Intergenerational transmission is not sufficient for positive long-term population growth

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Abstract

All leading long-term global population projections agree on continuing fertility decline, resulting in a rate of population size growth that will continue to decline towards zero and would eventually turn negative. However, scholarly and popular arguments have suggested that because fertility transmits intergenerationally (*i.e.*, higher-fertility parents tend to have higher-fertility children) and is heterogeneous within a population, long-term population growth must eventually be positive, as the high-fertility types come to dominate the population. In this research note, we show that intergenerational transmission of fertility is not sufficient for positive long-term population growth, for empirical and theoretical reasons. First, because transmission is imperfect, the combination of transmission rates and fertility rates may be quantitatively insufficient for long-term population growth: it may be that higher-fertility parents nevertheless produce too few children who retain higher-fertility preferences. Second, today even higher-fertility sub-populations show declining fertility rates, which may eventually be below replacement (and in some populations already are). Therefore, although different models of fertility transmission across generations reach different conclusions, depopulation is likely under any model if, in the future, even higher-fertility sub-populations prefer and achieve below-replacement fertility. These results highlight the plausibility of long-term global depopulation and the importance of understanding the possible consequences of depopulation.

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Introduction

Global fertility is projected to fall below replacement levels, which will eventually cause negative population growth (KC and Lutz, 2017; United Nations, Department of Economic and Social Affairs, Population Division, 2019; Vollset et al., 2020). Survey evidence confirms agreement among demographers that fertility will continue to fall (Gietel-Basten et al., 2014). However, some recent publications that apply models of intergenerational transmission from the mathematical biology literature propose that low fertility is unlikely to endure and global population growth is unlikely to become negative (Collins and Page, 2019; Burger and DeLong, 2016; Murphy and Wang, 2003; Ellis et al., 2017). These arguments also appear in popular-audience accounts of fertility (Kaufmann, 2010; Ingraham, 2015). Such authors reason that high fertility is intergenerationally transmissible, and so, if there are higher and lower fertility patterns exhibited within sub-populations, eventually the composition of the population will converge towards the higher-fertility pattern.[1]

Here we observe that—even granting the premise that higher fertility parents have higher fertility children with high probability—such intergenerational transmission is not sufficient for positive long-term population growth (LTPG).[2] One reason is that researchers should not conflate *higher* fertility within a heterogeneous population with *high* or *above-replacement* fertility: it is an empirical question whether future higher-fertility sub-populations will have above replacement fertility. If not, then population growth will be negative. There is strong historical and global evidence that even higher-fertility groups will trend to near or below replacement fertility. The second reason is that the existence of a sub-population with above-replacement fertility is not sufficient for positive LTPG, even with intergenerational transmission. This is because it might be that an insufficient number of children of high-fertility parents retain their parents' behaviors—that is, even if fertility is correlated within a family across generations, the correlation may be less than 1.0.

There are other convergent arguments in the literature. Boyd and Richerson (1988, p. 199-202), for example, detail mechanisms by which cultural and biological transmission could together cause sustained fertility decline. Kolk et al. (2014), in a point allied to but separate from ours, highlight that fertility heterogeneity and intergenerational transmission could be consistent with long-term

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[1] In fact, the model of Collins and Page (2019) implies the stronger claim that aggregate fertility rates are *always* increasing via this mechanism in post- demographic transition settings, counter to the experiences of developed countries over the previous half-century.

[2] Our note complements the empirical findings of Vogl (2020), who does not focus on long-term future population growth, but quantifies with survey data that any effect of intergenerational transmission on aggregate fertility has historically been small.

population decline, but only if new low-fertility groups ("types") are culturally invented again and again in the future. One of our contributions is to show that this is possible in a model with just two types.

Because we respond to a literature concerned with both genetic and social transmission and because we claim that intergenerational transmission is not sufficient for positive growth, we do not specify or limit our arguments to either genetic or social transmission of parental traits: Our arguments show that *neither* of these forms of transmission are sufficient for positive LTPG.

We formalize conditions under which intergenerational transmission does not cause positive LTPG: if enough children of high-fertility parents become low-fertility adults, long-term population growth can be negative even with both intergenerational transmission and an above-replacement-fertility sub-population. Whether modeled population growth is positive or negative in the long run depends on model structure, parameters, and initial conditions, but we show it is not guaranteed by the mere fact of intergenerational transmission.[3] We also show that depopulation is likely under *any* model if, in the future, even higher-fertility sub-populations prefer and achieve below-replacement fertility, as empirical facts suggest.

