-j; -2(BR^2-m)-m-j+1; -2(BR^2-m)-m-j+1 \end{matrix} \middle| 1, 1 \right) \quad (\text{B.11})$$

Finally, the series over s in the R.H.S of Eq.(B.10) can also be written in terms of a ${}_1F_1$ -series. This ends the proof. \square

APPENDIX C. PROOF OF EQ.(4.19)

First, we note that by using the asymptotic expansion Eq.(A.1)

$$\tau_{m,j}^{B,R} (2(BR^2-m)-1) \simeq \frac{(m \wedge j)!}{(m \vee j)!} (2BR^2)^{|m-j|+1} \quad \text{as } R \rightarrow \infty \quad (\text{C.1})$$

and

$$\frac{\Gamma(2(BR^2-m)-1+k)}{\Gamma(2(BR^2-m)+|m-j|+s+2k)} \simeq \frac{1}{(2BR^2)^{1+|m-j|+s+k}} \quad \text{as } R \rightarrow \infty. \quad (\text{C.2})$$

Next, we make us of the approximation formula which derive from Eq.(A.1)

$$(z)_a = z^a \left(1 + \frac{a(a-1)}{2z} + \mathcal{O}(z^{-2}) \right), \quad |z| \rightarrow \infty, \quad |\arg(z)| \leq \pi - \varepsilon, \quad 0 < \varepsilon < \pi \quad (\text{C.3})$$

to obtain

$$C_{j,k}^{B,R,m} \simeq \frac{(2(m \wedge j))!}{((m \wedge j)!)^2} \frac{(|m-j|+1)_{2(m \wedge j)} (-1)^k}{(2(m \wedge j)-k)!(|m-j|+1)_k} \times F_{2:2:2;2:1:1} \left(\begin{matrix} -2(m \wedge j)+k, -2BR^2; -m, -j; -m, -j \\ -2(m \wedge j), -m-j; -2BR^2; -2BR^2 \end{matrix} \middle| 1, 1 \right) \quad \text{as } R \rightarrow \infty. \quad (\text{C.4})$$

From the definition of $F_{l:m:n}^{p:q:k}$ in Eq.(2.13), we may rewrite Eq.(C.4) as

$$C_{j,k}^{B,R,m} \simeq \frac{(2(m \wedge j))!}{((m \wedge j)!)^2} \frac{(|m-j|+1)_{2(m \wedge j)} (-1)^k}{(2(m \wedge j) - k)! (|m-j|+1)_k} \quad (C.5)$$

$$\times F_{2:0:0}^{1:2:2} \left(\begin{matrix} -2(m \wedge j) + k; -m, -j; -m, -j \\ -2(m \wedge j), -m - j; -; - \end{matrix} \middle| 1, 1 \right) \quad \text{as } R \rightarrow \infty.$$

Using equations Eq.(C.1), Eq.(C.2) and Eq.(C.5) we get, after simplification, that

$$\phi_{\mathfrak{x}_m^{(R)}}(u) \simeq \frac{(m+j)!}{m!j!} \sum_{k=0}^{2(m \wedge j)} \binom{2(m \wedge j)}{k} F_{2:0:0}^{1:2:2} \left(\begin{matrix} -2(m \wedge j) + k; -m, -j; -m, -j \\ -2(m \wedge j), -m - j; -; - \end{matrix} \middle| 1, 1 \right)$$

$$\times \left(\frac{i u}{2B} \right)^k \sum_{s \geq 0} (|m-j|+k+1)_s \frac{\left(\frac{i u}{2B} \right)^s}{s!} \quad \text{as } R \rightarrow \infty \quad (C.6)$$

The infinite sum in Eq.(C.6) reduces to

$$\sum_{s \geq 0} \frac{1}{s!} (|m-j|+k+1)_s \left(\frac{i u}{2B} \right)^s = \left(1 - \frac{i u}{2B} \right)^{-|m-j|-k-1}, \quad u < 2B. \quad (C.7)$$

