微分几何作业

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5.1.4. 验证曲面的平均曲率$H=\dfrac{1}{2}b_{\alpha\beta}g^{\alpha\beta}$,且在参数变换

$$(u^{1'},u^{2'})\to(u^1,u^2),a^{\alpha}_{\alpha'}=\frac{\partial u^{\alpha}}{\partial u^{\alpha'}},\det(a^{\alpha}_{\alpha'})>0$$

下不变.

证明. 首先,$H=\dfrac{1}{2}\mathrm{tr}(b^{\beta}_{\alpha})=\dfrac{1}{2}b^{\alpha}_{\alpha}=\dfrac{1}{2}b_{\alpha\beta}g^{\beta\alpha}$.其次,此变换即正向参数变换,有$b_{\alpha\beta}=b'_{\alpha\beta}$,且$g_{\alpha\beta}=g'_{\alpha\beta}$,故$g^{\alpha\beta}=g'^{\alpha\beta}$.故得证. □

5.2.1. 在本节定理条件假设下,推导

$$\begin{aligned}
f_{\alpha\beta}(u)&=(\boldsymbol{r}^{(1)}_{\alpha}-\boldsymbol{r}^{(2)}_{\alpha})\cdot(\boldsymbol{r}^{(1)}_{\beta}-\boldsymbol{r}^{(2)}_{\beta}),\\
f_{\alpha}(u)&=(\boldsymbol{r}^{(1)}_{\alpha}-\boldsymbol{r}^{(2)}_{\alpha})\cdot(\boldsymbol{n}^{(1)}-\boldsymbol{n}^{(2)}),\\
f(u)&=(\boldsymbol{n}^{(1)}-\boldsymbol{n}^{(2)})^2
\end{aligned}$$

所满足的方程组

$$\begin{cases}
\dfrac{\partial f_{\alpha\beta}}{\partial u^{\gamma}}=\Gamma^{\delta}_{\gamma\alpha}f_{\delta\beta}+\Gamma^{\delta}_{\gamma\beta}f_{\alpha\delta}+b_{\gamma\alpha}f_{\beta}+b_{\gamma\beta}f_{\alpha},\\
\dfrac{\partial f_{\alpha}}{\partial u^{\gamma}}=-b^{\delta}_{\gamma}f_{\alpha\delta}+\Gamma^{\delta}_{\alpha\gamma}f_{\delta}+b_{\alpha\gamma}f,\\
\dfrac{\partial f}{\partial u^{\gamma}}=-2b^{\delta}_{\gamma}f_{\delta}.
\end{cases}$$

证明. 首先,由于两曲面的第一第二基本形式相同,故有同一个$(g_{\alpha\beta}),(b_{\alpha\beta})$,故也有相同的$\Gamma^{\gamma}_{\alpha\beta}$.因此:

$$\begin{aligned}
\frac{\partial f_{\alpha\beta}}{\partial u^{\gamma}}&=(\boldsymbol{r}^{(1)}_{\alpha}-\boldsymbol{r}^{(2)}_{\alpha})\cdot\frac{\partial(\boldsymbol{r}^{(1)}_{\beta}-\boldsymbol{r}^{(2)}_{\beta})}{u^{\gamma}}+\frac{\partial(\boldsymbol{r}^{(1)}_{\alpha}-\boldsymbol{r}^{(2)}_{\alpha})}{u^{\gamma}}\cdot(\boldsymbol{r}^{(1)}_{\beta}-\boldsymbol{r}^{(2)}_{\beta})\\
&=(\boldsymbol{r}^{(1)}_{\alpha}-\boldsymbol{r}^{(2)}_{\alpha})\cdot\Big(\Gamma^{\delta}_{\beta\gamma}\boldsymbol{r}^{(1)}_{\delta}+b_{\beta\gamma}\boldsymbol{n}^{(1)}-\Gamma^{\delta}_{\beta\gamma}\boldsymbol{r}^{(2)}_{\delta}-b_{\beta\gamma}\boldsymbol{n}^{(2)}\Big)\\
&+\Big(\Gamma^{\delta}_{\alpha\gamma}\boldsymbol{r}^{(1)}_{\delta}+b_{\alpha\gamma}\boldsymbol{n}^{(1)}-\Gamma^{\delta}_{\alpha\gamma}\boldsymbol{r}^{(2)}_{\delta}-b_{\alpha\gamma}\boldsymbol{n}^{(2)}\Big)\cdot(\boldsymbol{r}^{(1)}_{\beta}-\boldsymbol{r}^{(2)}_{\beta})\\
&=\Gamma^{\delta}_{\beta\gamma}(\boldsymbol{r}^{(1)}_{\alpha}-\boldsymbol{r}^{(2)}_{\alpha})\cdot(\boldsymbol{r}^{(1)}_{\delta}-\boldsymbol{r}^{(2)}_{\delta})+b_{\beta\gamma}(\boldsymbol{r}^{(1)}_{\alpha}-\boldsymbol{r}^{(2)}_{\alpha})\cdot(\boldsymbol{n}^{(1)}-\boldsymbol{n}^{(2)})\\
&+\Gamma^{\delta}_{\alpha\gamma}(\boldsymbol{r}^{(1)}_{\beta}-\boldsymbol{r}^{(2)}_{\beta})\cdot(\boldsymbol{r}^{(1)}_{\delta}-\boldsymbol{r}^{(2)}_{\delta})+b_{\alpha\gamma}(\boldsymbol{r}^{(1)}_{\beta}-\boldsymbol{r}^{(2)}_{\beta})\cdot(\boldsymbol{n}^{(1)}-\boldsymbol{n}^{(2)})\\
&=\Gamma^{\delta}_{\beta\gamma}f_{\alpha\delta}+\Gamma^{\delta}_{\alpha\gamma}f_{\beta\delta}+b_{\alpha\gamma}f_{\beta}+b_{\beta\gamma}f_{\alpha}.\\
\frac{\partial f_{\alpha}}{\partial u^{\gamma}}&=(\boldsymbol{r}^{(1)}_{\alpha}-\boldsymbol{r}^{(2)}_{\alpha})\cdot\frac{\partial(\boldsymbol{n}^{(1)}-\boldsymbol{n}^{(2)})}{u^{\gamma}}+\frac{\partial(\boldsymbol{r}^{(1)}_{\alpha}-\boldsymbol{r}^{(2)}_{\alpha})}{u^{\gamma}}\cdot(\boldsymbol{n}^{(1)}-\boldsymbol{n}^{(2)})\\
&=(\boldsymbol{r}^{(1)}_{\alpha}-\boldsymbol{r}^{(2)}_{\alpha})\cdot(-b^{\delta}_{\gamma}\boldsymbol{r}^{(1)}_{\delta}+b^{\delta}_{\gamma}\boldsymbol{r}^{(2)}_{\delta})+\Big(\Gamma^{\delta}_{\alpha\gamma}\boldsymbol{r}^{(1)}_{\delta}+b_{\alpha\gamma}\boldsymbol{n}^{(1)}-\Gamma^{\delta}_{\alpha\gamma}\boldsymbol{r}^{(2)}_{\delta}-b_{\alpha\gamma}\boldsymbol{n}^{(2)}\Big)\cdot(\boldsymbol{n}^{(1)}-\boldsymbol{n}^{(2)})\\
&=-b^{\delta}_{\gamma}f_{\alpha\delta}+\Gamma^{\delta}_{\alpha\gamma}(\boldsymbol{r}^{(1)}_{\delta}-\boldsymbol{r}^{(2)}_{\delta})(\boldsymbol{n}^{(1)}-\boldsymbol{n}^{(2)})+b_{\alpha\gamma}(\boldsymbol{n}^{(1)}-\boldsymbol{n}^{(2)})^2\\
&=-b^{\delta}_{\gamma}f_{\alpha\delta}+\Gamma^{\delta}_{\alpha\gamma}f_{\delta}+b_{\alpha\gamma}f.\\
\frac{\partial f}{\partial u^{\gamma}}&=2(\boldsymbol{n}^{(1)}-\boldsymbol{n}^{(2)})\cdot\frac{\partial(\boldsymbol{n}^{(1)}-\boldsymbol{n}^{(2)})}{u^{\gamma}}=2(\boldsymbol{n}^{(1)}-\boldsymbol{n}^{(2)})\cdot(-b^{\delta}_{\gamma}\boldsymbol{r}^{(1)}_{\delta}+b^{\delta}_{\gamma}\boldsymbol{r}^{(2)}_{\delta})=-2b^{\delta}_{\gamma}f_{\delta}.
\end{aligned}$$

□

5.4.1. 验证

$$f_{\alpha\beta}(u)=\boldsymbol{r}_\alpha\cdot\boldsymbol{r}_\beta-g_{\alpha\beta},\qquad f_\alpha(u)=\boldsymbol{r}_\alpha\cdot\boldsymbol{n},\qquad f(u)=\boldsymbol{n}^2-1$$

也满足上述方程组.