Evidence of fertility decline among high fertility groups

Figure 1 documents fertility trends in 48 different countries using data from the Demographic and Health Survey (DHS).[4] These 48 countries comprise nearly 45 percent of the world's population and 60 percent of births each year. These countries are primarily developing and emerging economies, which is useful for our purposes because these are, in general, the populations for which above-replacement fertility currently exists. In Figure 1, the horizontal axis is cohort (year of birth) binned into five-year increments from 1950 to 1989; the vertical axis is the average parity at age 30 of women in that cohort bin.[5] Each thin line represents a different country, and the thick line represents the average across countries. All but two of the 48 countries have decreasing fertility rates. The evidence offers no reason to conclude that the downward trend will stop above replacement levels. Indeed,

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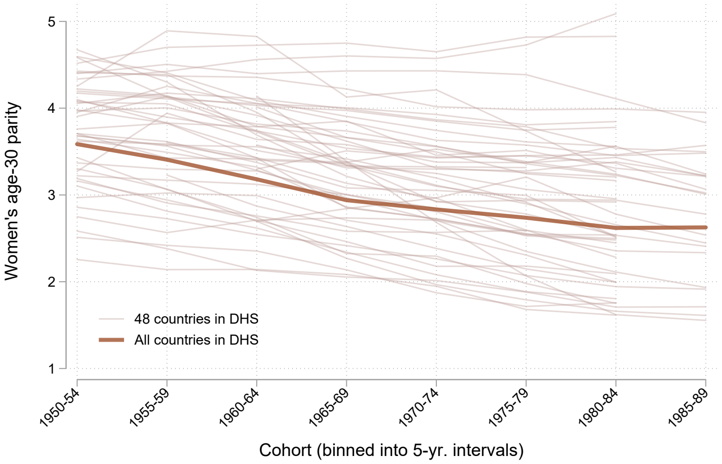
[3] Our purpose here is not to establish which is the correct model of intergenerational transmission of fertility heterogeneity.

[4] The set of 48 countries is the subset of all DHS countries for which at least 500 women are interviewed in at least 6 of the 8 cohort bins pictured.

[5] In principle, such cohort fertility rates could be declining over time merely because women are delaying fertility to later ages; however, the declining trends documented in this section also appear for later-age cohort fertility (restricting attention to earlier cohorts).

Figure 1: Fertility decline is found in diverse populations: DHS



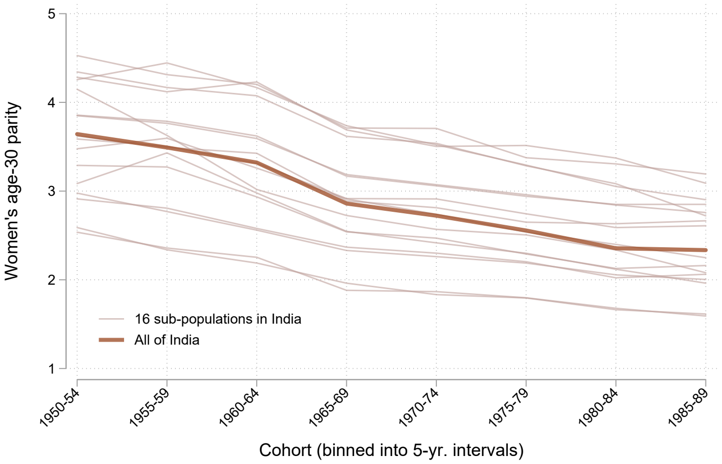
Note: This figure shows women's age-30 parity by birth cohort for 48 different countries. Data are taken from the Demographic and Health Surveys. The horizontal axis is cohort (year of birth) binned into five-year increments from 1950 to 1989; the vertical axis is the average parity at age 30 of women in that cohort bin. Each thin line represents a different country. The thick line represents the average across countries.

many of even these emerging and developing economies are already below or near replacement level.

Figure 2 focuses on India, which accounts for one-sixth of the world's population and has historically been a focus of global population policy debates (Connelly, 2010). India is home to diverse sub-populations, who differ, among other ways, in their average fertility levels. In Figure 2, we categorize women in India in the DHS into 16 non-overlapping groups by interacting indicators for: south-India/other-states, rural/urban, Muslim/not-Muslim, and no-education/some-education. For example, one of the thin lines represents South-Indian, rural, Muslim women with some education. The axes are as before: cohort and fertility, respectively. Each of the 16 groups demonstrates a clear downward slope. That is, as with nearly all countries in the DHS, all major sub-populations within India—including those with the highest levels of fertility—have declining fertility.

