第13周作业(高代A)

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第一次(20211109)

转专业补修

A

1. 证明:(1) $0\cdot x=(0+0)\cdot x=0\cdot x+0\cdot x$

$\Rightarrow 0\cdot x=0\cdot x-0\cdot x=0$

(2) $\lambda\cdot 0_V=\lambda\cdot(0_V+0_V)=\lambda\cdot 0_V+\lambda\cdot 0_V$

$\Rightarrow \lambda\cdot 0_V=0_V$

(3) $(-1)\cdot x=(0-1)\cdot x=0\cdot x-1\cdot x \xlongequal{\text{由(1)已证}} 0_V-1\cdot x=-x$

(4) $\lambda\cdot(-x)=\lambda\cdot(-1\cdot x)\xlongequal{\text{由(3)已证}}\lambda\cdot(-1)\cdot x=[\lambda\cdot(-1)]\cdot x=-\lambda\cdot x$

(5) $\lambda(x-y)=\lambda[x+(-y)]=\lambda\cdot x+\lambda(-y)\xlongequal{\text{由(4)已证}}\lambda\cdot x-\lambda\cdot y$

(6) $(\lambda-\mu)\cdot x=[\lambda+(-\mu)]x=\lambda\cdot x+(-\mu)\cdot x\xlongequal{\text{由(4)已证}}\lambda\cdot x-\mu\cdot x$

2. 证明:由于$\bigcap\limits_{i\in I}V_i$中任意运算都可看作在某个子空间$V_i$上进行,故各二元运算的运算律成立,故只需证下述4条:

(1) 零元存在性(这也说明$\bigcap\limits_{i\in I}V_i$非空):

$\because \forall i\in I$, V_i为子空间

$\therefore 0_V\in V_i$

$\therefore 0_V\in\bigcap\limits_{i\in I}V_i$

(2) 对加法封闭:设$\forall a,b\in\bigcap\limits_{i\in I}V_i$,则$\forall i\in I, a,b\in V_i$

由V_i为线性空间,知$a+b\in V_i$,$\forall i\in I$

$\therefore a+b\in\bigcap\limits_{i\in I}V_i$

(3) 对数乘量乘法封闭:$\forall\lambda\in K$, $a\in\bigcap\limits_{i\in I}V_i$,有$a\in V_i$, $\forall i\in I$

由V_i为线性空间,有:$\lambda a\in V_i$, $\forall i\in I$

$\therefore \lambda a\in\bigcap\limits_{i\in I}V_i$

(4) 负元存在性,由已证:$\forall a\in\bigcap\limits_{i\in I}V_i\subset V$, $-a=(-1)\cdot a$

由(3)已证:$(-1)\cdot a\in\bigcap\limits_{i\in I}V_i$

证毕

3. 证明:记$\bar V=V_1+V_2+\cdots+V_m$,则$0=0+0+\cdots+0\in\bar V$

$\therefore \bar V$非空

(1) 取$\forall x=x_1+\cdots+x_m\in\bar V$, $y=y_1+\cdots+y_m\in\bar V$

其中$x_i,y_i\in V_i$

则$x_i+y_i\in V_i$

故$x+y=x_1+\cdots+x_m+y_1+\cdots+y_m=(x_1+y_1)+\cdots+(x_m+y_m)\in\bar V$

(2) 取$\forall\lambda\in K$, $\forall x=x_1+\cdots+x_m\in\bar V$,其中$x_i\in V_i$

则$\lambda x_i\in V_i$

故$\lambda x=\lambda(x_1+\cdots x_m)=\lambda x_1+\cdots+\lambda x_m\in\bar V$

$\therefore \bar V$是V的子空间

4. 证明：设 W 有一组基 $\alpha_1, \cdots \alpha_r$，则 $\alpha_1, \alpha_2, \cdots \alpha_n$ 线性无关

故可将其扩充为 V 的一组基 $\alpha_1, \cdots \alpha_r, \alpha_{r+1}, \cdots \alpha_n$

$\therefore \dim_K W = r \leq n = \dim_K V$

且 $r=n \Leftrightarrow \alpha_1, \cdots \alpha_r$ 是 V 的一组基 $\Leftrightarrow W=V$

5. 证明：(1) 假设 $V_2 \subseteq V_1$ 和 $V_1 \subseteq V_2$ 均不成立

则 $\exists x_1 \in V_1, x_1 \notin V_2$

$\exists x_2 \notin V_1, x_2 \in V_2$

令 $x = x_1 + x_2$，[redacted]

则 $x - x_1 = x_2 \notin V_1$

$x - x_2 = x_1 \notin V_2$

若 $x \in V_1$，则 $x - x_1 \in V_1$，矛盾

若 $x \in V_2$，则 $x - x_2 \in V_2$，矛盾

$\therefore x \notin V_1 \cup V_2$

故 $x = x_1 + x_2 \notin V_1 \cup V_2$，即 $V_1 \cup V_2$ 不是线性空间

(2) 假设 $V_2 \subseteq V_1$ 或 $V_1 \subseteq V_2$ 成立，不妨设前者成立

则 $V_1 \cup V_2 = V_1$，为线性空间

6. 证明：(1) 只证 V_1 即可，V_2 同理，

① $(0,0) \in V_1$，故 V_1 非空

② $\forall (x_1,0), (x_2,0) \in V_1$，$(x_1,0)+(x_2,0) = (x_1+x_2, 0) \in V_1$

③ $\forall \lambda \in \mathbb{R}$，$(x,0) \in V_1$，$\lambda(x_1,0) = (\lambda x_1, 0) \in V_1$

$\therefore$ ~~[redacted]~~ 又 $V_1 \subset \mathbb{R}^2$

$\therefore V_1$ 是 $\mathbb{R}^2$ 的子空间

(2) 由于 $V_1 \subset V$，$V_2 \subset V$，故 $V_1 + V_2 \subset V$

故只需证 $V_1 + V_2 \supset V$：

设 $\forall (x,y) \in V$，则 $(x,y) = (x,0) + (0,y)$，$(x,0) \in V_1$，$(0,y) \in V_2$

$\therefore (x,y) \in V_1 + V_2$，故 $V_1 + V_2 \supset V$

$\therefore V_1 + V_2 = V$

(3) 设 $(1,1) \in \mathbb{R}^2$，则由 $1 \neq 0$ 知 $(1,1) \notin V_1$，$(1,1) \notin V_2$

$\therefore \mathbb{R}^2 \neq V_1 \cup V_2$

7. 证明：(1) 设 $\lambda_1, \lambda_2 \in \mathbb{R}$，$\lambda_1 \sin x + \lambda_2 \cos x = 0$

$$\begin{cases} 取\ x = \frac{\pi}{2}:\ \lambda_1 \cdot 1 + \lambda_2 \cdot 0 = 0 \\ 取\ x = 0:\ \lambda_1 \cdot 0 + \lambda_2 \cdot 1 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases}$$

$\therefore \sin x, \cos x \in V$ 线性无关

(2) 设 $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$，$\mu_1 \cdot 1 + \mu_2 \sin x + \mu_3 \cos x = 0$

$$\begin{cases} 取\ x = \frac{\pi}{2}:\ \mu_1 + \mu_2 = 0 \\ 取\ x = \frac{\pi}{4}:\ \mu_1 + \frac{\sqrt{2}}{2}\mu_2 + \frac{\sqrt{2}}{2}\mu_3 = 0 \\ 取\ x = 0:\ \mu_1 + \mu_3 = 0 \end{cases} \Rightarrow \begin{cases} \mu_1 = 0 \\ \mu_2 = 0 \\ \mu_3 = 0 \end{cases}$$

(3) 设 $k_1, k_2, \cdots k_n \in \mathbb{R}$，$\sum_{i=1}^{n} k_i \sin ix = 0$

$\therefore \int_{-\pi}^{\pi}(\sum_{i=1}^{n}k_i \sin ix)\sin jx\ dx = \sum_{i=1}^{n}k_i\int_{-\pi}^{\pi}(\sin ix)(\sin jx)\ dx$

$= \sum_{i=1}^{n}k_i\int_{-\pi}^{\pi}\frac{1}{2}[\cos(i-j)x+\cos(i+j)x]dx$

$= \frac{1}{2}\sum_{i=1}^{n}k_i[\int_{-\pi}^{\pi}\cos(i-j)x\,dx+\int_{-\pi}^{\pi}\cos(i+j)x\,dx]$ (*)

$\because i,j\in \mathbb{N}_+$

$\therefore \int_{-\pi}^{\pi}\cos(i+j)x\,dx = \frac{1}{i+j}\sin[(i+j)x]\Big|_{x=-\pi}^{x=\pi} = 0$

$i\neq j$时：

$\int_{-\pi}^{\pi}\cos(i-j)x\,dx = \frac{1}{i-j}\sin(i-j)x\Big|_{x=-\pi}^{x=\pi} = 0$

$i=j$时：

$\int_{-\pi}^{\pi}\cos(i-j)x\,dx = \int_{-\pi}^{\pi}1\,dx = 2\pi$

代回(*)得：$\int_{-\pi}^{\pi}(\sum_{i=1}^{n}k_i\sin ix)\sin jx\ dx = \frac{1}{2}\sum_{\substack{i=1\\i\neq j}}^{n}k_i(0+0)+\frac{1}{2}k_j(2\pi+0)$

$= \pi k_j$

$\because \sum_{i=1}^{n}k_i\sin ix = 0 \Rightarrow \int_{-\pi}^{\pi}(\sum_{i=1}^{n}k_i\sin ix)\sin jx\,dx = 0$

$\therefore \pi k_j = 0 \Rightarrow k_j = 0$，且由$j$任意性知$k_1=k_2=\cdots=k_n=0$

$\therefore \sin x, \sin 2x, \cdots \sin nx$ 线性无关

8. 证明：(1) 显然 $0\in V_1$，故V_1非空

且由定义易知$V_1\subset V$

取$\forall x,y\in V_1$，$\forall \lambda\in k$

则 $\begin{cases} l(x+y) = l(x)+l(y) = 0+0 = 0 \\ l(\lambda x) = \lambda l(x) = \lambda\cdot 0 = 0 \end{cases}$

$\therefore x+y\in V_1$，$\lambda x\in V_1$

综上，V_1为V的子空间

(2) 设$x_1,x_2,\cdots x_n\in V$为V的一组基，

$\forall x\in V$，$x = e_1^*(x)x_1+e_2^*(x)x_2+\cdots e_n^*(x)x_n$

$\therefore l(x) = \sum_{i=1}^{n}e_i^*(x)l(x_i)$，故$l(x)=0 \Leftrightarrow \sum_{i=1}^{n}e_i^*(x)l(x_i)=0$

$\because l$非零函数，故$l(x_1),\cdots l(x_n)$不全为0，故不妨$l(x_1)\neq 0$（否则$l(x_j)\neq 0$，将x_1与x_j互换得新基即可）

$\therefore l(x)=0 \overset{(*)}{\Leftrightarrow} e_1^*(x)+e_2^*(x)\frac{l(x_2)}{l(x_1)}+\cdots+e_n^*(x)\frac{l(x_n)}{l(x_1)}=0$ (*')

解该线性方程(*')得基础解系 $\eta_1 = \begin{pmatrix} -l(x_2) \\ l(x_1) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots; \eta_{n-1} = \begin{pmatrix} -l(x_n) \\ \vdots \\ \vdots \\ l(x_1) \end{pmatrix}$

设 $\begin{cases} y_1 = l(x_1)x_2 - l(x_2)x_1 \\ y_2 = l(x_1)x_3 - l(x_3)x_1 \\ \vdots \\ y_{n-1} = l(x_1)x_n - l(x_n)x_1 \end{cases}$ $\underbrace{\quad}_{\text{由向量空间同构知}}$ $y_1,y_2,\cdots y_{n-1}$为(*)的一组基础解系

亦即$y_1,y_2,\cdots y_{n-1}$为V_1的一组基

$\therefore \dim V_1 = n-1$

9. 证明：(1) 当 l_1, l_2 在 V^* 中线性相关时，存在不全为0的 $\lambda_1, \lambda_2 \in K$

使 $\lambda_1 l_1 + \lambda_2 l_2 = 0$，则不妨 $\lambda_1 \neq 0$，故 $l_1 = -\frac{\lambda_2}{\lambda_1} l_2$

∵ l_1, l_2 不全为0 ∴ $l_2 \neq 0$

∴ $\begin{cases} l_1(x)=0 \\ l_2(x)=0 \end{cases} \Leftrightarrow \begin{cases} -\frac{\lambda_2}{\lambda_1} l_2(x)=0 \\ l_2(x)=0 \end{cases} \Leftrightarrow l_2(x)=0$，故 $W = V_{l_2}$