Then for $u < 2B$, Eq.(C.6) becomes

$$\phi_{\mathfrak{x}_m^{(R)}}(u) \simeq \frac{(m+j)!}{m!j!} \left(1 - \frac{i u}{2B} \right)^{-|m-j|-1}$$

$$\times \sum_{k=0}^{2(m \wedge j)} \binom{2(m \wedge j)}{k} F_{2:0:0}^{1:2:2} \left(\begin{matrix} -2(m \wedge j) + k; -m, -j; -m, -j \\ -2(m \wedge j), -m - j; -; - \end{matrix} \middle| 1, 1 \right) \left(\frac{i u}{2B - i u} \right)^k, \quad (C.8)$$

as $R \rightarrow \infty$. Applying Proposition 6.1 for parameters $n = 2(m \wedge j)$, $a = -m$, $b = -j$, $c = -m - j$ and $x = \frac{i u}{2B - i u}$, Eq.(C.8) reduces to

$$\phi_{\mathfrak{x}_m^{(R)}}(u) \simeq \frac{(m+j)!}{m!j!} \left(1 - \frac{i u}{2B} \right)^{-(m+j+1)} {}_2F_1 \left(\begin{matrix} -m, -j \\ -m - j \end{matrix} \middle| 1 + \left(\frac{u}{2B} \right)^2 \right), \quad u < 2B \quad (C.9)$$

as $R \rightarrow \infty$ for $m \wedge j > 0$. Then, we recover the announced limiting formula in Eq.(4.19) for $m \wedge j > 0$ using the connexion formula in Eq.(4.6). Now for $m \wedge j = 0$, we get directly

$$\phi_{\mathfrak{x}_m^{(R)}}(u) = {}_1F_1 \left(\begin{matrix} m \vee j + 1 \\ 2(BR^2 - m) + m \vee j \end{matrix} \middle| i u R^2 \right) \simeq \left(1 - \frac{i u}{2B} \right)^{-(m \vee j + 1)}, \quad u < 2B \quad (C.10)$$

as $R \rightarrow \infty$, that coincides to Eq.(4.19) putting $m \wedge j = 0$. This completes the proof. \square

APPENDIX D. PROOF OF EQS.(4.16)-(4.17)

Let us write the characteristic function $\phi_{\mathfrak{x}_m^{(R)}}(u)$ as

$$\phi_{\mathfrak{x}_m^{(R)}}(u) = \sum_{k=0}^{2(m \wedge j)} \sum_{s \geq 0} \delta_{j,k,s}^{B,R,m} (i u R^2)^{s+k}$$

$$= \sum_{s \geq 0} \delta_{j,0,s}^{B,R,m} (i u R^2)^s + \sum_{s \geq 0} \delta_{j,1,s}^{B,R,m} (i u R^2)^{s+1} + \sum_{k=2}^{2(m \wedge j)} \sum_{s \geq 0} \delta_{j,k,s}^{B,R,m} (i u R^2)^{s+k} \quad (D.1)$$

where

$$\delta_{j,k,s}^{B,R,m} := \tau_{m,j}^{B,R} (2(BR^2 - m) - 1) C_{j,k}^{B,R,m} \frac{(-1)^k \Gamma(2(BR^2 - m) - 1 + k) \Gamma(|m-j| + s + k + 1)}{k! s! \Gamma(2(BR^2 - m) + |m-j| + s + 2k)} \quad (D.2)$$

$\tau_{m,j}^{B,R}$ and $C_{j,k}^{B,R,m}$ being given respectively by Eq.(4.4) and Eq.(4.15). Then, the first derivative of $\phi_{\mathfrak{X}_m^{(R)}}(u)$ reads

$$\begin{aligned} \frac{d}{du}\phi_{\mathfrak{X}_m^{(R)}}(u) &= \left(\delta_{j,0,1}^{B,R,m} + \delta_{j,1,0}^{B,R,m}\right)(iR^2) + \sum_{s \geq 2} \delta_{j,0,s}^{B,R,m} s(iR^2)(iuR^2)^{s-1} \\ &+ \sum_{s \geq 1} \delta_{j,1,s}^{B,R,m} (s+1)(iR^2)(iuR^2)^s + \sum_{k=2}^{2(m \wedge j)} \sum_{s \geq 0} \delta_{j,k,s}^{B,R,m} (s+k)(iR^2)(iuR^2)^{s+k-1}. \end{aligned} \quad (\text{D.3})$$