证明.

$$\begin{aligned}\frac{\partial f_{\alpha\beta}}{\partial u^\gamma}&=\frac{\partial \boldsymbol{r}_\alpha}{\partial u^\gamma}\boldsymbol{r}_\beta+\boldsymbol{r}_\alpha\frac{\partial \boldsymbol{r}_\beta}{\partial u^\gamma}-\frac{\partial g_{\alpha\beta}}{\partial u^\gamma}=\left(\Gamma^\delta_{\alpha\gamma}\boldsymbol{r}_\delta+b_{\alpha\gamma}\boldsymbol{n}\right)\boldsymbol{r}_\beta+\boldsymbol{r}_\alpha\left(\Gamma^\delta_{\beta\gamma}\boldsymbol{r}_\delta+b_{\beta\gamma}\boldsymbol{n}\right)-\frac{\partial g_{\alpha\beta}}{\partial u^\gamma}\\&=\Gamma^\delta_{\alpha\gamma}(f_{\beta\delta}+g_{\beta\delta})+\Gamma^\delta_{\beta\gamma}(f_{\alpha\delta}+g_{\alpha\delta})+b_{\alpha\gamma}f_\beta+b_{\beta\gamma}f_\alpha-\frac{\partial g_{\alpha\beta}}{\partial u^\gamma}\\&=\Gamma^\delta_{\beta\gamma}f_{\alpha\delta}+\Gamma^\delta_{\alpha\gamma}f_{\beta\delta}+b_{\alpha\gamma}f_\beta+b_{\beta\gamma}f_\alpha\end{aligned}$$

因为其中

$$\begin{aligned}\Gamma^\delta_{\alpha\gamma}g_{\beta\delta}+\Gamma^\delta_{\beta\gamma}g_{\alpha\delta}-\frac{\partial g_{\alpha\beta}}{\partial u^\gamma}&=\Gamma_{\beta\alpha\gamma}+\Gamma_{\alpha\beta\gamma}-\frac{\partial g_{\alpha\beta}}{\partial u^\gamma}\\&=\frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial u^\gamma}+\frac{\partial g_{\beta\gamma}}{\partial u^\alpha}-\frac{\partial g_{\alpha\gamma}}{\partial u^\beta}+\frac{\partial g_{\alpha\beta}}{\partial u^\gamma}+\frac{\partial g_{\alpha\gamma}}{\partial u^\beta}-\frac{\partial g_{\beta\gamma}}{\partial u^\alpha}-2\frac{\partial g_{\alpha\beta}}{\partial u^\gamma}\right)=0\end{aligned}$$

其次,

$$\begin{aligned}\frac{\partial f_\alpha}{\partial u^\gamma}&=\frac{\partial \boldsymbol{r}_\alpha}{\partial u^\gamma}\boldsymbol{n}+\boldsymbol{r}_\alpha\frac{\partial \boldsymbol{n}}{\partial u^\gamma}=\left(\Gamma^\delta_{\alpha\gamma}\boldsymbol{r}_\delta+b_{\alpha\gamma}\boldsymbol{n}\right)\boldsymbol{n}+\boldsymbol{r}_\alpha\left(-b^\delta_\gamma\boldsymbol{r}_\delta\right)\\&=\Gamma^\delta_{\alpha\gamma}f_\delta+b_{\alpha\gamma}(f+1)-b^\delta_\gamma(f_{\alpha\delta}+g_{\alpha\delta})=-b^\delta_\gamma f_{\alpha\delta}+\Gamma^\delta_{\alpha\gamma}f_\delta+b_{\alpha\gamma}f\\\frac{\partial f}{\partial u^\gamma}&=2\boldsymbol{n}\frac{\partial \boldsymbol{n}}{\partial u^\gamma}=-2b^\delta_\gamma\boldsymbol{r}_\delta\boldsymbol{n}=-2b^\delta_\gamma f_\delta\end{aligned}$$

□

5.5.1. 求具有下列第一基本形式的曲面的Gauss曲率,其中a,c均为常数.

1. $\mathbf{I}=\dfrac{\mathrm{d}u^2+\mathrm{d}v^2}{\left(1+\frac{c}{4}(u^2+v^2)\right)^2}$,
2. $\mathbf{I}=\dfrac{a^2}{v^2}(\mathrm{d}u^2+\mathrm{d}v^2), v>0$,
3. $\mathbf{I}=\dfrac{\mathrm{d}u^2+\mathrm{d}v^2}{u^2+v^2+c}, c>0$,
4. $\mathbf{I}=\mathrm{d}u^2+\mathrm{e}^{\frac{2u}{a}}\mathrm{d}v^2$,
5. $\mathbf{I}=\mathrm{d}u^2+\cosh^2\dfrac{u}{a}\mathrm{d}v^2$.