由8已证：$\dim_K W = n-1 > n-2$

(2) 当 l_1, l_2 在 V^* 中线性无关时：设 $x_1, x_2, \cdots x_n$ 为 V 的一组基

$(l_1(x_1), l_1(x_2) \cdots l_1(x_n))$ 与 $(l_2(x_1), l_2(x_2), \cdots l_2(x_n))$ 在 K^n 中线性无关

∴ $\exists x_i, x_j$，$i \neq j$，使 $\begin{vmatrix} l_1(x_i) & l_1(x_j) \\ l_2(x_i) & l_2(x_j) \end{vmatrix} \neq 0$

不妨为 $x_i = x_1$，$x_j = x_2$

令 $A = \begin{pmatrix} l_1(x_1) & l_1(x_2) & \cdots & l_1(x_n) \\ l_2(x_1) & l_2(x_2) & \cdots & l_2(x_n) \end{pmatrix}$

设 $x \in V$，则 $x = e_1^*(x) x_1 + \cdots + e_n^*(x) x_n$

∴ $\begin{cases} l_1(x)=0 \\ l_2(x)=0 \end{cases}$ $(*)$ $\Leftrightarrow \begin{cases} \sum_{i=1}^{n} e_i^*(x) l_1(x_i) = 0 \\ \sum_{i=0}^{n} e_i^*(x) l_2(x_i) = 0 \end{cases} \Leftrightarrow A \begin{pmatrix} e_1^*(x) \\ \vdots \\ e_n^*(x) \end{pmatrix} = 0$ $(*')$

解方程组 $(*')$，由 $\mathrm{rank}(A) = 2$ 知 $(*')$ 有一基础解系 $\eta_1, \eta_2, \cdots \eta_{n-2}$

② $\forall \lambda \in \mathbb{R}$，$(x_1, 0) \in V_1$，$\lambda(x_1, 0) = (\lambda x_1, 0) \in V_1$

记 $y_i = (x_1, x_2, \cdots x_n)\eta_i$

由同构知 $y_1, y_2, \cdots y_{n-2}$ 即为 $(*)$ 的一个基础解系

进而 $y_1, y_2, \cdots y_{n-2}$ 为 W 的一组基

∴ $\dim_K W = n-2$

~~综上~~

综合(1)(2)：$\dim_K W \geq n-2$，且"="成立 $\Leftrightarrow$ $l_1, l_2 \in V^*$ 线性无关

10. 证明：(1) ① 零函数 $l_0 \in V_x^*$，故 V_x^* 非空，而显然 $V_x^* \subset V^*$

② $\forall l_1, l_2 \in V_x^*$，$\lambda \in K$，有：

$(l_1 + l_2)(x) = l_1(x) + l_2(x) = 0$

$(\lambda l_1)(x) = \lambda l_1(x) = 0$

∴ $l_1 + l_2 \in V_x^*$，$\lambda l_1 \in V_x^*$

∴ V_x^* 为 V^* 的子空间

(2) 设 $x_1, x_2, \cdots x_n$ 为 V 的一组基

则对应得 V^* 的一组基 $e_1^*(x), \cdots e_n^*(x)$

∴ $l = l(x_1) e_1^* + l(x_2) e_2^* + \cdots + l(x_n) e_n^*$

由 $x \neq 0$ 知 $e_1^*、e_2^*、\cdots e_n^*$ 不全为零函数

∴ 不妨设 e_1^* 不为零函数

则 $l(x) = 0 \overset{(*)}{\Leftrightarrow} l(x_1) e_1^*(x) + \cdots + l(x_n) e_n^*(x) = 0 \Leftrightarrow \overbrace{l(x_1) + l(x_2)\frac{e_2^*(x)}{e_1^*(x)} + \cdots + l(x_n)\frac{e_n^*(x)}{e_1^*(x)} = 0}^{(*)}$

线性方程组(*')的基础解系 $\eta_1=\begin{pmatrix}-e_2^*(x)\\ e_1^*(x)\\ 0\\ \vdots\\ 0\end{pmatrix},\cdots,\eta_{n-1}=\begin{pmatrix}-e_n^*(x)\\ 0\\ 0\\ \vdots\\ e_1^*(x)\end{pmatrix}$

记 $\begin{cases} l_1=e_1^*(x)e_2^*-e_2^*(x)e_1^* \\ \vdots \\ l_{n-1}=e_1^*(x)e_n^*-e_n^*(x)e_1^*\end{cases}$ 由同构知 则 $l_1,\cdots l_{n-1}$ 线性无关且为(*)的基础解系

$\therefore l_1,\cdots,l_{n-1}$ 为 V_x^* 的一组基，故 $\dim_K V_x^*=n-1$

11、证明：(1) ① 设 ~~…~~ $\forall x=x_1+x_2\in V_1+V_2$，$x_1\in V_1$，$x_2\in V_2$

则由 $\alpha_1,\cdots\alpha_m,\beta_1,\cdots\beta_t$ 为 V_1 的基，

知 $\exists\lambda_1,\cdots\lambda_m,\lambda_{m+1},\cdots\lambda_{m+t}\in K$

使 $x_1=\lambda_1\alpha_1+\cdots+\lambda_m\alpha_m+\lambda_{m+1}\beta_1+\cdots+\lambda_{m+t}\beta_t$

同理 $\exists\mu_1,\cdots,\mu_m,\mu_{m+1},\cdots\mu_{m+s}\in K$

使 $x_2=\mu_1\alpha_1+\cdots+\mu_m\alpha_m+\mu_{m+1}\gamma_1+\cdots+\mu_{m+s}\gamma_s$

$\therefore x=x_1+x_2=(\lambda_1+\mu_1)\alpha_1+\cdots+(\lambda_m+\mu_m)\alpha_m+\lambda_{m+1}\beta_1\cdots\lambda_{m+t}\beta_t$

$+\mu_{m+1}\gamma_1+\cdots+\mu_{m+s}\gamma_s$

且 $\lambda_1+\mu_1,\cdots\lambda_m+\mu_m,\lambda_{m+1},\cdots,\lambda_{m+t},\mu_{m+1},\cdots\mu_{m+s}\in K$

即 $\forall x\in V_1+V_2$，x 可由 $\alpha_1,\cdots\alpha_m,\beta_1,\cdots\beta_t,\gamma_1,\cdots\gamma_s$ 线性表出

② 设 $\exists k_1,\cdots k_m,k_1',\cdots k_t',k_1'',\cdots k_s''\in K$

使 $k_1\alpha_1+\cdots+k_m\alpha_m+k_1'\beta_1+\cdots+k_t'\beta_t+k_1''\gamma_1+\cdots k_s''\gamma_s=0$ (*)

则：$k_1''\gamma_1+\cdots+k_s''\gamma_s=-(k_1\alpha_1+\cdots+k_m\alpha_m+k_1'\beta_1+\cdots+k_t'\beta_t)\in V_1$

又 $k_1''\gamma_1+\cdots+k_s''\gamma_s\in V_2$

$\therefore k_1''\gamma_1+\cdots+k_s''\gamma_s\in V_1\cap V_2$

$\therefore k_1''\gamma_1+\cdots+k_s''\gamma_s$ 可由 $\alpha_1,\cdots\alpha_m$ 线性表出

不妨 $k_1''\gamma_1+\cdots+k_s''\gamma_s=h_1\alpha_1+\cdots+h_m\alpha_m$，$h_1,\cdots,h_m\in K$

$\therefore(-h_1)\alpha_1+\cdots+(-h_m)\alpha_m+k_1''\gamma_1+\cdots+k_s''\gamma_s=0$

$\because\alpha_1,\cdots\alpha_m,\gamma_1,\cdots\gamma_s$ 为 V_2 的一组基

$\therefore -h_1=\cdots=-h_m=k_1''=\cdots=k_s''=0$，将其代回(*)得：

$k_1\alpha_1+\cdots+k_m\alpha_m+k_1'\beta_1+\cdots+k_t'\beta_t=0$

又 $\alpha_1,\cdots,\alpha_m,\beta_1,\cdots\beta_t$ 为 V_1 的一组基

$\therefore k_1=\cdots=k_m=k_1'=\cdots=k_t'=0$

故：$k_1=\cdots=k_m=k_1'=\cdots=k_t'=k_1''=\cdots=k_s''=0$

$\therefore\alpha_1,\cdots\alpha_m,\beta_1,\cdots,\beta_t,\gamma_1,\cdots,\gamma_s$ 线性无关

故综合①②：$\alpha_1,\cdots,\alpha_m,\beta_1,\cdots,\beta_t,\cdots\gamma_1,\cdots\gamma_s$ 是 V_1+V_2 的一组基

(2) 证明：由(1)知：$\dim_K V_1=m+t$，$\dim_K V_2=m+s$，$\dim_K(V_1\cap V_2)=m$

$\dim_K(V_1+V_2)=m+t+s$

$\therefore \dim_K(V_1+V_2) = m+t+s = (m+t)+(m+s)-m = \dim_K V_1 + \dim_K V_2 - \dim_K(V_1 \cap V_2)$

12. 证明：~~因~~ 先证(2)，后由"命题等价⇔逆命题等价"即可推得(1)

设 $k_1, \cdots k_n \in K$

由 $\alpha_i = (e_1, \cdots e_n) A^{(i)} = e_1^*(\alpha_i) e_1 + \cdots + e_n^*(\alpha_i) e_n$

知：$k_1\alpha_1 + \cdots + k_n\alpha_n = \cancel{[k_1 e_1^*(\alpha_1) + \cdots k_n e_1^*(\alpha_n)]}$

$= [k_1 e_1^*(\alpha_1) + \cdots + k_n e_1^*(\alpha_n)] e_1 + \cdots [k_1 e_n^*(\alpha_1) + \cdots + k_n e_n^*(\alpha_n)] e_n$

$\therefore k_1\alpha_1 + \cdots + k_n\alpha_n = 0 \in K \Rightarrow \begin{cases} k_1 e_1^*(\alpha_1) + \cdots + k_n e_1^*(\alpha_n) = 0 \\ \vdots \\ k_1 e_n^*(\alpha_1) + \cdots + k_n e_n^*(\alpha_n) = 0 \end{cases}$

$\Leftrightarrow k_1 \begin{pmatrix} e_1^*(\alpha_1) \\ \vdots \\ e_n^*(\alpha_1) \end{pmatrix} + \cdots + k_n \begin{pmatrix} e_1^*(\alpha_n) \\ \vdots \\ e_n^*(\alpha_n) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

$\Leftrightarrow k_1 A^{(1)} + \cdots + k_n A^{(n)} = 0 \in K^n$

显 故 $\alpha_1, \cdots \alpha_n$ 线性无关 $\Leftrightarrow k_1\alpha_1 + \cdots + k_n\alpha_n = 0 \in V$ 当且仅当 $k_1 = \cdots = k_n = 0$

$\Leftrightarrow k_1 A^{(1)} + \cdots + k_n A^{(n)} = 0 \in K^n$ 当且仅当 $k_1 = \cdots = k_n = 0$

$\Leftrightarrow A^{(1)}, \cdots, A^{(n)} \in K^n$ 线性无关

证毕

第13周作业(高代A)

第二次(2021.11.10)

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转专业补修

1. 解：(1) 设方程组系数阵为A，由于方程组齐次，故对A进行初等行变换：

$$A=\begin{pmatrix}1&2&1&-3&2\\2&1&1&1&-3\\1&1&3&2&-2\\2&3&-5&-17&10\end{pmatrix}\to\begin{pmatrix}1&2&1&-3&2\\0&-3&-1&7&-7\\0&-1&2&5&-4\\0&-1&-7&-11&6\end{pmatrix}$$

$$\to\begin{pmatrix}1&2&1&-3&2\\0&1&-2&-5&4\\0&3&1&-7&7\\0&1&7&11&-6\end{pmatrix}\to\begin{pmatrix}1&2&1&-3&2\\0&1&-2&-5&4\\0&0&7&8&-5\\0&0&9&16&-10\end{pmatrix}$$

$$\to\begin{pmatrix}1&2&1&-3&2\\0&1&-2&-5&4\\0&0&7&8&-5\\0&0&-5&0&0\end{pmatrix}\to\begin{pmatrix}1&2&1&-3&2\\0&1&-2&-5&4\\0&0&-5&0&0\\0&0&7&8&-5\end{pmatrix}$$

$$\to\begin{pmatrix}1&2&0&-3&2\\0&1&0&-5&4\\0&0&1&0&0\\0&0&0&8&-5\end{pmatrix}\to\begin{pmatrix}1&0&0&7&-6\\0&1&0&-5&4\\0&0&1&0&0\\0&0&0&8&-5\end{pmatrix}$$

$$\to\begin{pmatrix}1&0&0&7&-6\\0&1&0&-5&4\\0&0&1&0&0\\0&0&0&1&-\frac{5}{8}\end{pmatrix}\to\begin{pmatrix}1&0&0&0&-\frac{13}{8}\\0&1&0&0&\frac{7}{8}\\0&0&1&0&0\\0&0&0&1&-\frac{5}{8}\end{pmatrix}$$