The mean value of $\mathfrak{X}_m^{(R)}$ is then given by

$$\mathbb{E}\left(\mathfrak{X}_m^{(R)}\right) := -i \frac{d}{du} \phi_{\mathfrak{X}_m^{(R)}} \Big|_{u=0} = \left(\delta_{j,0,1}^{B,R,m} + \delta_{j,1,0}^{B,R,m}\right) R^2. \quad (\text{D.4})$$

where

$$\delta_{j,0,1}^{B,R,m} = \tau_{m,j}^{B,R} \frac{(|m-j|+1)! \Gamma(2(BR^2-m))}{\Gamma(2(BR^2-m) + |m-j| + 1)} C_{j,0}^{B,R,m} \quad (\text{D.5})$$

and

$$\delta_{j,1,0}^{B,R,m} = -\tau_{m,j}^{B,R} \frac{(|m-j|+1)! \Gamma(2(BR^2-m))}{\Gamma(2(BR^2-m) + |m-j| + 2)} (2(BR^2-m) - 1) C_{j,1}^{B,R,m}. \quad (\text{D.6})$$

This completes the proof of Eq.(4.16). For $m=0$, it is clear that $k=0$ and $C_{j,0}^{B,R,0} = 1$. Then

$$\mathbb{E}\left(\mathfrak{X}_0^{(R)}\right) = \kappa_j^{B,R,0} \frac{(j+1)! \Gamma(2BR^2)}{\Gamma(2BR^2 + j + 1)} R^2 = \frac{(j+1)R^2}{2BR^2 + j}. \quad (\text{D.7})$$

Now, for the variance, we search for the second derivative of $\phi_{\mathfrak{X}_m^{(R)}}(u)$. We rewrite Eq.(D.3) as

$$\begin{aligned} \frac{d}{du}\phi_{\mathfrak{X}_m^{(R)}}(u) &= \left(\delta_{j,0,1}^{B,R,m} + \delta_{j,1,0}^{B,R,m}\right)(iR^2) + \left(\delta_{j,0,2}^{B,R,m} + \delta_{j,1,1}^{B,R,m} + \delta_{j,2,0}^{B,R,m}\right) 2(iR^2)(iuR^2) \\ &+ \sum_{s \geq 3} \delta_{j,0,s}^{B,R,m} s(iR^2)(iuR^2)^{s-1} + \sum_{s \geq 2} \delta_{j,1,s}^{B,R,m} (s+1)(iR^2)(iuR^2)^s \\ &+ \sum_{s \geq 1} \delta_{j,2,s}^{B,R,m} (s+2)(iR^2)(iuR^2)^{s+1} + \sum_{k=3}^{2(m \wedge j)} \sum_{s \geq 0} \delta_{j,k,s}^{B,R,m} (s+k)(iR^2)(iuR^2)^{s+k-1}. \end{aligned} \quad (\text{D.8})$$

Then,

$$\begin{aligned} \frac{d^2}{du^2}\phi_{\mathfrak{X}_m^{(R)}}(u) &= \left(\delta_{j,0,2}^{B,R,m} + \delta_{j,1,1}^{B,R,m} + \delta_{j,2,0}^{B,R,m}\right) 2(iR^2)^2 + \sum_{s \geq 3} \delta_{j,0,s}^{B,R,m} (s-1)s(iR^2)^2(iuR^2)^{s-2} \\ &+ \sum_{s \geq 2} \delta_{j,1,s}^{B,R,m} s(s+1)(iR^2)^2(iuR^2)^{s-1} + \sum_{s \geq 1} \delta_{j,2,s}^{B,R,m} (s+1)(s+2)(iR^2)^2(iuR^2)^s \\ &+ \sum_{k=3}^{2(m \wedge j)} \sum_{s \geq 0} \delta_{j,k,s}^{B,R,m} (s+k-1)(s+k)(iR^2)^2(iuR^2)^{s+k-2}. \end{aligned} \quad (\text{D.9})$$