These empirical facts cast doubt on a necessary condition for positive LTPG: that *higher*-fertility sub-populations will sustain *high* (that is, above replacement) levels of fertility. The necessity of this condition can be seen clearly by considering the mathematical biology model of Kolk et al. (2014). Their Model 1 describes the long-term evolution of the composition of a population with two fertility types: higher and lower. In their model, the composition of the population converges entirely to the higher-type. But the quantitative fertility level of the higher-fertility types is an unconstrained,

Figure 2: Fertility decline is found in diverse sub-populations: India



Note: This figure shows women's age-30 parity by cohort for 16 different sub-populations in India. The 16 non-overlapping groups are generated by interacting indicators for: north-India/south-India, rural/urban, Muslim/not-Muslim, and no-education/some-education. Data are taken from the Demographic and Health Survey. The horizontal axis of this figure is cohort (year of birth) binned into five-year increments from 1950 to 1989; the vertical axis is the average parity at age 30 of women in that cohort bin. Each thin line represents a different sub-population. The thick line represents the average for India overall.

exogenous parameter of their model.[6] If higher-fertility types have below-replacement fertility (as Figures 1 and 2 suggest may someday happen) then the model of Kolk et al. (2014) would project a depopulating world, even as the composition shifts to higher-fertility types.

Figures 1 and 2 focus on developing countries, where fertility is generally highest today. But discussions of intergenerational transmission in fertility are often motivated by reference to higher fertility among religious sub-populations in developed countries, such as the United States (Kaufmann, 2010; Ingraham, 2015; Ellis et al., 2017). Although these observations are not essential to our argument, we note two important facts about demographic patterns in the United States. First, fertility rates among religious Americans, despite a consistently higher level than among non-religious Americans, are falling approximately in parallel to fertility rates for the whole population (Perry and Schleifer, 2019). For example, the National Survey of Family Growth (NSFG) shows that, between cohorts born in the 1940s and cohorts born in the 1970s, completed fertility for both religious

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[6] In contrast, the post-demographic transition model of Collins and Page (2019) does not contain any parameter for average desired fertility levels. Instead, the fertility of the next generation is a function (representing what they call "heritability") only of the fertility of the last generation. Because such a model does not admit cultural, social, or economic influences on desired and achieved fertility other than through this narrow intergenerational transmission channel, we interpret it as difficult to reconcile with the empirical facts of Figures 1 and 2, which reflect children of high-fertility parents transitioning to radically lower-fertility behavior over the course of only a few generations. Therefore, we depart from their model by modeling the empirically-relevant possibility that post-demographic-transition fertility could, in part, reflect changes (such as in preferences) beyond or in addition to intergenerational transmission.

and non-religious women (operationalized as religious service attendance) fell by approximately 20 percent. Second, such group identities—whether defined by religion, educational attainment, rural/urban status, or cultural conservatism—are not transmitted perfectly across generations. The NSFG shows that the fraction of Americans who report being religious is falling over time, from 57 percent in 1988 to 43 percent in 2019. We take up the implications of such imperfect intergenerational transmission next.

A stylized two-type model of population dynamics

So far we have provided empirical evidence that many groups are trending towards below-replacement fertility. However, some authors have argued that if even one group remains above replacement, then this high-fertility group would eventually drive population fertility towards their high rate (e.g. Collins and Page, 2019). In this section, we show analytically that the existence of such a sub-group is not a sufficient condition for positive LTPG. Instead, we demonstrate in a two-type model that long-run population decline can exist in a world with a sub-group that has both above-replacement fertility and (imperfect) vertical transmission of fertility from mother to daughter. The intuition is this: If high-fertility parents have children at above-replacement levels, but only some fraction of those children receive the high-fertility type, then the size of the high fertility group (and the overall population) can nonetheless decline.