故得基础解系 $\eta=8\begin{pmatrix}\frac{13}{8}\\-\frac{7}{8}\\0\\\frac{5}{8}\\1\end{pmatrix}=\begin{pmatrix}13\\-7\\0\\5\\8\end{pmatrix}$

∴方程组解为 $X=k\eta\quad(\forall k\in\mathbb{R})$

(2) 设方程组系数阵为A，增广阵为$\bar{A}$

则 $$\bar{A}=\left(\begin{array}{cccc|c}1&2&6&-2&-7\\-2&-5&-10&3&10\\1&2&4&0&0\\0&1&3&-3&-10\end{array}\right)\to\left(\begin{array}{cccc|c}1&2&6&-2&-7\\0&-1&2&1&-4\\0&0&-2&2&7\\0&1&3&-3&-10\end{array}\right)$$

$$\to\left(\begin{array}{cccc|c}1&2&6&-2&-7\\0&1&3&-3&-10\\0&-1&2&1&-4\\0&0&-2&2&7\end{array}\right)\to\left(\begin{array}{cccc|c}1&2&6&-2&-7\\0&1&3&-3&-10\\0&0&5&-2&-14\\0&0&-2&2&7\end{array}\right)$$

$$\to\left(\begin{array}{cccc|c}1&2&6&-2&-7\\0&1&3&-3&-10\\0&0&1&2&0\\0&0&-2&2&7\end{array}\right)\to\left(\begin{array}{cccc|c}1&2&6&-2&-7\\0&1&3&-3&-10\\0&0&1&2&0\\0&0&0&6&7\end{array}\right)$$

$$\to\left(\begin{array}{cccc|c}1&2&6&-2&-7\\0&1&3&-3&-10\\0&0&1&2&0\\0&0&0&1&\frac{7}{6}\end{array}\right)\to\left(\begin{array}{cccc|c}1&2&6&0&-\frac{14}{3}\\0&1&3&0&-\frac{13}{2}\\0&0&1&0&-\frac{7}{3}\\0&0&0&1&\frac{7}{6}\end{array}\right)$$

$$\to\left(\begin{array}{cccc|c}1&2&0&0&\frac{28}{3}\\0&1&0&0&\frac{1}{2}\\0&0&1&0&-\frac{7}{3}\\0&0&0&1&\frac{7}{6}\end{array}\right)\to\left(\begin{array}{cccc|c}1&0&0&0&\frac{25}{3}\\0&1&0&0&\frac{1}{2}\\0&0&1&0&-\frac{7}{3}\\0&0&0&1&\frac{7}{6}\end{array}\right)$$

∴ $r(A)=r(\bar{A})=4$

∴方程组解为 $X=\cancel{\begin{pmatrix}\frac{25}{3}\\\frac{1}{2}\\-\frac{7}{3}\\\frac{7}{6}\end{pmatrix}}$

2. 解：设 $f(x)=ax^3+bx^2+cx+d$，$a\neq 0$

则 $f(-2)=-8a+4b-2c+d=1$

$f(-1)=-a+b-c+d=3$

$f(1)=a+b+c+d=13$

$f(2)=8a+4b+2c+d=33$

设该线性方程组系数矩阵为A，增广矩阵为 $\bar{A}$

则 $\bar{A}=\left(\begin{array}{cccc|c} -8 & 4 & -2 & 1 & 1 \\ -1 & 1 & -1 & 1 & 3 \\ 1 & 1 & 1 & 1 & 13 \\ 8 & 4 & 2 & 1 & 33 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 13 \\ -1 & 1 & -1 & 1 & 3 \\ 8 & 4 & 2 & 1 & 33 \\ -8 & 4 & -2 & 1 & 1 \end{array}\right)$

$\to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 13 \\ 0 & 2 & 0 & 2 & 16 \\ 8 & 4 & 2 & 1 & 33 \\ 0 & 8 & 0 & 2 & 34 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 13 \\ 0 & 1 & 0 & 1 & 8 \\ 8 & 4 & 2 & 1 & 33 \\ 0 & 4 & 0 & 1 & 17 \end{array}\right)$

$\to \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 5 \\ 0 & 1 & 0 & 1 & 8 \\ 8 & 0 & 2 & 0 & 16 \\ 0 & 4 & 0 & 1 & 17 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 5 \\ 0 & 1 & 0 & 1 & 8 \\ 4 & 0 & 1 & 0 & 8 \\ 0 & 4 & 0 & 1 & 17 \end{array}\right)$

$\to \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 5 \\ 0 & 1 & 0 & 1 & 8 \\ 0 & 0 & -3 & 0 & -12 \\ 0 & 0 & 0 & -3 & -15 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 5 \\ 0 & 1 & 0 & 1 & 8 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right)$

$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right)$ $\therefore r(A)=r(\bar{A})=4$

即方程组解为 $\begin{cases} a=1 \\ b=3 \\ c=4 \\ d=5 \end{cases}$

$\therefore f(x)=x^3+3x^2+4x+5$

3. 解：(1) 线性组合形式：

$x_1\begin{pmatrix}1\\3\\4\\3\end{pmatrix}+x_2\begin{pmatrix}1\\5\\5\\8\end{pmatrix}+x_3\begin{pmatrix}-2\\6\\-2\\24\end{pmatrix}+x_4\begin{pmatrix}2\\-4\\3\\-19\end{pmatrix}=\begin{pmatrix}1\\1\\3\\2\end{pmatrix}$

设其系数阵为A，

(2) 其增广阵：$\bar{A}=\left(\begin{array}{cccc|c} 1 & 1 & -2 & 2 & 1 \\ 3 & 5 & 6 & -4 & 1 \\ 4 & 5 & -2 & 3 & 3 \\ 3 & 8 & 24 & -19 & 2 \end{array}\right)$

$\to \left(\begin{array}{cccc|c} 1 & 1 & -2 & 2 & 1 \\ 0 & 2 & 12 & -10 & -2 \\ 0 & 1 & 6 & -5 & -1 \\ 0 & 5 & 30 & -25 & -1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & -2 & 2 & 1 \\ 0 & 1 & 6 & -5 & -1 \\ 0 & 1 & 6 & -5 & -1 \\ 0 & 1 & 6 & -5 & -\frac{1}{5} \end{array}\right)$

$\to \left(\begin{array}{cccc|c} 1 & 1 & -2 & 2 & 1 \\ 0 & 1 & 6 & -5 & -1 \\ 0 & 0 & 0 & 0 & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & -2 & 2 & 1 \\ 0 & 1 & 6 & -5 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$，故 $r(\bar{A})=3$，$r(A)=2$

1. 解: 统设 $A=(\alpha_1^T,\alpha_2^T,\alpha_3^T)$, $\tilde{A}=(A\vdots\beta^T)$

则: (1) $\tilde{A}=\left(\begin{array}{ccc|c}1&2&0&5\\0&1&-2&4\\-1&0&1&-2\\3&1&1&4\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}1&2&0&5\\0&1&-2&4\\0&2&1&3\\0&-5&1&-11\end{array}\right)$

$\rightarrow\left(\begin{array}{ccc|c}1&2&0&5\\0&1&-2&4\\0&0&5&-5\\0&0&-9&9\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}1&2&0&5\\0&1&-2&4\\0&0&1&-1\\0&0&1&-1\end{array}\right)$

$\rightarrow\left(\begin{array}{ccc|c}1&2&0&5\\0&1&0&2\\0&0&1&-1\\0&0&0&0\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&2\\0&0&1&-1\\0&0&0&0\end{array}\right)$

$\therefore r(A)=r(\tilde{A})=3$

故而线性方程组 $x_1\alpha_1^T+x_2\alpha_2^T+x_3\alpha_3^T=\beta^T$ 有唯一解 $\begin{cases}x_1=1\\x_2=2\\x_3=-1\end{cases}$

即 $\beta=\alpha_1+2\alpha_2-\alpha_3$

(2) $\tilde{A}=\left(\begin{array}{ccc|c}-1&4&3&1\\2&0&2&1\\-1&1&0&1\\1&-1&0&1\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}-1&4&3&1\\0&8&8&3\\0&-3&-3&0\\0&3&3&2\end{array}\right)\rightarrow\left(\begin{array}{ccc|c}1&-4&-3&-1\\0&1&1&0\\0&0&0&2\\0&0&0&3\end{array}\right)$

$\rightarrow\left(\begin{array}{ccc|c}1&0&1&-1\\0&1&1&0\\0&0&0&1\\0&0&0&0\end{array}\right)$

$\therefore$ ~~$r(\tilde{A})=2$~~ $r(A)=2<3=r(\tilde{A})$

故而线性方程组 $x_1\alpha_1^T+x_2\alpha_2^T+x_3\alpha_3^T=\beta^T$ 无解

即 β 不能用 $\alpha_1,\alpha_2,\alpha_3$ 线性表示

5. 证明: (1) 若 $r>0$: 由引理 3.2.2, 通过初等行变换可将 A 化成:

$$A_1=\begin{pmatrix}0\cdots0&1&\bar a_{1(i_1+1)}&\cdots&\bar a_{1i_2}&\cdots&0&\bar a_{1i_r}&\cdots&\bar a_{1n}\\0\cdots0&0&0&\cdots&1&\cdots&0&\bar a_{2i_r}&\cdots&\bar a_{2n}\\\vdots&\vdots&\vdots&&\vdots&&\vdots&\vdots&&\vdots\\0\cdots0&0&0&\cdots&0&\cdots&1&\cdots&&\bar a_{rn}\\0\cdots0&0&0&\cdots&0&\cdots&0&\cdots&&0\\\vdots&&&&&&&&&\vdots\\0\cdots0&0&0&\cdots&0&\cdots&0&\cdots&&0\end{pmatrix}$$

现将 A_1 通过初等列变换变形:

$$A_1\xrightarrow[k\geq i_1+1]{c_k-\bar a_{1k}c_{i_1}}\begin{pmatrix}0\cdots0&1&0&\cdots&0&\cdots&0&0&\cdots&0\\0\cdots0&0&0&\cdots&1&\cdots&0&\bar a_{2i_r}&\cdots&\bar a_{2n}\\\vdots&&&&&&&&&\vdots\\0\cdots0&0&0&\cdots&0&\cdots&1&\cdots&&\bar a_{rn}\\0\cdots0&0&0&\cdots&0&\cdots&0&\cdots&&0\\\vdots&&&&&&&&&\vdots\\0\cdots0&0&0&\cdots&0&\cdots&0&\cdots&&0\end{pmatrix}$$

$$\xrightarrow{\text{同理}}\begin{pmatrix}0\cdots0&1&0\cdots&0&\cdots&0&0\cdots0\\0\cdots0&0&0\cdots&1&\cdots&0&0\cdots0\\\vdots&&&&&&\vdots\\0\cdots0&0&0\cdots&0&\cdots&1&0\cdots0\\0\cdots0&0&0\cdots&0&\cdots&0&0\cdots0\\\vdots&&&&&&\vdots\\0\cdots0&0&0\cdots&0&\cdots&0&0\cdots0\end{pmatrix}$$

$$\xrightarrow[\text{且}k<r]{c_{i_k}\leftrightarrow c_k}\begin{pmatrix}1&0&0&\cdots&0&0&\cdots&0\\0&1&0&\cdots&0&0&\cdots&0\\0&0&1&\cdots&0&0&\cdots&0\\\vdots&\vdots&\vdots&\ddots&&\vdots&&\vdots\\0&0&0&\cdots&1&0&\cdots&0\\0&0&0&\cdots&0&0&\cdots&0\\\vdots&&&&&&&\vdots\\0&0&0&\cdots&0&0&\cdots&0\end{pmatrix}(r\neq1)=\bar{A}$$

即 A 可由初等行变换, 列变换化为 $\bar{A}$

6. 证明: 设 $k_0,k_1\cdots k_{n-1}\in\mathbb{R}$, 使:

$$k_0\alpha+k_1(\alpha+\beta_1)+\cdots+k_{n-1}(\alpha+\beta_{n-1})=0\ (*)$$

记 $A=(A^{(1)},\cdots,A^{(n)})$，则 $A\alpha=b$，~~$A(\beta_1$~~ $A\beta_1=\cdots=A\beta_{n-r}=0$

且 $r(A)=r$

$\therefore$ ~~(*)式两边左乘A~~：$(*)\Leftrightarrow(\sum_{i=0}^{n-r}k_i)\alpha+k_1\beta_1+\cdots+k_{n-r}\beta_{n-r}=0$ $(*')$