The evaluation at $u=0$ with the mean value gives the variance of $\mathfrak{X}_m^{(R)}$ as

$$\text{Var}\left(\mathfrak{X}_m^{(R)}\right) := -\frac{d^2}{du^2}\phi_{\mathfrak{X}_m^{(R)}}(u) \Big|_{u=0} - \mathbb{E}\left(\mathfrak{X}_m^{(R)}\right)^2 = \left(\delta_{j,0,2}^{B,R,m} + \delta_{j,1,1}^{B,R,m} + \delta_{j,2,0}^{B,R,m}\right) 2R^4 - \mathbb{E}\left(\mathfrak{X}_m^{(R)}\right)^2. \quad (\text{D.10})$$

Replacing

$$\delta_{j,0,2}^{B,R,m} = \tau_{m,j}^{B,R} \frac{(|m-j|+2)! \Gamma(2(BR^2-m))}{2\Gamma(2(BR^2-m) + |m-j| + 2)} C_{j,0}^{B,R,m}, \quad (\text{D.11})$$

$$\delta_{j,1,1}^{B,R,m} = -\tau_{m,j}^{B,R} \frac{(|m-j|+2)! \Gamma(2(BR^2-m))}{\Gamma(2(BR^2-m)+|m-j|+3)} (2(BR^2-m)-1) C_{j,1}^{B,R,m} \quad (\text{D.12})$$

and

$$\delta_{j,2,0}^{B,R,m} = \tau_{m,j}^{B,R} \frac{(|m-j|+2)! \Gamma(2(BR^2-m)+1)}{2\Gamma(2(BR^2-m)+|m-j|+4)} (2(BR^2-m)-1) C_{j,2}^{B,R,m} \quad (\text{D.13})$$

by their expressions into Eq.(D.10) gives the announced formula in Eq.(4.17). Like above, we find for $m=0$

$$\begin{aligned} \text{Var} \left(\mathfrak{X}_0^{(R)} \right) &= \kappa_j^{B,R,0} \frac{(j+2)! \Gamma(2BR^2)}{\Gamma(2BR^2+j+2)} \left(C_{j,0}^{B,R,0} \right) R^4 - \left(\kappa_j^{B,R,0} \frac{(j+1)! \Gamma(2BR^2)}{\Gamma(2BR^2+j+1)} R^2 \right)^2, \\ &= \frac{(j+1)(2BR^2-1)R^4}{(2BR^2+j)^2(2BR^2+j+1)}. \end{aligned} \quad (\text{D.14})$$

This ends the proof. \square

APPENDIX E. PROOF OF EQ.(5.24)

With the approximation Eq.(A.3) we find

$$\left(\frac{1 - \frac{z\bar{z}}{R^2}}{1 - \frac{z\bar{z}}{R^2} e^{-\beta}} \right)^{2BR^2} = \left(1 - \frac{2Bz\bar{z}}{2BR^2} \right)^{2BR^2} \left(1 + \frac{2Bz\bar{z}e^{-\beta}}{-2BR^2} \right)^{-2BR^2} \simeq \exp(-2Bz\bar{z}(1-e^{-\beta})) \quad (\text{E.1})$$

as $R \rightarrow \infty$. Direct calculation gives

$$\left(\frac{(\frac{z\bar{z}}{R^2} - e^{-\beta}) (1 - \frac{z\bar{z}}{R^2} e^{-\beta})}{(1 - \frac{z\bar{z}}{R^2})^2} \right)^m \simeq (-1)^m e^{-\beta m} \quad \text{as } R \rightarrow \infty, \quad (\text{E.2})$$