Denote high- and low-fertility types $i \in \{H, L\}$ with reproductive rates $F_H > 1 > F_L$, respectively, where we have simplified to a single-sex environment such that a reproductive rate 1 is replacement level.[7] This two-type model with transmissible fertility from parents to offspring builds on the structure employed by Kolk et al. (2014).[8] In our model, we emphasize that children's received fertility type, which is a function solely of their parent's type, is imperfectly transmitted. In particular, the offspring of type i retain their parent's fertility preferences with probability $p_{i\to i}$ and switch types with probability $(1 - p_{i\to i})$.

For simplicity, we focus on the case in which the fertility rate and transition probabilities

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[7] This single-sex model allows us to ignore assortative mating and is equivalent to assuming that only females' types matter. The most literal reading of our model is that the probability that a daughter is high or low type is a function only of whether her mother is high or low type. Our explicit assumption of a single-sex model serves to clarify our departure from a model where genetic transmission depends, through parental matching, on the proportion of a trait in the population—though see Model D in the online Supplementary Materials for an example of negative LTPG in a two-sex model with matching and transmission dependent on the proportion of the trait in the population.

[8] Cavalli-Sforza and Feldman (1981) also use a simplified two-type model of vertical transmission.

$(F_i, p_{i\to i})$ are constant for each type. Such a special case with fixed transmission probabilities is called a Markov model. Because we are not interested in deriving or characterizing constant (or other) equilibrium *shares* of the population,[9] but instead are interested in the asymptotic *total size* of the population, we need not assume fixed Markov probabilities: It would be sufficient for our purposes if transition probabilities fluctuate, are linked to the proportion of the trait in the population, or otherwise evolve but are bounded by the inequality we derive below (Condition 2).[10] Fixed probabilities would be incompatible with standard models of genetic inheritance in which genetic transmission is dependent on the proportion of a trait in a population.[11] However, for simplicity we assume fixed Markov probabilities, knowing that our model can be immediately relaxed in this way.

In this setting the evolution of types can be written as follows, where N_i is the number of types in each period.

$$\mathbf{N}_{t+1} = \mathbf{A}\mathbf{N}_t \tag{1}$$

where:

$$\mathbf{N} = \begin{bmatrix} N_H \\ N_L \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} p_{H\to H}F_H & (1-p_{L\to L})F_L \\ (1-p_{H\to H})F_H & p_{L\to L}F_L \end{bmatrix},$$

and the transition matrix $\mathbf{A}$ specifies how types evolve. The number of high types at $t+1$ equals $N_{H,t}(p_{H\to H}F_H) + N_{L,t}((1-p_{L\to L})F_L)$, which could be greater or less than the number of high types at t, depending on model parameters. Because, by construction, the low-types cannot sustain their numbers without inflow ($F_L < 1$), it is straightforward to show that there is positive LTPG only if there is long-run growth in the subset of high-types.[12]

To further simplify the initial exposition of high-type dynamics, assume that $p_{L\to L} = 1$. That is, children of low types receive that type with certainty. This exact assumption is not necessary for

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[9] Compare Preston and Campbell (1993), who use a Markov model of differential fertility to study the share of the population with intergenerationally transmissible traits. Lam (1993) observes in response that their argument for convergence to a steady-state composition of the population is driven by the setting of fixed Markov transition probabilities. Because we have a different theoretical goal, we need not assume a fixed Markov matrix, but we do so for ease of exposition.

[10] For example, it could be the case that $p_{H\to H}$ increases over the generations but F_H decreases, so that Condition (2), introduced below, is always met.

[11] For example, in population geneticists' simple Hardy-Weinberg model of a randomly-mating, stable population, the observed proportion of a genotype trait is a non-linear function of its genetic prevalence because parents of a type must match. See details in Falconer (1960).

[12] In the case where high-types grow, temporary population decline is possible if the low-types shrink sufficiently fast at the start; eventually, however, only high-types are left and their population growth necessarily takes over.

the model's main qualitative conclusions, and we relax it in the appendix, but it generates a simple and intuitive condition for long-term decline. Combining this one-way switching assumption with the general property of Markov processes that $\mathbf{N}_t = \mathbf{A}^t\mathbf{N}_0$, it can be shown that the population of high-types evolves according to $N_{H,t} = (p_{H\to H} \times F_H)^t N_{H,0}$. The high-types decline—and therefore the long-run aggregate population size declines—if

$$p_{H\to H} \times F_H < 1. \tag{2}$$

Condition (2) is the essential, intuitive requirement for negative LTPG. Even if transition probabilities are not fixed in a Markov sense, LTPG is negative if there is a time after which Condition (2) is always true.