$(*')$ 两边左乘 A：$(\sum_{i=0}^{n-r}k_i)b+0+\cdots+0=0$

$\Rightarrow(\sum_{i=0}^{n-r}k_i)b=0$

$\because$ $b\neq 0$

$\therefore \sum_{i=0}^{n-r}k_i=0$

$\therefore$ 代回 $(*')$ 有：$k_1\beta_1+\cdots+k_{n-r}\beta_{n-r}=0$

由 $\beta_1,\beta_2,\cdots\beta_{n-r}$ 为基：

$k_1=\cdots=k_{n-r}=0$

$\therefore \sum_{i=0}^{n-r}k_i=k_0=0$

故 $k_0=k_1=\cdots=k_{n-r}=0$

$\therefore \alpha,\alpha+\beta_1,\cdots\alpha+\beta_{n-r}$ 线性无关

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第一次 2021.11.17 转专业补修

A

显然 φ 为线性映射

1. 证明：取 $f: V \to K^n$，则由已知，f 为同构

$x \mapsto \begin{pmatrix} e_1^*(x) \\ \vdots \\ e_n^*(x) \end{pmatrix}$

又令 $\psi: K^n \to K^n$

$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} e_1^*(\sum_{i=1}^n x_i\varphi(\alpha_i)) \\ \vdots \\ e_n^*(\sum_{i=1}^n x_i\varphi(\alpha_i)) \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n x_i e_1^*(\varphi(\alpha_i)) \\ \vdots \\ \sum_{i=1}^n x_i e_n^*(\varphi(\alpha_i)) \end{pmatrix} = B\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

其中 $B = \begin{pmatrix} e_1^*(\varphi(\alpha_1)) & \cdots & e_1^*(\varphi(\alpha_n)) \\ \vdots & & \vdots \\ e_n^*(\varphi(\alpha_1)) & \cdots & e_n^*(\varphi(\alpha_n)) \end{pmatrix}$

则 $\varphi \cdot f = f \cdot \psi \Rightarrow \varphi = f \cdot \psi \cdot f^{-1}$

且观察发现：$B = A_\varphi$

采用 (1)⇒(2)⇒(3)⇒(1) 的方式证明：

(1)⇒(2)：∵ $\varphi(\alpha_1), \cdots \varphi(\alpha_n)$ 为 V 的一组基

∴ $\alpha_1, \cdots \alpha_n$ 可由其线性表示

不妨设 $(\alpha_1, \cdots, \alpha_n) = (\varphi(\alpha_1), \cdots \varphi(\alpha_n))S$

则 $(\alpha_1, \cdots \alpha_n) = (\alpha_1, \cdots \alpha_n)A_\varphi S$

∴ $A_\varphi S = I_n$（单位阵

同理 $(\varphi(\alpha_1)\cdots\varphi(\alpha_n)) = (\alpha_1 \cdots \alpha_n)A_\varphi = (\varphi(\alpha_1)\cdots\varphi(\alpha_n))SA_\varphi$

得 $SA_\varphi = I_n$

∴ $SA_\varphi S = SA_\varphi = I_n$，故 A_φ 可逆

(2)⇒(3)：只需证 φ 是双射，而由开头的分析：f, f^{-1} 均为同构，线为双射，而 $\varphi = f \cdot \psi \cdot f^{-1}$，故只需证 ψ 是双射

由 $B = A_\varphi$ 知 B 可逆

∴ ①：$\forall \alpha = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in K^n$，$\exists \beta = B^{-1}\alpha \in K^n$

使得 $\psi(\beta) = B(B^{-1}\alpha) = \alpha$，故 ψ 为满射

② $\forall \alpha = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in K^n$，$\alpha' = \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} \in K^n$

若 $\psi(\alpha) = \psi(\alpha')$ 即 $B\alpha = B\alpha'$

则 $B^{-1}(B\alpha) = B^{-1}(B\alpha') \Rightarrow \alpha = \alpha'$，故 ψ 为单射

综合①② 有 ψ 为双射

(3)⇒(1)：只需证 $\varphi(\alpha_1), \cdots, \varphi(\alpha_n)$ 线性无关

设 $k_1, \cdots k_n \in K$，$k_1\varphi(\alpha_1) + \cdots + k_n\varphi(\alpha_n) = 0 \in V$

由 φ 为线性映射：$\varphi(k_1\alpha_1 + \cdots + k_n\alpha_n) = 0 \in V$

而由 φ 为同构 及 $\varphi(0) = 0$

有 $k_1\alpha_1 + \cdots + k_n\alpha_n = 0$，而 $\alpha_1, \cdots, \alpha_n$ 为 V 的一组基

∴ $k_1 = \cdots = k_n = 0$

∴ $\varphi(\alpha_1), \cdots \varphi(\alpha_n)$ 线性无关

综合上述 证毕

2. ~~证明~~ 解：(1) $\because \varphi(x^i)=(1-x)^i=\sum_{k=0}^{i}C_i^k(-x)^k=\sum_{k=0}^{i}(-1)^kC_i^kx^k$

~~$\therefore(\varphi(1),\varphi(x),\cdots\varphi(x^{n-1}))=(1,x,\cdots x^{n-1})$~~

$$\therefore(\varphi(1),\varphi(x),\cdots\varphi(x^{n-1}))=(1,x,\cdots x^{n-1})\begin{pmatrix}1&1&1&\cdots&1\\0&-1&-C_2^1&\cdots&-C_n^1\\0&0&1&\cdots&C_n^2\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&(-1)^{n-1}\end{pmatrix}$$

$$\text{即 } A_\varphi=\begin{pmatrix}1&1&1&\cdots&1\\0&-1&-C_2^1&\cdots&-C_n^1\\0&0&1&\cdots&C_n^2\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&(-1)^{n-1}\end{pmatrix}$$

$$\therefore|A_\varphi|=\prod_{k=0}^{n-1}(-1)^k=(-1)^{\sum_{k=0}^{n-1}k}=(-1)^{\frac{n(n-1)}{2}}\neq0$$

故A_φ可逆

(2) $\because\varphi(x^i)=x\frac{d}{dx}(x^i)=(ix^{i-1})x=ix^i$

$i\geq1$时：

$i=0$时：$\varphi(1)=x\frac{d1}{dx}=0=0\cdot1$

$$\therefore(\varphi(1),\varphi(x),\cdots\varphi(x^{n-1}))=(1,x,\cdots x^{n-1})\begin{pmatrix}0&0&0&\cdots&0\\0&1&0&\cdots&0\\0&0&2&\cdots&0\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&n\end{pmatrix}$$

$$\text{即 } A_\varphi=\begin{pmatrix}0&0&0&\cdots&0\\0&1&0&\cdots&0\\0&0&2&\cdots&0\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&n-1\end{pmatrix},\quad\text{故 }|A_\varphi|=0\times1\times2\times\cdots\times(n-1)=0$$

~~证毕~~ 故A_φ不可逆

3. 证明：

(1) 设$\forall x\in V$，$\forall\lambda\in K$，$\forall l_1,l_2\in V^*$

则：$\varphi(l_1+l_2)=(l_1+l_2)(x)=l_1(x)+l_2(x)=\varphi(l_1)+\varphi(l_2)$

$\varphi(\lambda l_1)=\lambda l_1(x)=\lambda\varphi(l_1)$

$\therefore\varphi$是$V^*\to K$的线性函数

(2) ①先证φ为线性映射：

设$\forall x,x'\in V$，$\forall\lambda\in K$

则$\varphi(x+x')=\varphi_{x+x'}$

故而$\forall l\in V^*$，$\varphi_{x+x'}(l)=$自$l(x+x')\xlongequal{l\text{为线性函数}}l(x)+l(x')=\varphi_x(l)+\varphi_{x'}(l)$

$=(\varphi_x+\varphi_{x'})(l)$

$\therefore\varphi(x+x')=\varphi_{x+x'}=\varphi_x+\varphi_{x'}=\varphi(x)+\varphi(x')$

同理$\varphi(\lambda x)=\lambda\varphi(x)$

②再证φ为同构：设$x_1,x_2,\cdots x_n$为V的一组基，则$\dim V=n$

则$\dim V^*=n$，故由递推$\dim V^{**}=n$

而$\varphi(x_1),\cdots\varphi(x_n)\in V^{**}$，~~故~~要证$\varphi$为同构只需证$\varphi(x_1),\cdots\varphi(x_n)$为$V^{**}$的一组基，由$\dim V^{**}=n$，只需证$\varphi(x_1),\cdots\varphi(x_n)$线性无关

设$\lambda_1,\lambda_2,\cdots\lambda_n\in K$，$\lambda_1\varphi(x_1)+\cdots+\lambda_n\varphi(x_n)=\lambda_1\varphi_{x_1}+\cdots+\lambda_n\varphi_{x_n}=0\in V^{**}$

$l\in V^*$，则 $(\lambda_1\varphi_{x_1}+\cdots+\lambda_n\varphi_{x_n})(l)=0(l)=0\in K$

故：$\lambda_1\varphi_{x_1}(l)+\cdots+\lambda_n\varphi_{x_n}(l)=0$

$\Rightarrow \lambda_1 l(x_1)+\cdots+\lambda_n l(x_n)=0$

$\Rightarrow l(\lambda_1x_1+\cdots+\lambda_nx_n)=0$

分别取 $l=e_1^*$，$l=e_2^*$，$\cdots$ $l=e_n^*$

则：$e_1^*(\lambda_1x_1+\cdots+\lambda_nx_n)=0$

$e_2^*(\lambda_1x_1+\cdots+\lambda_nx_n)=0$

$\vdots$

$e_n^*(\lambda_1x_1+\cdots+\lambda_nx_n)=0$

不妨记 $x=\lambda_1x_1+\cdots+\lambda_nx_n$

~~$\therefore \lambda_1x_1+$~~

则 $x=e_1^*(x)x_1+\cdots+e_n^*(x)x_n=0\in V$

$\therefore \lambda_1x_1+\cdots+\lambda_nx_n=0$

又 $x_1,\cdots x_n$ 为 V 的一组基，故 $\lambda_1,\cdots\lambda_n=0\in K$

$\therefore \varphi(x_1),\cdots,\varphi(x_n)$ 线性无关

故由前分析知 φ 为同构，证毕

4. 解：(1) 当 $k\neq i,j$ 时：

e_k 的 x_i,x_j 处分量皆为0，故 $\varphi_{ij}(e_k)=e_k$

又 $\varphi_{ij}(e_i)=e_j$

$\varphi_{ij}(e_j)=e_i$

$\therefore (\varphi(e_1),\cdots\varphi(e_i)\cdots\varphi(e_j)\cdots\varphi(e_n))=(e_1\cdots e_i\cdots e_j\cdots e_n)\begin{pmatrix}1&&&&\\&0&\cdots&1&\\&\vdots&&\vdots&\\&1&\cdots&0&\\&&&&1\end{pmatrix}$

$$\therefore A_\varphi=\begin{array}{c}\begin{array}{cccccc} & & i列 & & j列 & \end{array}\\ \begin{pmatrix}1&&&&\\&0&\cdots&1&\\&\vdots&&\vdots&\\&1&\cdots&0&\\&&&&1\end{pmatrix}\begin{array}{l}\\ i行\\ \\ j行\\ \\ \end{array}\end{array}$$

即互换单位阵的 i,j 行

(2) 当 $k\neq i$ 时，$\varphi_{ij}^{(\lambda)}(e_k)=e_k$ （e_k在 x_i处分量为0）

而 $\varphi_{ij}^{(\lambda)}(e_i)=(0,0,\cdots,\overset{i}{1},\cdots,\overset{j}{\lambda},\cdots0)=$ ~~$e_i+\lambda e_j$~~ $e_i+\lambda e_j$

$\therefore (\varphi_{ij}^{(\lambda)}(e_1)\cdots\varphi_{ij}^{(\lambda)}(e_i),\cdots\varphi_{ij}^{(\lambda)}(e_n))=(e_1,\cdots e_i\cdots e_j\cdots e_n)\begin{array}{c}\begin{array}{cccc} & 第i列 & 第j列 & \end{array}\\ \begin{pmatrix}1&&&\\&1&&\\&\lambda&1&\\&&&1\end{pmatrix}\begin{array}{l}\\ 第i行\\ 第j行\\ \\ \end{array}\end{array}$

$$\therefore A_\varphi=\begin{array}{c}\begin{array}{cccc} & 第i列 & 第j列 & \end{array}\\ \begin{pmatrix}1&&&\\&1&&\\&\lambda&1&\\&&&1\end{pmatrix}\begin{array}{l}\\ 第i行\\ 第j行\\ \\ \end{array}\end{array}$$