and

$$P_m^{(2(BR^2-m)-1,0)} \left(1 + \frac{2e^{-\beta} (1 - \frac{z\bar{z}}{R^2})^2}{(\frac{z\bar{z}}{R^2} - e^{-\beta}) (1 - \frac{z\bar{z}}{R^2} e^{-\beta})} \right) \simeq (-1)^m P_m^{(0,2BR^2)}(1), \quad \text{as } R \rightarrow \infty \quad (\text{E.3})$$

that becomes

$$P_m^{(2(BR^2-m)-1,0)} \left(1 + \frac{2e^{-\beta} (1 - \frac{z\bar{z}}{R^2})^2}{(\frac{z\bar{z}}{R^2} - e^{-\beta}) (1 - \frac{z\bar{z}}{R^2} e^{-\beta})} \right) \simeq (-1)^m L_m^{(0)} \left(-8Bz\bar{z} \sinh^2 \frac{\beta}{2} \right), \quad \text{as } R \rightarrow \infty \quad (\text{E.4})$$

by using the limiting formula Eq.(A.5). Then, it follow from Eq.(E.1), Eq.(E.2) and Eq.(E.4) that

$$Q_\beta^{B,R,m}(z) \simeq (1 - e^{-\beta}) \exp(-2Bz\bar{z}(1 - e^{-\beta})) e^{-\beta m} L_m^{(0)} \left(-8Bz\bar{z} \sinh^2 \frac{\beta}{2} \right), \quad \text{as } R \rightarrow \infty. \quad (\text{E.5})$$

Comparing Eq.(E.5) to [4, Eq.(3.14)], we recover the limiting formula $Q_\beta^{B,R,m}(z) \rightarrow q_m(|\sqrt{2B}z|^2, \beta)$ as $R \rightarrow \infty$. \square

APPENDIX F. PROOF OF EQ.(5.28)

Identify $\alpha = 2(BR^2 - m) - 1$ and $\xi = e^{-\beta}$ and set $x = \frac{\lambda}{R^2}$. Then, the characteristic function reads from Eq.(5.21) as

$$\begin{aligned} \phi_{\mathfrak{Y}_m^{(R)}}(u) &= \alpha(1-\xi) \int_0^1 e^{iuxR^2} \left(\frac{1-x}{1-x\xi} \right)^{\alpha+2m+1} \left(\frac{(x-\xi)(1-x\xi)}{(1-x)^2} \right)^m \\ &\quad \times P_m^{(\alpha,0)} \left(1 + \frac{2\xi(1-x)^2}{(x-\xi)(1-x\xi)} \right) \frac{dx}{(1-x)^2}. \end{aligned} \quad (\text{F.1})$$

By using the Jacobi polynomial definition in Eq.(2.9) one obtains the contraction

$$\left(\frac{(x-\xi)(1-x\xi)}{(1-x)^2}\right)^m P_m^{(\alpha,0)}\left(1+\frac{2\xi(1-x)^2}{(x-\xi)(1-x\xi)}\right) = \xi^m \sum_{k=0}^m \binom{m+\alpha}{k} \binom{m}{k} \left(\frac{(1-\xi)^2}{\xi}\right)^k \times \left(\frac{x}{(1-x)^2}\right)^k. \quad (\text{F.2})$$

Then, Eq.(F.1) becomes after reorganization

$$\phi_{\mathfrak{Y}_m^{(R)}}(u) = \frac{\alpha(1-\xi)\xi^m}{(-\xi)^{\alpha+2m+1}} \sum_{k=0}^m \binom{m+\alpha}{k} \binom{m}{k} \left(\frac{(1-\xi)^2}{\xi}\right)^k \int_0^1 e^{iuxR^2} x^k (1-x)^{\alpha+2m-2k-1} (x-\xi^{-1})^{-\alpha-2m-1} dx. \quad (\text{F.3})$$