证明. 1.$\lambda=\left(1+\dfrac{c}{4}(u^2+v^2)\right)^{-1}, \ln\lambda=-\ln\left(1+\dfrac{c}{4}(u^2+v^2)\right), \Delta\ln\lambda=-\dfrac{c}{(1+\frac{c}{4}(u^2+v^2))^2}, K=-\dfrac{\Delta\ln\lambda}{\lambda^2}=c$.

2.$\lambda=\dfrac{|a|}{v}, \ln\lambda=\ln|a|-\ln v, \Delta\ln\lambda=v^{-2}, K=-\dfrac{v^{-2}}{a^2v^{-2}}=-a^{-2}$.

3.$\lambda^2=(u^2+v^2+c)^{-1}, \ln\lambda=-\dfrac{1}{2}\ln(u^2+v^2+c), \Delta\ln\lambda=-\dfrac{2c}{(u^2+v^2+c)^2}, K=\dfrac{2c(u^2+v^2+c)^{-2}}{(u^2+v^2+c)^{-1}}=\dfrac{2c}{u^2+v^2+c}$.

4.$\sqrt{E}=1, \sqrt{G}=\mathrm{e}^{\frac{u}{a}}$,因此

$$K=-\frac{1}{\sqrt{EG}}\left(\partial_v\left(\frac{\partial_v\sqrt{E}}{\sqrt{G}}\right)+\partial_u\left(\frac{\partial_u\sqrt{G}}{\sqrt{E}}\right)\right)=-\mathrm{e}^{-\frac{u}{a}}\left(0+\partial_u\left(\frac{\mathrm{e}^{\frac{u}{a}}}{a}\right)\right)=-\mathrm{e}^{-\frac{u}{a}}\frac{\mathrm{e}^{\frac{u}{a}}}{a^2}=-a^{-2}$$

5.$\sqrt{E}=1, \sqrt{G}=\cosh\dfrac{u}{a}$,因此

$$K=-\frac{1}{\sqrt{EG}}\left(\partial_v\left(\frac{\partial_v\sqrt{E}}{\sqrt{G}}\right)+\partial_u\left(\frac{\partial_u\sqrt{G}}{\sqrt{E}}\right)\right)=-\frac{1}{\cosh(\frac{u}{a})}\left(0+\partial_u\left(\frac{\sinh(\frac{u}{a})}{2a\cosh(\frac{u}{a})}\right)\right)=-\frac{\cosh(\frac{u}{a})a^{-2}}{\cosh(\frac{u}{a})}=-a^{-2}$$

□

5.5.4. 设曲面S和$\bar{S}$的第一基本形式分别为

$$\mathbf{I}=\mathrm{e}^{2v}\big(\mathrm{d}u^2+a^2(1+u^2)\mathrm{d}v^2\big)\,,\qquad \bar{\mathbf{I}}=\mathrm{e}^{2\bar{v}}\big(\mathrm{d}\bar{u}^2+b^2(1+\bar{u}^2)\mathrm{d}\bar{v}^2\big)$$

其中$a^2\neq b^2$.证明在$\bar{u}=u,\bar{v}=v$的对应下,S和$\bar{S}$有相同的Gauss曲率,但该对应不是保长对应.

证明. $\sqrt{E}=\mathrm{e}^v$, $\sqrt{G}=|a|\mathrm{e}^v\sqrt{1+u^2}$,因此

$$\begin{aligned}K&=-\frac{1}{|a|\mathrm{e}^{2v}\sqrt{1+u^2}}\left(\partial_v\left(\frac{\mathrm{e}^v}{|a|\mathrm{e}^v\sqrt{1+u^2}}\right)+\partial_u\left(\frac{|a|u\mathrm{e}^v}{\sqrt{1+u^2}}\cdot\frac{1}{\mathrm{e}^v}\right)\right)\\&=-\frac{1}{|a|\mathrm{e}^{2v}\sqrt{1+u^2}}\left(0+|a|\partial_u\frac{u}{\sqrt{1+u^2}}\right)=-\frac{1}{\mathrm{e}^{2v}\sqrt{1+u^2}}\frac{1}{(1+u^2)^{3/2}}=-\frac{\mathrm{e}^{2v}}{(1+u^2)^2}\end{aligned}$$

这与系数a无关,因此两曲面有相同的Gauss曲率.但显然$\mathbf{I}\neq\bar{\mathbf{I}}$,因此不是保长对应. □

6.1.2. 证明:旋转面上纬线的测地曲率是常数,其倒数为过纬线上一点的经线的切线从切点到切线与旋转轴交点间的长度.