即将单位阵第 i 行 λ 倍加到第 j 行

(3) 当 $k\neq i$ 时，$\varphi_i(e_k)=e_k$

而 $\varphi_i(e_i)=(0,0,\cdots\overset{i}{\lambda},0\cdots0)=\lambda e_i$

$\therefore (\varphi_i^{(\lambda)}(e_1),\cdots\varphi_i^{(\lambda)}(e_i)\cdots\varphi_i^{(\lambda)}(e_n))=(e_1,\cdots e_i,\cdots e_n)\begin{array}{c}\begin{array}{ccc} & 第i列 & \end{array}\\ \begin{pmatrix}1&&\\&\lambda&\\&&1\end{pmatrix}第i行\end{array}$

$$\therefore A_\varphi=\begin{array}{c}\begin{array}{ccc} & 第i行 & \end{array}\\ \begin{pmatrix}1&&\\&\lambda&\\&&1\end{pmatrix}第i列\end{array}$$

即单位阵第 i 行乘 λ

5. 证明：~~不~~不妨设所取的基为 $x_1, x_2, \cdots x_s \in K^s$

$y_1, y_2, \cdots y_m \in K^m$

$w_1, w_2, \cdots w_n \in K^n$

~~又~~ 又设 $A=(a_{ij}),\ B=(b_{ij})$

(1)

则 $\varphi_{AB}(w_k)=\varphi_A\varphi_B(w_k)=\varphi_A(b_{1k}x_1+\cdots+b_{sk}x_s)$

$=b_{1k}\varphi_A(x_1)+\cdots+b_{sk}\varphi_A(x_s)=\sum_{p=1}^{s}b_{pk}\varphi_A(x_p)$

$=\sum_{p=1}^{s}b_{pk}(a_{1p}y_1+a_{2p}y_2+\cdots+a_{mp}y_m)$

$=\sum_{p=1}^{s}b_{pk}\sum_{q=1}^{m}a_{qp}y_q$

$=\sum_{q=1}^{m}(\sum_{p=1}^{s}a_{qp}b_{pk})y_q$

$\therefore(\varphi_{AB}(w_1)\cdots\varphi_{AB}(w_n))=(y_1,\cdots y_m)\begin{pmatrix}\sum_{p=1}^{s}a_{1p}b_{p1} & \cdots & \sum_{p=1}^{s}a_{1p}b_{pn}\\ \vdots & & \vdots\\ \sum_{p=1}^{s}a_{mp}b_{p1} & \cdots & \sum_{p=1}^{s}a_{mp}b_{pn}\end{pmatrix}$

$=(y_1,\cdots y_m)\begin{pmatrix}a_{11}&\cdots&a_{1s}\\ \vdots&&\vdots\\ a_{m1}&\cdots&a_{ms}\end{pmatrix}\begin{pmatrix}b_{11}&\cdots&b_{1n}\\ \vdots&&\vdots\\ b_{s1}&\cdots&b_{sn}\end{pmatrix}$

设 φ_{AB} 坐标矩阵为 D，则 $D=AB$

(2) 设 K^s、K^m、K^n 中元素 α, y, w 在所取基下坐标分别为 C_α, C_y, C_w

则 $y\in$ ~~Im~~ $\mathrm{Im}\,\varphi_{AB}\Leftrightarrow C_y\in\{ABC_\alpha\mid\alpha\in K^n\}$

$w\in\ker\varphi_{AB}\Leftrightarrow ABC_w=0\in K^m$

同理得 $y\in\mathrm{Im}(\varphi_A)$、$w\in\ker\varphi_B$ 的充要条件

$\therefore y\in\mathrm{Im}\,\varphi_{AB}\Leftrightarrow C_y\in\{ABC_\alpha\mid\alpha\in K^n\}\xRightarrow{BC_\alpha\in K^s}C_y\in\{AC_\beta\mid\beta\in K^s\}$

$\Leftrightarrow y\in\mathrm{Im}\,\varphi_A$

$w\in\ker\varphi_B\Leftrightarrow BC_w=0\Rightarrow ABC_w=0\Leftrightarrow w\in\ker\varphi_{AB}$

$\therefore\begin{cases}\mathrm{Im}\,\varphi_{AB}\subset\mathrm{Im}\,\varphi_A\\ \ker\varphi_B\subset\ker\varphi_{AB}\end{cases}$

(3) 由(2)已证：$\begin{cases}\mathrm{Im}\,\varphi_{AB}\subset\mathrm{Im}\,\varphi_A\\ \ker\varphi_B\subset\ker\varphi_{AB}\end{cases}\Rightarrow\begin{cases}\dim_K(\mathrm{Im}\,\varphi_{AB})\le\dim_K(\mathrm{Im}\,\varphi_A)\\ \dim_K(\ker\varphi_{AB})\ge\dim_K(\ker\varphi_B)\end{cases}$

~~$\therefore r(AB)$~~

$\therefore\begin{cases}r(AB)=\dim_K(\mathrm{Im}\,\varphi_{AB})\le\dim_K(\mathrm{Im}\,\varphi_A)\le r(A)\\ \dim_K K^n-r(AB)=\dim_K(\ker\varphi_{AB})\ge\dim_K(\ker\varphi_B)\\ \qquad=\dim_K K^n-r(B)\end{cases}$

$\Rightarrow\begin{cases}r(AB)\le r(A)\\ r(AB)\le r(B)\end{cases}$

$\therefore r(AB)\le\min\{r(A),r(B)\}$

(4) ① 证明 ~~若~~ f 为线性映射：

由于 φ_A 为线性映射，故只需证 U 对运算封闭 即证U为一子空间

显然 $0\in U$，故U非

(2) 设出坐标，则：$\cancel{\quad}$ $\forall x, x' \in U = Im\,\varphi_B$

$\Leftrightarrow C_x, C_{x'} \in \{BC_\gamma \mid \gamma \in K^n\}$

$\therefore$ 不妨 $C_x = BC_1$，$C_{x'} = BC_1'$，$C_1, C_1' \in K^n$

$\therefore C_x + C_{x'} = B(C_1 + C_1')$，

$\forall \lambda \in K$，$\lambda C_x = \lambda BC_1 = B(\lambda C_1)$

由 K^n 为向量空间：$C_1 + C_1' \in K^n$，$\lambda C_1 \in K^n$

$\therefore C_x + C_{x'}$，$\lambda C_x \in \{BC_\gamma \mid \gamma \in K^n\}$

又 $\begin{cases} C_{x+x'} = C_x + C_{x'} \\ C_{\lambda x} = \lambda C_x \end{cases}$

$\therefore x + x'$，$\lambda x \in U$

$\therefore U$ 为子空间

② 证明 $Im f = V$：<1> 设 $\forall x \in U \subset K^s \Leftrightarrow C_x \in \{BC_\gamma \mid C_\gamma \in K^n\}$，不妨 $C_x = BC_w$ ($C_w \in K^n$ 为 w 坐标)

则 $f(x) = \varphi_A(x)$，记 $y = \varphi_A(x)$

则：$C_y = AC_x = ABC_w = (AB)C_w$，$C_1 \in K^n$

$\therefore y = \varphi_{AB}(w) \in Im\,\varphi_{AB} = V$

$\therefore Im f \subset V$

<2> 设 $\forall y' \in V \Leftrightarrow C_{y'} \in \{ABC_\gamma \mid C_\gamma \in K^n\}$，不妨 $C_{y'} = ABC_{w'}$ ($C_{w'} \in K^n$ 为 w' 坐标，且 $x' \in U$)

则 $C_{y'} = ABC_{w'} = A(BC_{w'})$

而 $BC_{w'} \in K^s$，设 $C_{x'} = BC_{w'} \in K^s$ 为 x' 坐标，则由 $x' = \varphi_B(w')$

则 $y' = \varphi_A(x') \overset{x' \in U}{=\!=} f(x)$，故 $y' \in Im f$

$\therefore V \subset Im f$

综合 <1><2>，$Im f = V$

(5) 证明：$\because x \in \ker f \Leftrightarrow \begin{cases} x \in U \\ f(x) = 0 \end{cases} \Leftrightarrow \begin{cases} x \in U \\ \varphi_A(x) = 0 \end{cases} \Leftrightarrow \begin{cases} x \in U \\ x \in \ker \varphi_A \end{cases} \Leftrightarrow x \in U \cap \ker \varphi_A$

$\therefore \ker f = U \cap (\ker \varphi_A)$

$\therefore \ker f \subset \ker \varphi_A$

(6) 由(4)已证：$Im f = Im\,\varphi_{AB}$

而 $\dim_K(Im f) = \dim_K U - \dim_K(\ker f)$

$\Rightarrow \dim_K(Im\,\varphi_{AB}) = \dim_K(Im\,\varphi_B) - \dim_K(\ker f)$ ①

由定理 3.3.1 (3)：$\dim_K(Im\,\varphi_{AB}) = r(AB)$ ②

$\dim_K(Im\,\varphi_B) = r(B)$ ③

而 $\ker f \subset \ker \varphi_A \Rightarrow \dim_K(\ker f) \le \dim(\ker \varphi_A)$

$= \dim_K K^s - \dim(Im\,\varphi_A)$

$= s - r(A)$ ④

②③④ 代入①：

$r(AB) \ge r(B) - (s - r(A)) \Rightarrow r(A) + r(B) - s \le r(AB)$

证毕

6. 证明：易知 φ_{A+B} 坐标矩阵为 $A+B$

设所取基为 $x_1, x_2, \cdots x_n \in K^n$

$y_1, y_2, \cdots, y_m \in K^m$

(1) 设 $\forall x \in K^n$，$y \in K^m$ 在上述基下有坐标 C_x，C_y

$\because y \in Im(\varphi_{A+B}) \Leftrightarrow C_y \in \{(A+B)C_x \mid x \in K^n\}$

$\Leftrightarrow C_y \in \{AC_x + BC_x \mid x \in K^n\}$

$\Rightarrow C_y \in \{AC_x \mid x \in K^n\} + \{BC_x \mid x \in K^n\}$

$\Leftrightarrow y \in Im(\varphi_A) + Im(\varphi_B)$

$\therefore$ 即 $y \in Im(\varphi_{A+B}) \Rightarrow y \in Im(\varphi_A) + Im(\varphi_B)$

$\therefore Im(\varphi_{A+B}) \subset Im(\varphi_A) + Im(\varphi_B)$

(2) $\because Im(\varphi_{A+B}) \subset Im(\varphi_A) + Im(\varphi_B)$

$\Rightarrow dim_K Im(\varphi_{A+B}) \le dim_K[Im(\varphi_A) + Im(\varphi_B)]$

$= dim_K Im(\varphi_A) + dim_K Im(\varphi_B) - dim_K\{[Im(\varphi_A)] \cap [Im(\varphi_B)]\}$

$\le dim_K Im(\varphi_A) + dim_K Im(\varphi_B)$

$\Rightarrow r(A+B) \le r(A) + r(B)$

7. 证明：由 5.(6) 已证：

$r(AB) + r(C) \le n + r(ABC) = n + r(O) = n + 0 = n$ ①

$\cancel{r(AB) \le}$ $r(A) + r(B) \le n + r(AB)$ ②

① + ② 即得：$r(A) + r(B) + r(C) \le n + n = 2n$

即 $r(A) + r(B) + r(C) \le 2n$

8. 证明：φ 是单射 $\Leftrightarrow \forall x_1, x_2$ 若 $\varphi(x_1) = \varphi(x_2)$ 则 $x_1 = x_2$

$\Leftrightarrow \forall x_1, x_2$，若 $\varphi(x_1) - \varphi(x_2) = 0$，则 $x_1 - x_2 = 0$

$\xLeftrightarrow{\varphi\text{是线性映射}} \forall x_1, x_2$，若 $\varphi(x_1 - x_2) = 0$，则 $x_1 - x_2 = 0$

$\Leftrightarrow \forall x_1, x_2$，若 $x_1 - x_2 \in ker\varphi$，则 $x_1 - x_2 = 0$

$\Leftrightarrow ker\varphi = \{0\}$

即 φ 是单射 $\Leftrightarrow ker\varphi = \{0\}$

9. 证明：(1) 设 $\forall f(x), g(x) \in \mathbb{R}[x]_n$，$\forall \lambda \in \mathbb{R}$

则 $l_k(f(x)+g(x)) = \dfrac{(f+g)^{(k)}(0)}{k!}$

$= \dfrac{f^{(k)}(0) + g^{(k)}(0)}{k!}$

$= \dfrac{f^{(k)}(0)}{k!} + \dfrac{g^{(k)}(0)}{k!}$

$= l_k(f(x)) + l_k(g(x))$

$l_k(\lambda f(x)) = \dfrac{(\lambda f)^{(k)}(0)}{k!} = \dfrac{\lambda f^{(k)}(0)}{k!} = \lambda \dfrac{f^{(k)}(0)}{k!} = \lambda l_k(f(x))$