Thanks to the formula ([46, p.328]):

$$\int_0^a x^{\tau-1} (a-x)^{\beta-1} (x+z)^{-\rho} e^{-px} dx = \mathcal{B}(\tau, \beta) z^{-\rho} a^{\tau+\beta-1} \Phi_1\left(\begin{matrix} \tau, \rho \\ \tau + \beta \end{matrix} \middle| -\frac{a}{z}, ap\right) \quad (\text{F.4})$$

where $a, \Re\tau, \Re\beta > 0; |\arg(1+a/z)| < \pi$. It application on the integral at the R.H.S of Eq.(F.3) for $a = 1, z = -\xi^{-1}, \tau = k + 1, \beta = \alpha + 2m - 2k, \rho = \alpha + 2m + 1$ and $p = -iuR^2$ gives after simplification

$$\phi_{\mathfrak{Y}_m^{(R)}}(u) = \alpha(1-\xi)\xi^m \sum_{k=0}^m \binom{m+\alpha}{k} \binom{m}{k} \left(\frac{(1-\xi)^2}{\xi}\right)^k \mathcal{B}(k+1, \alpha+2m-2k) \Phi_1\left(\begin{matrix} k+1, \alpha+2m+1 \\ \alpha+2m-k+1 \end{matrix} \middle| \xi, -iuR^2\right). \quad (\text{F.5})$$

Replacing α and ξ by their expressions ends the proof. \square

APPENDIX G. PROOF OF EQS.(5.29)-(5.30)

Due to the definition of the Humbert series in Eq.(2.14), its derivative reads

$$\frac{\partial}{\partial z} \Phi_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| w, z\right) = \frac{a}{c} \Phi_1\left(\begin{matrix} a+1, b \\ c+1 \end{matrix} \middle| w, z\right). \quad (\text{G.1})$$

Then, the derivative of $\phi_{\mathfrak{Y}_m^{(R)}}(u)$ in Eq.(F.5) is given by

$$\frac{d}{du} \phi_{\mathfrak{Y}_m^{(R)}}(u) = \alpha(1-\xi)\xi^m (-iR^2) \sum_{k=0}^m \binom{m+\alpha}{k} \binom{m}{k} \left(\frac{(1-\xi)^2}{\xi}\right)^k \mathcal{B}(k+2, \alpha+2m-2k) \Phi_1\left(\begin{matrix} k+2, \alpha+2m+1 \\ \alpha+2m-k+2 \end{matrix} \middle| \xi, -iuR^2\right). \quad (\text{G.2})$$

The evaluation at $u = 0$ taking into account the formula Eq.(2.15) gives the mean value of $\mathfrak{Y}_m^{(R)}$ as

$$\begin{aligned} \mathbb{E}\left(\mathfrak{Y}_m^{(R)}\right) &= -i \frac{d}{du} \phi_{\mathfrak{Y}_m^{(R)}}(u) \Big|_{u=0} \\ &= \alpha(\xi-1)\xi^m R^2 \sum_{k=0}^m \binom{m+\alpha}{k} \binom{m}{k} \left(\frac{(1-\xi)^2}{\xi}\right)^k \mathcal{B}(k+2, \alpha+2m-2k) {}_2F_1\left(\begin{matrix} k+2, \alpha+2m+1 \\ \alpha+2m-k+2 \end{matrix} \middle| \xi\right). \end{aligned} \quad (\text{G.3})$$

For the variance, we search for the second derivative of $\phi_{\mathfrak{Y}_m^{(R)}}(u)$ that reads from Eq.(G.1) and Eq.(G.2) as

$$\begin{aligned} \frac{d^2}{du^2} \phi_{\mathfrak{Y}_m^{(R)}}(u) &= -\alpha(1-\xi)\xi^m R^4 \sum_{k=0}^m \binom{m+\alpha}{k} \binom{m}{k} \left(\frac{(1-\xi)^2}{\xi}\right)^k \\ &\quad \mathcal{B}(k+3, \alpha+2m-2k) \Phi_1 \left(\begin{matrix} k+3, \alpha+2m+1 \\ \alpha+2m-k+3 \end{matrix} \middle| \xi, -iuR^2 \right). \end{aligned} \quad (\text{G.4})$$