In the online appendix, we relax the assumption that $p_{L\to L} = 1$. We prove, with fixed transmission probabilities, that positive LTPG will not occur if

$$p_{H\to H} \times F_H < \frac{1 - F_H F_L + F_L(F_H - 1)p_{L\to L}}{1 - F_L}, \tag{3}$$

which can be satisfied by a range of feasible and empirically plausible parameters. This condition simplifies to Condition (2) if $p_{L\to L} = 1$ or $F_L = 0$.

For example, consider a world in which the average high-fertility woman has 1.2 female children and the average low-fertility woman has 0.33 female children. Assume the higher fertility group is culturally attractive, so that a child of a high-fertility parent has an 80% chance of becoming a high-fertility adult (and 20% chance of becoming a low-fertility adult), and a child of a low-fertility parent has only a 75% chance of becoming a low-fertility adult (and 25% chance of becoming a high-fertility adult). In this world, the intergenerational correlation of parents' and children's fertility would be high, but Condition (3) is met, so the size of the population would decrease over time.[13]

Note that if this model were the true data generating process in some population, then a regression of one's parent's fertility on own-fertility in that population would return a positive coefficient. This is exactly the type of empirical correlation pointed to (incorrectly) as evidence that high fertility patterns will come to dominate a population and lead to long-term population growth. Here, the

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[13] See the online Supplementary Materials for the calculations corresponding to this numerical example (Model A). These materials also contain examples with transmission probabilities that vary exogenously over time (Model B) and transmission probabilities that are a function of the proportion of the trait in the population (Model C for a single-sex case, and Model D for a two-sex case with matching dynamics). B, C, and D all generate negative LTPG despite the fact that $F_H > 1$ and $p_{t,H\to H} > p_{t,L\to L}$ for all generations, t.

correlation arises in a statistical process in which long-term fertility and population both asymptote to zero. This is the key error in this literature: the existence of (high, positive) mother-daughter fertility correlations and a higher-fertility subgroup are not enough to infer whether population size will increase over time.

Finally, to return to the point of our empirical section, F_H may itself decline below replacement, which would ensure that $F_H \times p_{H \to H} < 1$. In fact, although the model of Kolk et al. (2014) differs from ours in assuming that transmission probabilities depend upon the composition of the population (we model them as fixed), their model and ours agree on this implication, in the empirical case where *higher* fertility becomes *low*. Especially as fertility rates become low, fertility outcomes are importantly shaped by fertility preferences, choices, and intentions (*e.g.* Pritchett, 1994; Goldstein et al., 2003; Gietel-Basten, 2019; Yeatman et al., 2020). One key way in which human population dynamics differ from the mathematical dynamics of non-human populations is the importance of fertility determinants such as culture, economics, preferences, and contraception (Kohler and Rodgers, 2003).

Discussion

Negative population growth, if it occurs, may have many consequences for societies and economies (Morgan, 2003; Jones, 2020). Here we respond to a literature motivated by mathematical biology that intends to cast doubt on projections of depopulation. Human fertility is unlike other animals' because it is shaped by culture, economics, and intention. Building on the work of Kolk et al. (2014), we have shown here that intergenerational transmission of fertility is not sufficient to prevent long-run population decline.

To generate positive LTPG in the simple model above, the number of children who retain the high fertility preferences of their parents must exceed replacement; it is not sufficient merely that higher-fertility types have above-replacement fertility, even with transmissibility. The condition for positive LTPG is met through some combination of *both* high fertility rates and low net outflow. Empirical evidence presented here and elsewhere suggests that even "higher-fertility" types of the future may prefer and achieve fertility rates near (or even below) replacement. This would leave little (or no) room for positive LTPG in the presence of any intergenerational outflow under any theoretical model.

Positive LTPG depends on the facts: there may yet arise high-fertility groups with sufficiently low intergenerational outflow to meet the required conditions. Evidence broadly suggests that most socio-economic properties show imperfect intergenerational correlation, including important examples like religious practice, political affiliation, and income (Vogl and Freese, 2020; Chetty et al., 2014). Fertility itself has been recently examined in post-demographic-transition populations and has been shown to have positive but low intergenerational correlation, providing evidence that the transmission parameter above, $p_{H \to H}$, is likely low (Vogl, 2020).