$\therefore l_k$ 为线性函数

(2) 证明：设 $\forall f(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} \in \mathbb{R}[x]_n$

$k!a_k + \frac{(k+1)!}{1!}a_{k+1}x + \cdots \frac{(n-1)!}{(n-k-1)!}x^{n-k-1}$

$\therefore f^{(k)}(0) = k!a_k \Rightarrow a_k = \frac{f^{(k)}(0)}{k!} = l_k(f(x))$, $0 \le k \le n-1$

$\therefore f(x) = l_0(f(x))\cdot 1 + l_1(f(x))\cdot x + \cdots + l_{n-1}(f(x))\cdot x^{n-1}$

故 $l_0, l_1, \cdots l_{n-1}$ 为 $1, x, \cdots x^{n-1}$ 的对偶基

10. 证明：由于 $l_1, l_2, \cdots l_m$ 的顺序不影响证明

故不妨设 $l_1, l_2, \cdots l_s$ 为 $l_1, l_2, \cdots l_m$ 的极大无关组

则 $\langle l_1, l_2, \cdots l_m\rangle = \langle l_1, l_2, \cdots l_s\rangle := W^*$, $\dim_K W^* = s$

$\therefore x \in \ker\varphi \Leftrightarrow l_1(x) = \cdots = l_m(x) = 0 \Leftrightarrow l_1(x) = \cdots = l_s(x) = 0$

取 V 的一组基 $x_1, x_2, \cdots x_n$

则 $(l_1(x_1) \cdots l_1(x_n))$、$\cdots (l_s(x_1), \cdots l_s(x_n)) \in K^s$ 线性无关

设 $x = a_1x_1 + \cdots + a_nx_n \in \ker\varphi$

则 $l_1(x) = a_1 l_1(x_1) + \cdots + a_n l_1(x_n) = 0$

$\vdots$

$l_s(x) = a_1 l_s(x_1) + \cdots + a_n l_s(x_n) = 0$

即有线性方程组：$\begin{pmatrix} l_1(x_1) & \cdots & l_1(x_n) \\ \vdots & & \vdots \\ l_s(x_1) & \cdots & l_s(x_n) \end{pmatrix}\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

记为 $LA = 0$，由前分析 $r(L) = s$

$\therefore$ 线性方程组解空间 $N(A)$ 维数 $\dim_K N(A) = n - r(L) = s$

设 $\eta_1, \eta_2, \cdots \eta_{n-s}$ 为其一个基础解系

则 $\begin{cases} \alpha_1 = (x_1, \cdots x_n)\eta_1 \\ \vdots \\ \alpha_{n-s} = (x_1, \cdots x_n)\eta_{n-s} \end{cases}$ 为 $\ker\varphi$ 一组基

$\therefore \dim_K(\ker\varphi) = n - s = n - \dim_K W^*$

代入 W^* 即有：$\dim_K(\ker\varphi) = n - \dim_K(\langle l_1, \cdots l_m\rangle)$

第16周作业(高代A)

第一次(2021.12.01)

A

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1. 解: (a) 原式 $= \begin{vmatrix} 1 & -1 & 1 & -2 \\ 0 & 4 & -2 & 5 \\ 0 & -2 & 5 & 1 \\ 0 & -3 & -5 & -19 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & -7 & -14 \\ 0 & -2 & 5 & 1 \\ 0 & -3 & -5 & -19 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & -7 & -14 \\ 0 & 0 & -9 & -27 \\ 0 & 0 & -26 & -61 \end{vmatrix}$

$= 9\begin{vmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & -7 & -14 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 26 & 61 \end{vmatrix} = 9\begin{vmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & -7 & -14 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -17 \end{vmatrix}$

$= 9\times(-17) = -153$

(b) 原式 $= \begin{vmatrix} 1 & n-1 & n-1 & \cdots & n-1 & n-1 \\ n & 2-n & 0 & \cdots & 0 & 0 \\ n & 0 & 3-n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 0 & 0 & \cdots & -1 & 0 \\ n & 0 & 0 & \cdots & 0 & 0 \end{vmatrix} = (n-1)\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ n & 2-n & 0 & \cdots & 0 & 0 \\ n & 0 & 3-n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n & 0 & 0 & \cdots & -1 & 0 \\ n & 0 & 0 & \cdots & 0 & 0 \end{vmatrix}$

$= n(n-1)\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 2-n & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3-n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{vmatrix} = -n(n-1)\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2-n & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3-n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix}$

$= -n(n-1)\cdot(-1)^{n-2}(n-2)! = (-1)^{n-1}n!$

(c). 原式 $= \begin{vmatrix} a+(n-1)b & b & \cdots & b & b \\ a+(n-1)b & a & \cdots & b & b \\ \vdots & \vdots & & \vdots & \vdots \\ a+(n-1)b & b & \cdots & a & b \\ a+(n-1)b & b & \cdots & b & a \end{vmatrix} = [a+(n-1)b]\begin{vmatrix} 1 & b & \cdots & b & b \\ 1 & a & \cdots & b & b \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & b & \cdots & a & b \\ 1 & b & \cdots & b & a \end{vmatrix}$

$= [a+(n-1)b]\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & a-b & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & \cdots & a-b & 0 \\ 1 & 0 & \cdots & 0 & a-b \end{vmatrix} = (a-b)^{n-1}[a+(n-1)b]$

2. 证明: $\begin{vmatrix} 1 & 7 & 9 & 8 \\ 2 & 1 & 3 & 9 \\ 3 & 2 & 5 & 5 \\ 4 & 8 & 6 & 7 \end{vmatrix} \xlongequal{C_4+10C_3+100C_2+1000C_1} \begin{vmatrix} 1 & 7 & 9 & 1798 \\ 2 & 1 & 3 & 2139 \\ 3 & 2 & 5 & 3255 \\ 4 & 8 & 6 & 4867 \end{vmatrix}$

不妨设 $D = \begin{vmatrix} 1 & 7 & 9 & 8 \\ 2 & 1 & 3 & 9 \\ 3 & 2 & 5 & 5 \\ 4 & 8 & 6 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 7 & 9 & 1798 \\ 2 & 1 & 3 & 2139 \\ 3 & 2 & 5 & 3255 \\ 4 & 8 & 6 & 4867 \end{vmatrix}$

再设 $d = (1798, 2139, 3255, 4867)$

$1798 = dd_1$

$2139 = dd_2$

$3255 = dd_3$

$4867 = dd_4$

则 $D = \begin{vmatrix} 1 & 7 & 9 & dd_1 \\ 2 & 1 & 3 & dd_2 \\ 3 & 2 & 5 & dd_3 \\ 4 & 8 & 6 & dd_4 \end{vmatrix} = d\begin{vmatrix} 1 & 7 & 9 & d_1 \\ 2 & 1 & 3 & d_2 \\ 3 & 2 & 5 & d_3 \\ 4 & 8 & 6 & d_4 \end{vmatrix} \Rightarrow d \mid D$

证毕

3.解：(1) $f(x,y)$是K^n上的对称双线性函数，下证之：

显然 $f(x,y)$是双线性函数

而 $\forall x,y\in K^n$，${}^t x\cdot y \in K$，故 $({}^t x\cdot y)^t = {}^t x\, y$

$\Rightarrow y^t x = {}^t x\, y$

即 $f(x,y)=f(y,x)$

∴ $f(x,y)$是K^n上的对称双线性函数

(2) $f(A,B)$是$M_{m\times n}(K)$上的对称双线性函数，下证之：

①：$\forall \lambda,\mu\in K$，$A_1,A_2,B_1,B_2\in M_{m\times n}(K)$：

$$f(A_1,\lambda B_1+\mu B_2) = tr({}^tA_1\cdot(\lambda B_1+\mu B_2))$$
$$= tr(\lambda\, {}^tA_1 B_1 + \mu\, {}^tA_1 B_2)$$
$$= \lambda\, tr({}^tA_1B_1) + \mu\, tr({}^tA_1B_2)$$
$$= \lambda f(A_1,B_1)+\mu f(A_1,B_2)$$

同理 $f(\lambda A_1+\mu A_2, B_1) = \lambda f(A_1,B_1)+\mu f(A_2,B_1)$

∴ f是$M_{m\times n}(K)$上的双线性函数

② ∵ $\forall A,B\in M_{m\times n}(K)$，${}^tA\cdot B\in M_n(K)$

∴ $tr({}^tA\cdot B) = tr({}^t({}^tA\cdot B)) = tr({}^tB\cdot A)$

即 $f(A,B)=f(B,A)$ 对称

∴由①②：$f(A,B)$是$M_{m\times n}(K)$上的双线性函数

(3) ~~$f(A,B)$不是~~ 当$n=1$时，显然f是$M_n(K)$上的对称双线性函数；

当$n\geq 2$时，设集类 $\mathcal{F}_1 = \{W \mid W\subset M_n(K)$为子空间，$\forall A\in W$，$\det A=0\}$

$\mathcal{F}_2 = \{W\mid W\subset M_n(K)$为子空间，$\exists A_0\in W$，$\det A_0\neq 0\}$

（注：$\{0\}\in\mathcal{F}_1$，$M_n(K)\in\mathcal{F}_2$，故$\mathcal{F}_1$、$\mathcal{F}_2$均非空）

设$V_1\subset M_n(K)$为子空间，则：① $V_1\in\mathcal{F}_1$时 f是V_1上的对称双线性函数

② $V_1\in\mathcal{F}_2$时 f不是V_1上的对称双线性函数

证明：∵ $\forall A,B\in M_n(K)$

$$f(A,B)=\det(AB)=(\det A)(\det B)=(\det B)(\det A)=\det(BA)=f(B,A)$$

∴ f为V_1上对称双线性函数 $\Leftrightarrow$ f为V_1上双线性函数

①若$V_1\in\mathcal{F}_1$：∵ $\forall A,B\in V_1$，$f(A,B)=\det(AB)=(\det A)(\det B)=0$

∴ $f(x)$为V_1上的零函数，显然是双线性函数，进而是对称双线性函数

②若$V_1\in\mathcal{F}_2$：设$A_0\in W$，$\det A_0\neq 0$

则$2A_0\in W$，$3A_0\in W$ 且 $\det(2A_0)$、$\det(3A_0)$至少有一个非0

（后者 char $K=2$时非零）

只分析前者：

假设f在V_1上是双线性函数

则 $f(2A_0,A_0)=2f(A_0,A_0) \Rightarrow \det(2A_0^2)=2\det(A_0^2)$

$\Rightarrow 2^n(\det A_0)^2 = 2(\det A_0)^2 \Rightarrow 2^{n-1}=1$，这是不可能的

综合①②，证毕

设集类 $\mathcal{F}=\{W \mid W\subset M_n(K)$ 为子空间 $\forall A,B\in W,\ AB=BA=(AB)^T\}$

由于 $W=\{kI_n \mid \forall k\in K\}\in\mathcal{F}$，故 $\mathcal{F}\neq\phi$

~~则于是~~ 则 $\forall V\in\mathcal{F}$，f 是 V 上的对称双线性函数

但 f 不是 $M_n(K)$ 上的对称双线性函数

由于 $(A+B)C=AC+BC$，故 f 是对称双线性函数 $\Leftrightarrow f(B,A)=f(A,B)$

证明之：① 设 $AB=C$，则 $BA=C$ 且 $C=C^T$

$\therefore f(A,B)=C_{ij}\xlongequal{C=C^T}C_{ji}=f(B,A)$

$\therefore f$ 是 V 上的对称双线性函数

② 设 $A=\begin{pmatrix}1&1\\-1&-1\end{pmatrix}$，$B=\begin{pmatrix}0&1\\-1&1\end{pmatrix}$

取 $i=1$，$j=2$

则 $f(A,B)=ent_{12}\begin{pmatrix}-1&2\\1&-2\end{pmatrix}=2$

$f(B,A)=ent_{12}\begin{pmatrix}-1&-1\\-2&-2\end{pmatrix}=-1$

$\therefore f(A,B)\neq f(B,A)$

(5) 记 $V=\{g(x) \mid g(x)\in C[a,b]\}$，显然 V 为 $\mathbb{R}$ 上线性空间

而：$\forall u_1(x),u_2(x),v_1(x),v_2(x)\in V$，$\forall\lambda,\mu\in\mathbb{R}$

$f(\lambda u_1(x)+\mu u_2(x),v_1(x))=\int_a^b(\lambda u_1(x)+\mu u_2(x))v_1(x)x^2dx$