The evaluation at $u=0$ taking into account the formula Eq.(2.15) gives

$$\begin{aligned} \left. \frac{d^2}{du^2} \phi_{\mathfrak{Y}_m^{(R)}}(u) \right|_{u=0} &= -\alpha(1-\xi)\xi^m R^4 \sum_{k=0}^m \binom{m+\alpha}{k} \binom{m}{k} \left(\frac{(1-\xi)^2}{\xi}\right)^k \\ &\quad \mathcal{B}(k+3, \alpha+2m-2k) {}_2F_1 \left(\begin{matrix} k+3, \alpha+2m+1 \\ \alpha+2m-k+3 \end{matrix} \middle| \xi \right). \end{aligned} \quad (\text{G.5})$$

The variance of $\mathfrak{Y}_m^{(R)}$ that is defined by

$$\text{Var} \left(\mathfrak{Y}_m^{(R)} \right) := -\frac{d^2}{du^2} \phi_{\mathfrak{Y}_m^{(R)}}(u) - \left(\mathbb{E} \left(\mathfrak{Y}_m^{(R)} \right) \right)^2 \quad (\text{G.6})$$

reads then as

$$\begin{aligned} \text{Var} \left(\mathfrak{Y}_m^{(R)} \right) &= \\ &\alpha(1-\xi)\xi^m R^4 \sum_{k=0}^m \binom{m+\alpha}{k} \binom{m}{k} \left(\frac{(1-\xi)^2}{\xi}\right)^k \mathcal{B}(k+3, \alpha+2m-2k) {}_2F_1 \left(\begin{matrix} k+3, \alpha+2m+1 \\ \alpha+2m-k+3 \end{matrix} \middle| \xi \right) - \\ &\left[\alpha(1-\xi)\xi^m R^2 \sum_{k=0}^m \binom{m+\alpha}{k} \binom{m}{k} \left(\frac{(1-\xi)^2}{\xi}\right)^k \mathcal{B}(k+2, \alpha+2m-2k) {}_2F_1 \left(\begin{matrix} k+2, \alpha+2m+1 \\ \alpha+2m-k+2 \end{matrix} \middle| \xi \right) \right]^2. \end{aligned} \quad (\text{G.7})$$

This ends the proof replacing α and ξ by their expressions. \square

APPENDIX H. PROOF OF EQ.(5.31)

By using the formula Eq.(A.1), we obtain

$$\binom{2BR^2 - m - 1}{k} \mathcal{B}(k+1, 2BR^2 - 2k - 1) = \frac{\Gamma(2BR^2 - m)}{\Gamma(2BR^2 - m - k)} \frac{\Gamma(2BR^2 - 2k - 1)}{\Gamma(2BR^2 - k)} \simeq \frac{1}{2BR^2} \quad (\text{H.1})$$

as $R \rightarrow +\infty$, that yields

$$(2(BR^2 - m) - 1) \binom{2BR^2 - m - 1}{k} \mathcal{B}(k+1, 2BR^2 - 2k - 1) \simeq 1, \quad \text{as } R \rightarrow \infty. \quad (\text{H.2})$$

Next, by the definition of Φ_1 in Eq.(2.14) and the use of the approximation formula Eq.(C.3) we obtain after simplification

$$\Phi_1 \left(\begin{matrix} k+1, 2BR^2 \\ 2BR^2 - k \end{matrix} \middle| e^{-\beta}, -iuR^2 \right) \simeq \sum_{s,t=0}^{+\infty} (k+1)_{s+t} \frac{e^{-s\beta}}{s!} \frac{\left(\frac{iu}{2B}\right)^t}{t!}, \quad \text{as } R \rightarrow \infty. \quad (\text{H.3})$$

By the use of the fact $(a)_{n+m} = (a)_n (a+n)_m$, the series at the R.H.S of Eq.(H.3) becomes

$$\sum_{s,t=0}^{+\infty} (k+1)_{s+t} \frac{e^{-s\beta}}{s!} \frac{\left(\frac{iu}{2B}\right)^t}{t!} = \sum_{s=0}^{+\infty} (k+1)_s \frac{e^{-s\beta}}{s!} \sum_{t=0}^{+\infty} (k+1+s)_t \frac{\left(\frac{iu}{2B}\right)^t}{t!}. \quad (\text{H.4})$$