The 20th century was characterized by uniquely rapid population growth (Lam, 2011). Understanding the implications of a switch to population decline, or even merely stabilization, is of clear importance. Contrary to some arguments in the literature, empirical facts and models of intergenerational transmission do not provide reason to conclude that positive population growth is bound to continue.

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Online Appendix for "Intergenerational Transmission Is Not Sufficient for Positive Long-Term Population Growth"

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Proof for case where $p_{L\to L} < 1$

For compactness in this appendix, write p_L for $p_{L\to L}$ and p_H for $p_{H\to H}$. One condition that is sufficient to ensure that positive LTPG does not take place is if the Markov transition matrix $\mathbf{A}$ is a convergent matrix, or a matrix that converges to the zero matrix as the exponent on the matrix goes to infinity; in that case, overall population would eventually converge to zero. This occurs if the spectral radius of $\mathbf{A}$, or the largest absolute value of its eigenvalues, is less than 1. For a 2x2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the eigenvalues are given by:

$$\lambda = \frac{1}{2}\left(a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}\right)$$

and in the context of our matrix $\mathbf{A}$, this gives us:

$$\begin{aligned}\lambda &= \frac{1}{2}\left(p_H F_H + p_L F_L \pm \sqrt{(p_H F_H + p_L F_L)^2 - 4(p_H F_H p_L F_L - (1-p_L)F_L(1-p_H)F_H)}\right) \\ &= \frac{1}{2}\left(p_H F_H + p_L F_L \pm \sqrt{(p_H F_H + p_L F_L)^2 - 4(p_H + p_L - 1)F_H F_L}\right).\end{aligned}$$

It is possible to show that the term under the square root is always positive: that term decreases with p_L, and yet it is always positive even if $p_L = 1$ (algebra available upon request). Therefore, we do not need to concern ourselves with imaginary numbers, and the largest eigenvalue (in absolute value) will be the one with the plus sign in front of the square root; as a result, the condition for $\mathbf{A}$ to be a convergent matrix is:

$$X \equiv p_H F_H + p_L F_L + \sqrt{(p_H F_H + p_L F_L)^2 - 4(p_H + p_L - 1)F_H F_L} < 2.$$

It is then possible to show that $\frac{\partial X}{\partial p_H} \geq 0$ (algebra available upon request), with a strict inequality unless $p_L = 1$ and $p_H F_H < F_L$; but in the latter case, $X < 2$, and as p_H increases, it will eventually reach a point at which $p_H F_H > F_L$, at which point $\frac{\partial X}{\partial p_H} > 0$. This means that for any combination of values of p_L, F_L, and F_H, there is one unique value of p_H for which $X = 2$, and for any value of p_H below this critical value the matrix $\mathbf{A}$ is convergent.

It remains only for us to characterize this critical value of p_H, which we call $\widehat{p_H}$:

$$\begin{gathered}\widehat{p_H} F_H + p_L F_L + \sqrt{(\widehat{p_H} F_H + p_L F_L)^2 - 4(\widehat{p_H} + p_L - 1)F_H F_L} = 2 \\ \therefore (\widehat{p_H} F_H + p_L F_L)^2 - 4(\widehat{p_H} + p_L - 1)F_H F_L = (2 - \widehat{p_H} F_H - p_L F_L)^2 \\ \therefore (\widehat{p_H} F_H + p_L F_L)^2 - 4(\widehat{p_H} + p_L - 1)F_H F_L = 4(1 - \widehat{p_H} F_H - p_L F_L) + (\widehat{p_H} F_H + p_L F_L)^2 \\ \therefore (1 - \widehat{p_H} - p_L)F_H F_L = (1 - \widehat{p_H} F_H - p_L F_L) \\ \therefore \widehat{p_H} = \frac{1 - F_H F_L + F_L(F_H - 1)p_L}{F_H(1 - F_L)}.\end{gathered}$$

Therefore, if $p_H < \widehat{p_H}$, positive LTPG will not take place, and the population will converge to zero in the long run. If $p_L = 1$ as previously assumed, this condition simplifies to $\widehat{p_H} = \frac{1}{F_H}$, which demonstrates that the earlier result for long-run population decline with $p_L = 1$ is just a special case of this general solution. It is also important to note that, if $F_H F_L > 1$, the condition above could generate negative critical values for $\widehat{p_H}$, in which case it is impossible for $\mathbf{A}$ to be a convergent matrix; however, for any value of $p_L \in [0, 1]$, there exist parameter configurations such that positive LTPG will not take place.