$=\lambda\int_a^b u_1(x)v_1(x)x^2dx+\mu\int_a^b u_2(x)v_1(x)x^2dx$

$=\lambda f(u_1(x),v_1(x))+\mu f(u_2(x),v_1(x))$

同理 $f(u_1(x),\lambda v_1(x)+\mu v_2(x))=\lambda f(u_1(x),v_1(x))+\mu f(u_1(x),v_2(x))$

且 $\forall u(x),v(x)\in V$，有 $f(u(x),v(x))=\int_a^b u(x)v(x)x^2dx=\int_a^b v(x)u(x)x^2dx=f(v(x),u(x))$

$\therefore f$ 是 V 上的对称双线性函数

(6) f 不是 (5) 中 V 上的对称双线性函数，下举一反例：

令 $u(x)=x$，$x\in[a,b]$，显然 $u(x)\in V$

则 $f(2u(x),u(x))=\int_a^b(2u(x)+u(x))^2dx=9\int_a^b x^2dx$

$f(u(x),u(x))=\int_a^b(u(x)+u(x))^2dx=4\int_a^b x^2dx$

故 $f(2u(x),u(x))-2f(u(x),u(x))\neq0$，即 f 不是 V 上的双线性函数

进而不可能是对称双线性函数

(7) 设 $\forall u_1(x),u_2(x),v_1(x),v_2(x)\in K[x]$，$\forall\lambda,\mu\in K$

则 $f(\lambda u_1(x)+\mu u_2(x),v_1(x))=[(\lambda u_1+\mu u_2)\cdot v_1](a)$

$=\lambda(u_1v_1)(a)+\mu(u_2v_2)(a)=\lambda f(u_1,v_1)+\mu f(u_2,v_2)$

同理 $f(u_1,\lambda v_1+\mu v_2)=\lambda f(u_1,v_1)+\mu f(u_1,v_2)$

又 $\forall u,v\in K[x]$，$f(u,v)=(u\cdot v)(a)=(v\cdot u)(a)$

$\therefore f(x)$ 是 $K[x]$ 上的对称双线性函数

(8) 形式导数满足：$\frac{d(uv)}{dx}=(\frac{du}{dx})v+u(\frac{dv}{dx})$

而 $\frac{du}{dx}$、$\frac{dv}{dx}\in K[x]$

设 $\forall u_1(x), u_2(x), v_1(x), v_2(x)\in K[x]$，$\forall \lambda, \mu\in K$

则 $f(\lambda u_1+\mu u_2, v_1)=\left[\frac{d(\lambda u_1+\mu u_2)}{dx}v_1+(\lambda u_1+\mu u_2)\frac{dv_1}{dx}\right](a)$

$=\lambda\left[(\frac{du_1}{dx})v_1+u_1(\frac{dv_1}{dx})\right](a)+\mu\left[(\frac{du_2}{dx})v_1+u_2(\frac{dv_1}{dx})\right](a)$

$=\lambda f(u_1,v_1)+\mu f(u_2,v_1)$

同理 $f(u_1,\lambda v_1+\mu v_2)=\lambda f(u_1,v_1)+\mu f(u_1,v_2)$

而 $f(u_1,v_1)=\left[(\frac{du_1}{dx})v_1+u_1(\frac{dv_1}{dx})\right](a)=\left[(\frac{dv_1}{dx})u_1+v_1(\frac{du_1}{dx})\right](a)$

$=f(v_1,u_1)$

$\therefore f$ 是 $K[x]$ 上对称双线性函数

4. 证明：(1) 只需证 $f_1(x)$，$f_2(x)$ 同理可证

设 $\forall \alpha,\beta\in V$，$\lambda,\mu\in k$

则 $f_1(x)(\lambda\alpha+\mu\beta)=f(x,\lambda\alpha+\mu\beta)\xlongequal{f双线性}\lambda f(x,\alpha)+\mu f(x,\beta)$

$=\lambda f_1(x)(\alpha)+\mu f_1(x)(\beta)$

$\therefore f_1(x)$ 为 V 上的线性函数

~~证毕~~

(2) 只需证 f_1，f_2 同理可证

设 $\forall \alpha, x_1, x_2\in V$，$\forall\lambda,\mu\in K$

则 $f_1(\lambda x_1+\mu x_2)(\alpha)=f(\lambda x_1+\mu x_2,\alpha)\xlongequal{f双线性}\lambda f(x_1,\alpha)+\mu f(x_2,\alpha)$

$=\lambda f_1(x_1)(\alpha)+\mu f_1(x_2)(\alpha)$

$\Rightarrow f_1(\lambda x_1+\mu x_2)=\lambda f_1(x_1)+\mu f_1(x_2)$

$\therefore f_1$ 为线性映射

(3) $f_1=f_2\Leftrightarrow\forall x\in V$，$f_1(x)=f_2(x)$

$\Leftrightarrow\forall x\in V$，$\forall\alpha\in V$，$f_1(x)(\alpha)=f_2(x)(\alpha)$

$\Leftrightarrow\forall x\in V$，$\forall\alpha\in V$，$f(x,\alpha)=f(\alpha,x)$

$\Leftrightarrow f$ 对称

证毕

(4) 由(2)已证 f_1、f_2 均为线性映射，故 f_1、f_2 是同构 $\Leftrightarrow f_1$、f_2 为双射

① 先证一结论：若 f 为 $V_1\to V_2$ 的线性映射且 $\dim_K V_1=\dim_K V_2$，则 f 为单射 $\Leftrightarrow f$ 为双射

"$\Leftarrow$" 显然，故只证 "$\Rightarrow$"：

$\because f$ 为线性映射

$\therefore \dim_K \ker f+\dim_K \mathrm{Im} f=\dim_K V_1$

$\because f$ 为单射 $\Rightarrow \ker f=\{0\}$，$\dim_K V_1=\dim_K V_2$

$\therefore \dim_K \ker f=0$，故 $\dim_K \mathrm{Im} f=\dim_K V_2$

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$\therefore \operatorname{Im} f = V_2$，故f是满射，由又f是单射 故f是双射

②现证 f_1, f_2 为同构 $\Leftrightarrow$ f非退化：

f_1, f_2 为同构 $\Leftrightarrow$ f_1, f_2 为双射 $\xLeftrightarrow{(1)已证}$ f_1, f_2 为单射

$\Leftrightarrow \ker f_1 = \ker f_2 = \{0\}$

而 $\ker f_1 = \{0\} \Leftrightarrow \forall x \neq 0 \in V, f_1(x) \neq 0 \in V^*$

$\Leftrightarrow \forall x \neq 0 \in V, \exists \alpha \in V, f_1(x)(\alpha) \neq 0 \in V$

$\Leftrightarrow \forall x \neq 0 \in V, \exists \alpha \in V, f(x, \alpha) \neq 0$

$\xLeftrightarrow{f为双线性函数}$ f非退化

同理 $\ker f_2 = \{0\} \Leftrightarrow$ f非退化

$\therefore$ ~~f_1, f_2为同构~~ $\ker f_1 = \ker f_2 = \{0\} \Leftrightarrow$ f非退化

$\therefore f_1, f_2$ 为同构 $\Leftrightarrow$ f非退化

5、证明：采用 $(2) \Rightarrow (1) \Rightarrow (3) \Rightarrow (2)$ 的证明方法

令 $A = [a_{ij}]$

证 $(2) \Rightarrow (1)$：构造矩阵 $A^* = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$

其中 A_{ij} 为 a_{ij} 的代数余子式

则 $AA^* = A^*A = |A|E$

又 $\because |A| \neq 0$

$\therefore (\frac{1}{|A|}A^*)A = A(\frac{1}{|A|}A^*) = E$

故A可逆且 $A^{-1} = \frac{1}{|A|}A^*$

证 $(1) \Rightarrow (3)$：$\because$ A可逆

$\therefore$ 齐次线性方程组 $AX = 0$ 只有零解

由~~线性~~齐次线性方程组解所对应条件知 $r(A) = r_r(A) = n$

证 $(3) \Rightarrow (2)$：由于初等行变化下秩不改变

故对A进行初等行变化将其化为阶梯形：

$A \xrightarrow[行变化]{初等} \begin{bmatrix} a_{11}' & a_{12}' & \cdots & a_{1n}' \\ 0 & a_{22}' & \cdots & a_{2n}' \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn}' \end{bmatrix}$

其中 $a_{11}', a_{22}', \cdots, a_{nn}' \neq 0$，否则与 $r(A) = n$ 矛盾

$\therefore |A| = (-1)^q a_{11}' a_{22}' \cdots a_{nn}' \neq 0$

综上，命题证毕

6、证明：显然 $\forall x, y \in V, f_A(x, y) \in K$

$\therefore {}^t f_A(x, y) = f_A(x, y)$

$\Rightarrow {}^t[{}^t[e^*(x)] A [e^*(y)]] = {}^t[e^*(x)] A [e^*(y)]$

$\Rightarrow {}^t[e^*(y)] {}^tA [e^*(x)] = {}^t[e^*(x)] A [e^*(y)]$

其中 $e^*(x)$ 处X为 $\begin{pmatrix} e_1^*(x) \\ \vdots \\ e_n^*(x) \end{pmatrix}$，$e^*(y)$ 同理

∴ f 对称 $\Leftrightarrow \forall x,y\in V,\ f(x,y)=f(y,x)$

$\Leftrightarrow \forall x,y\in V,\ {}^t[e^*(x)]A[e^*(y)] = {}^t[e^*(y)]A[e^*(x)]$

$\xLeftrightarrow{\text{前已证}} \forall x,y\in V,\ {}^t[e^*(x)]A[e^*(y)] = {}^t[e^*(x)]\,{}^tA[e^*(y)]$

$\Leftrightarrow A={}^tA$

同理 f 反对称 $\Leftrightarrow A=-{}^tA$

证毕

7. 解：(a) $q(x_1,x_2)={}^t\begin{pmatrix}x_1\\x_2\end{pmatrix}\begin{pmatrix}1&\frac{3}{2}\\ \frac{3}{2}&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$

(b) $q(x_1,x_2,x_3)={}^t\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\begin{pmatrix}1&\frac{1}{2}&\sqrt{2}\\ \frac{1}{2}&-1&0\\ \sqrt{2}&0&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

8. 证明：∵ ${}^tA=A$

∴ ∃正交阵 Q 使 ${}^tQAQ=\mathrm{diag}(\lambda_1,\cdots\lambda_n)$ 其中 λ_i 为 A 的特征值

∵ A 可逆，故 $\lambda_i\neq 0$

∴ A^{-1} 的特征值为 $\frac{1}{\lambda_1},\frac{1}{\lambda_2},\cdots\frac{1}{\lambda_n}$

∴ $Q^{-1}A^{-1}({}^tQ^{-1})=\mathrm{diag}(\frac{1}{\lambda_1},\frac{1}{\lambda_2},\cdots\frac{1}{\lambda_n})$

又即 ${}^tQA^{-1}Q=\mathrm{diag}(\frac{1}{\lambda_1},\frac{1}{\lambda_2},\cdots\frac{1}{\lambda_n})$

不妨 $\lambda_1,\cdots\lambda_p>0$，$\lambda_{p+1}\cdots\lambda_n<0$

则 $\frac{1}{\lambda_1},\cdots\frac{1}{\lambda_p}>0$，$\frac{1}{\lambda_{p+1}}\cdots\frac{1}{\lambda_n}<0$

~~故A与~~

∴ $f_A(x_1,\cdots,x_n)$ 和 $f_{A^{-1}}(x_1,\cdots,x_n)$ 有相同的正惯性指数

9. 证明：(1) 设 $A=[a_{ij}]$，$B=[b_{ij}]$，$A,B\in M_n(k)$

则 $tr(AB)=\sum_{j=1}^n(\sum_{i=1}^n a_{ji}b_{ij})=\sum_{j=1}^n\sum_{i=1}^n b_{ij}a_{ji}=\sum_{i=1}^n\sum_{j=1}^n b_{ij}a_{ji}$

$=\sum_{i=1}^n(\sum_{j=1}^n b_{ij}a_{ji})=tr(BA)$

∵ $\forall A_1,A_2,B_1,B_2\in M_n(k)$，$\forall\lambda,\mu\in K$

有 $(\lambda A_1+\mu A_2)B_1=\lambda A_1B_1+\mu A_2B_1$

$A_1(\lambda B_1+\mu B_2)=\lambda A_1B_1+\mu A_1B_2$

∴ $f(\lambda A_1+\mu A_2,B_1)=tr(\lambda A_1B_1+\mu A_2B_1)=\lambda tr(A_1B_1)+\mu tr(A_2B_1)$