Identifying the series (over t) at the R.H.S of Eq.(H.4) as a ${}_1F_0$, it follow for $u < 2B$

$$\begin{aligned} \sum_{s,t=0}^{+\infty} (k+1)_{s+t} \frac{e^{-s\beta} \left(\frac{i u}{2B}\right)^t}{s! t!} &= \sum_{s=0}^{+\infty} (k+1)_s \frac{e^{-s\beta}}{s!} \left(1 - \frac{i u}{2B}\right)^{-k-1-s} \\ &= \left(1 - \frac{i u}{2B}\right)^{-k-1} \sum_{s=0}^{+\infty} (k+1)_s \frac{[(1 - \frac{i u}{2B})^{-1} e^{-\beta}]^s}{s!} \end{aligned} \quad (\text{H.5})$$

that becomes identifying twice ${}_1F_0$ as

$$\begin{aligned} \sum_{s,t=0}^{+\infty} (k+1)_{s+t} \frac{e^{-s\beta} \left(\frac{i u}{2B}\right)^t}{s! t!} &= \left(1 - \frac{i u}{2B}\right)^{-k-1} \left(1 - (1 - \frac{i u}{2B})^{-1} e^{-\beta}\right)^{-k-1} \\ &= \left(1 - e^{-\beta} - \frac{i u}{2B}\right)^{-k-1}. \end{aligned} \quad (\text{H.6})$$

Then, Eq.(H.3) becomes

$$\Phi_1 \left(\begin{matrix} k+1, 2BR^2 \\ 2BR^2 - k \end{matrix} \middle| e^{-\beta}, -i u R^2 \right) \simeq \left(1 - e^{-\beta} - \frac{i u}{2B}\right)^{-k-1}, \quad \text{as } R \rightarrow \infty \quad (\text{H.7})$$

for $u < 2B$. Thus, from Eq.(H.2) and Eq.(H.7), $\phi_{\mathfrak{Y}_m^{(R)}}(u)$ has as limit when $R \rightarrow \infty$

$$\phi_{\mathfrak{Y}_m^{(R)}}(u) \simeq (1 - e^{-\beta}) e^{-m\beta} \sum_{k=0}^m \binom{m}{k} \left(\frac{(1 - e^{-\beta})^2}{e^{-\beta}}\right)^k \left(1 - e^{-\beta} - \frac{i u}{2B}\right)^{-k-1}, \quad u < 2B \quad (\text{H.8})$$

that reads successively

$$\begin{aligned} \phi_{\mathfrak{Y}_m^{(R)}}(u) &\simeq \frac{(1 - e^{-\beta}) e^{-m\beta}}{1 - e^{-\beta} - \frac{i u}{2B}} \sum_{k=0}^m \binom{m}{k} \left(\frac{(1 - e^{-\beta})^2}{e^{-\beta}}\right)^k \left(1 - e^{-\beta} - \frac{i u}{2B}\right)^{-k-1} \\ &\simeq \frac{(1 - e^{-\beta}) e^{-m\beta}}{1 - e^{-\beta} - \frac{i u}{2B}} \left(1 + \frac{(1 - e^{-\beta})^2}{e^{-\beta} (1 - e^{-\beta} - \frac{i u}{2B})}\right)^m \\ &\simeq \frac{(1 - e^{-\beta})}{1 - e^{-\beta} - \frac{i u}{2B}} \left(\frac{1 - e^{-\beta} - \frac{i u}{2B} e^{-\beta}}{1 - e^{-\beta} - \frac{i u}{2B}}\right)^m, \quad u < 2B. \end{aligned} \quad (\text{H.9})$$

This ends the proof by comparing Eq.(H.9) with ([4, Eq.(3.24)]). \square

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Email address: *mouayn@usms.ma, ^bchhaiba.hassan@gmail.com [‡]hamidoukass@gmail.com, [†]Kayupepatrick@gmail.com