证明. 旋转面$\boldsymbol{r}(u,v)=(f(u)\cos v,f(u)\sin v,g(u))$的纬线$u=u_0,v=v$有$s=f(u_0)v$使其为弧长参数.旋转面的第一类基本量为$E=f'^2+g'^2,F=0,G=f^2$,因此$v$-曲线上

$$\kappa_g=\frac{\partial_u\ln G}{2\sqrt{E}}=\frac{f'(u_0)}{f(u_0)\sqrt{f'^2(u_0)+g'^2(u_0)}}$$

□

6.2.6. 已知曲面的第一基本形式如下,求曲面上的测地线:1.$\mathbf{I}=v(\mathrm{d}u^2+\mathrm{d}v^2)$;2.$\mathbf{I}=\dfrac{a^2}{v^2}(\mathrm{d}u^2+\mathrm{d}v^2)$.

证明. 仅需解微分方程组

$$\begin{cases}\dfrac{\mathrm{d}u}{\mathrm{d}s}=\dfrac{\cos\theta}{\sqrt{E}}\\\dfrac{\mathrm{d}v}{\mathrm{d}s}=\dfrac{\sin\theta}{\sqrt{G}}\\\dfrac{\mathrm{d}\theta}{\mathrm{d}s}=\dfrac{\partial_v\ln E}{2\sqrt{G}}\cos\theta-\dfrac{\partial_u\ln G}{2\sqrt{E}}\sin\theta\end{cases}$$

1.$E=G=v$,因此

$$\frac{\mathrm{d}u}{\mathrm{d}s}=v^{-\frac12}\cos\theta,\qquad\frac{\mathrm{d}v}{\mathrm{d}s}=v^{-\frac12}\sin\theta,\qquad\frac{\mathrm{d}\theta}{\mathrm{d}s}=\frac{v^{-\frac32}}{2}\cos\theta$$

显然

$$\frac{\mathrm{d}\theta}{\mathrm{d}v}=\frac{1}{2v\tan\theta},\tan\theta\mathrm{d}\theta=\frac{\mathrm{d}v}{2v},-\ln\cos\theta=\frac{\ln v}{2}+C',\cos\theta=\frac{C_1}{\sqrt{v}},\sin\theta=\sqrt{1-\frac{C_1^2}{v}}$$

代入有

$$\frac{\mathrm{d}v}{\mathrm{d}u}=\tan\theta=\frac{\sqrt{1-C_1^2/v}}{C_1/\sqrt{v}}=\sqrt{\frac{v}{C_1}-1},\qquad u=\int\frac{\mathrm{d}v}{\sqrt{v/C_1-1}}=2C_1\sqrt{\frac{v}{C_1}-1}+C_2$$

故$v=\dfrac{(u-C_2)^2}{4C_1}+C_1$.

2.$E=G=\dfrac{a^2}{v^2}$,因此有

$$\frac{\mathrm{d}u}{\mathrm{d}s}=\frac{v}{a}\cos\theta,\qquad\frac{\mathrm{d}v}{\mathrm{d}s}=\frac{v}{a}\sin\theta,\qquad\frac{\mathrm{d}\theta}{\mathrm{d}s}=-\frac{\cos\theta}{a}.$$

因此

$$\frac{\mathrm{d}\theta}{\mathrm{d}v}=-\frac{1}{v\tan\theta},\tan\theta\mathrm{d}\theta=-\frac{\mathrm{d}v}{v},-\ln\cos\theta=C'-\ln v,\cos\theta=C_1v,\sin\theta=\sqrt{1-C_1^2v^2}$$

代入有

$$\frac{\mathrm{d}v}{\mathrm{d}u}=\tan\theta=\frac{\sqrt{1-C_1^2v^2}}{C_1v},\qquad u=\int\frac{C_1v\mathrm{d}v}{\sqrt{1-C_1^2v^2}}=C_2-\frac{\sqrt{1-C_1^2v^2}}{C_1}$$

故$v=\sqrt{C_1^{-2}-(C_2-u)^2}=\sqrt{-u^2+Au+B}$. □