Hall's Theorem

Let G be a bipartite graph with vertex sets V_1 and V_2 and edge set E. A *complete matching* $M \subset E$ from V_1 to V_2 is a set of $m = |V_1|$ independent edges in G. In a complete matching M, each vertex in V_1 is incident with precisely one edge from M.

For a subset $S \subset V_1$, write

$$\Gamma(S) = \{v \in V_2 : uv \in E \text{ for some } u \in V_1\} \subset V_2.$$

Theorem 1. *A bipartite graph G with vertex sets V_1 and V_2 contains a complete matching from V_1 to V_2 if and only if it satisfies* Hall's condition

$$|\Gamma(S)| \geq |S| \text{ for every } S \subset V_1.$$

Proof. First, we observe that Hall's condition is clearly *necessary*. To prove that it is also *sufficient*, we use induction on m. The theorem is true for $m = 1$, so assume that G satisfies Hall's condition and that $m = |V_1| \geq 2$.

Case 1. Suppose that for **all** *proper* subsets $S \neq \emptyset$ of V_1,

$$|\Gamma(S)| \geq |S| + 1.$$

Then we can start with an arbitrary edge $e = v_1v_2 \in E$, and put e in M. The graph $G' = G - \{v_1, v_2\}$ satisfies Hall's condition, so we can complete the matching by induction.

Case 2. Suppose that for **some** proper subset $T \neq \emptyset$ of V_1,

$$|\Gamma(T)| = |T|.$$

Applying the induction hypothesis to $G' = G[T \cup \Gamma(T)]$ and $G'' = G[(V_1 \setminus T) \cup (V_2 \setminus \Gamma(T))]$ (you should check for yourself that G'' satisfies Hall's condition), we obtain two disjoint matchings containing $|T|$ and $m - |T|$ edges respectively, whose union is a complete matching from V_1 to V_2. $\square$

One of my friends summed up this proof as follows: "either it's trivial, or it's trivial".