$=\lambda f(A_1,B_1)+\mu f(A_2,B_1)$

同理 $f(A_1,\lambda B_1+\mu B_2)=\lambda f(A_1,B_1)+\mu f(A_1,B_2)$

∴ f 是 $M_n(k)\times M_n(k)\to K$ 的一个对称双线性函数

而 $\forall A\in M_n(k)$，$\exists A^T\in M_n(k)$，$tr(AA^T)=\sum_{j=1}^n\sum_{i=1}^n a_{ij}a_{ij}=\sum_{i=1,j=1}^n a_{ij}^2$

若 $A\neq 0$，则 $\exists a_{st}\neq 0$，故 $f(A,A^T)=tr(AA^T)=\sum_{i=1,j=1}^n a_{ij}^2\geq a_{st}^2>0$

$M_n(k)\times M_n(k)\longrightarrow k$ 的一个非退化对称双线性函数

取 $M_n(k)$ 的 组标准基 $E_{11}, E_{12},\cdots E_{1n},\cdots E_{n1}, E_{n2},\cdots E_{nn}$

其中 E_{ij} 定义为: $entst\ E_{ij}=\begin{cases}0 & s\neq i 或 t\neq j\\ 1 & s=i, t=j\end{cases}$

则 $\underbrace{M}_{\forall M=[M_{ij}]\in M_n(k)} E_{ij}=[0,0,\cdots \overset{第j列}{M^{(i)}},0\cdots 0]$

$\therefore tr(ME_{ij})=m_{ji}$，故 $m_{ij}=tr(ME_{ji})$

令 $a_{ij}=l(E_{ji})$，$A=[a_{ij}]\in M_n(k)$

则由上分析: $tr(AE_{ij})=a_{ji}=l(E_{ij})$

而由(1)已证: $f: M_n(k)\times M_n(k)\to k$，$(A_i,B_i)\mapsto tr(A_iB_i)$

为非退化对称双线性函数

$\therefore$ 由4(1)已证: $\bar f: M_n(k)\to k$，$X\mapsto f(AX)$ 为线性函数

下证 $\bar f=l$: 由上述分析: $\bar f(E_{ij})=tr(AE_{ij})=l(E_{ij})$

$\forall S\in M_n(k)$，$\underbrace{\quad}_{S=[s_{ij}]}$ 可由 $\{E_{ij}\}$ 线性表示，即 $S=\sum_{\substack{i=1\\j=1}}^{n} s_{ij}E_{ij}$

又 $\bar f$、l 均为线性函数，故 $\bar f(S)=\bar f(\sum_{\substack{i=1\\j=1}}^{n} s_{ij}E_{ij})=\sum_{\substack{i=1\\j=1}}^{n} s_{ij}\bar f(E_{ij})$

$=\sum_{\substack{i=1\\j=1}}^{n} s_{ij}l(E_{ij})=l(\sum_{\substack{i=1\\j=1}}^{n} s_{ij}E_{ij})=l(S)$

$\therefore \bar f=l$

即 $\exists A$ 使 ~~$=f(AX)$~~ $l(X)=f(AX)=tr(AX)$，证毕

10. 证明: (1) 设 $\forall f,g\in \mathbb{R}[x_1,\cdots x_n]$，$\forall\lambda\in\mathbb{R}$

则 $L(f+g)=\sum_{i,j}a_{ij}\frac{\partial^2(f+g)}{\partial x_i\partial x_j}=\sum_{i,j}a_{ij}\left(\frac{\partial^2 f}{\partial x_i\partial x_j}+\frac{\partial^2 g}{\partial x_i\partial x_j}\right)$

$=\sum_{i,j}a_{ij}\frac{\partial^2 f}{\partial x_i\partial x_j}+\sum_{i,j}a_{ij}\frac{\partial^2 g}{\partial x_i\partial x_j}=L(f)+L(g)$

同理 $L(\lambda f)=\lambda L(f)$

$\therefore L$ 是一个 $\mathbb{R}$-线性映射

(2) 先证一引理: $\forall f\in\mathbb{R}[x_1,\cdots x_n]$，$\frac{\partial f}{\partial x_i}=\sum_{j=1}^{n}\frac{\partial f}{\partial y_j}\cdot\frac{\partial y_j}{\partial x_i}$

同(1)可证 $D_i(f)=\frac{\partial f}{\partial x_i}$ 为线性映射

故只需对 $\underbrace{f=x_1^{t_1}\cdots x_n^{t_n}}_{单项式}$ 进行证明即可

而 $\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}=C\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$，$C$ 可逆，故不妨 $\underbrace{D}_{C=[c_{ij}],}=[d_{ij}]=C^{-1}$

则 $\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=D\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}$，故 $x_i^{t_i}=(d_{i1}y_1+\cdots+d_{in}y_n)^{t_i}$

$\therefore \frac{\partial f}{\partial x_1}=t_1x_1^{t_1-1}x_2^{t_2}\cdots x_n^{t_n}$

$\frac{\partial f}{\partial y_j}=\sum_{k=1}^{n}\left[(t_k-1)x_k^{t_k-1}\left(\prod_{\substack{s\neq k\\ s=1}}x_s^{t_s}\right)d_{kj}\right]$

$\frac{\partial y_j}{\partial x_1}=c_{j1}$

$\therefore \sum_{j=1}^{n} \frac{\partial f}{\partial y_j}\frac{\partial y_j}{\partial x_1} = \sum_{j=1}^{n}\left(\sum_{k=1}^{n} t_k x_k^{t_k-1}\left(\prod_{\substack{s=1\\ s\neq k}}^{n} x_s^{t_s}\right) d_{kj}\, c_{j1}\right)$

$= \sum_{k=1}^{n}\left\{ t_k x_k^{t_k-1}\left(\prod_{\substack{s=1\\ s\neq k}}^{n} x_s^{t_s}\right)\left[\sum_{j=1}^{n} d_{kj} c_{j1}\right]\right\}$

而 $\sum_{j=1}^{n} d_{kj} c_{j1} = ({}^tD)^{(k)}\cdot C^{(1)} = ent_{k1}(DC)$

又 $DC = I_n$，故 $\sum_{j=1}^{n} d_{kj} c_{j1} = \begin{cases} 1 & k=1 \\ 0 & k\neq 1 \end{cases}$

$\therefore \sum_{j=1}^{n} \frac{\partial f}{\partial y_j}\frac{\partial y_j}{\partial x_1} = t_1 x_1^{t_1-1} x_2^{t_2}\cdots x_n^{t_n} = \frac{\partial f}{\partial x_1}$

同理 $\forall i$，$\frac{\partial f}{\partial x_i} = \sum_{j=1}^{n}\frac{\partial f}{\partial y_j}\frac{\partial y_j}{\partial x_i}$

$\therefore \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial}{\partial x_i}\left(\sum_{s=1}^{n}\frac{\partial f}{\partial y_s}\cdot\frac{\partial y_s}{\partial x_j}\right)$

$= \sum_{s=1}^{n}\left\{\left[\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial y_s}\right)\right]\cdot\frac{\partial y_s}{\partial x_j} + \left(\frac{\partial f}{\partial y_s}\right)\left[\frac{\partial}{\partial x_i}\left(\frac{\partial y_s}{\partial x_j}\right)\right]\right\}$

$= \sum_{s=1}^{n}\left(\sum_{t=1}^{n}\frac{\partial^2 f}{\partial y_t \partial y_s}\cdot\frac{\partial y_t}{\partial x_i}\frac{\partial y_s}{\partial x_j} + \frac{\partial f}{\partial y_s}\cdot\frac{\partial c_{sj}}{\partial x_i}\right)$

$= \sum_{s=1}^{n}\sum_{t=1}^{n}\frac{\partial^2 f}{\partial y_t \partial y_s} c_{ti} c_{sj} + 0$

$= \sum_{s=1}^{n}\sum_{t=1}^{n}\frac{\partial^2 f}{\partial y_t \partial y_s} c_{ti} c_{sj}$

$\therefore L(f) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{n}\sum_{t=1}^{n} a_{ij}\frac{\partial^2 f}{\partial y_t \partial y_s} c_{ti} c_{sj}$

$= \sum_{t=1}^{n}\sum_{s=1}^{n}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ti} a_{ij} c_{sj}\right)\frac{\partial^2 f}{\partial y_t \partial y_s}$

$\therefore b_{ts} = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ti} a_{ij} c_{sj}$

$= \sum_{i=1}^{n} c_{ti}\sum_{j=1}^{n} a_{ij} c_{sj}$

$\xlongequal{{}^tA = A \Rightarrow a_{ij} = a_{ji}} \sum_{i=1}^{n} c_{ti}\sum_{j=1}^{n} a_{ji} c_{sj}$

$= \sum_{i=1}^{n} c_{ti}\, p_{si} = \sum_{i=1}^{n} p_{si}\, c'_{it}$

显然 $p_{si} = ent_{si}(CA)$

$c'_{it} = c_{ti} = ent_{it}({}^tC)$

$\therefore b_{ts} = \sum_{i=1}^{n} p_{si}\, c'_{it} = ent_{st}(CA\,{}^tC)$

$= ent_{ts}\left({}^t(CA\,{}^tC)\right)$

$= ent_{ts}(C\,{}^tA\,{}^tC)$

$\xlongequal{{}^tA = A} ent_{ts}(CA\,{}^tC)$

$\therefore L(f) = \sum_{t=1}^{n}\sum_{s=1}^{n} b_{ts}\frac{\partial^2 f}{\partial y_t \partial y_s}$

$= \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\frac{\partial^2 f}{\partial y_i \partial y_j}$

其中 $B = [b_{ij}] = CA\,{}^tC$

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第17周作业（高代A）

第一次（2021|1209）

A

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转专业补修

1. 证明：∵ $\forall s\in N$，$(TAT^{-1})^s = TAT^{-1}TAT^{-1}\cdots TAT^{-1}$

$$= TA(T^{-1}T)A(T^{-1}T)\cdots(T^{-1}T)AT^{-1}$$

$$= TA^sT^{-1}$$

∴ 不妨 $f(x)=a_0+\cdots+a_mx^m$

则 $f(TAT^{-1}) = a_0(TT^{-1})+a_1(TAT^{-1})+\cdots+a_m(TAT^{-1})^m$

$$= a_0(TT^{-1})+a_1(TAT^{-1})+\cdots+a_m(TA^mT^{-1})$$

$$= T(a_0I_n+a_1A+\cdots+a_mA^m)T^{-1}$$

$$= Tf(A)T^{-1}$$

2. 解：(1) ∵ $f(x)=(x-1)(x^2-x-1)$

∴ $f(A)=(A-I_3)(A^2-A-I_3)$

∵ $A-I_3=\begin{pmatrix}2&1&0\\0&2&0\\1&1&1\end{pmatrix}-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&1&0\\0&1&0\\1&1&0\end{pmatrix}$

$A^2-A-I_3=\begin{pmatrix}2&1&0\\0&2&0\\1&1&1\end{pmatrix}^2-\begin{pmatrix}2&1&0\\0&2&0\\1&1&1\end{pmatrix}-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&3&0\\0&1&0\\2&4&-1\end{pmatrix}$

∴ $f(A)=\begin{pmatrix}1&1&0\\0&1&0\\1&1&0\end{pmatrix}\begin{pmatrix}1&3&0\\0&1&0\\2&4&-1\end{pmatrix}=\begin{pmatrix}1&4&0\\0&1&0\\1&4&0\end{pmatrix}$

(2) ∵ $f(x)=(x-1)^2(x+2)$ ∴ $f(A)=(A-I_3)^2(A+2I_3)$

而 $A-I_3=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$，$A+2I_3=\begin{pmatrix}4&1&1\\1&4&1\\1&1&4\end{pmatrix}$

∴ $f(A)=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}^2\times\begin{pmatrix}4&1&1\\1&4&1\\1&1&4\end{pmatrix}=18\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$

3. 证明：不妨 $A=[a_{ij}]_{n\times n}$

则 $E_{ii}A=\begin{bmatrix}0&0&\cdots&0\\\vdots&\vdots&&\vdots\\0&0&\cdots&0\\a_{i1}&a_{i2}&\cdots&a_{in}\\0&0&\cdots&0\\\vdots&\vdots&&\vdots\\0&0&\cdots&0\end{bmatrix}$，$AE_{ii}=\begin{bmatrix}0&\cdots&0&a_{1i}&0&\cdots&0\\0&\cdots&0&a_{2i}&0&\cdots&0\\\vdots&&\vdots&\vdots&\vdots&&\vdots\\0&\cdots&0&a_{ni}&0&\cdots&0\end{bmatrix}$

而 $E_{ii}A=AE_{ii}$，对比有

$a_{1i}=\cdots=a_{(i-1)i}=a_{(i+1)i}=\cdots=a_{ni}=0$

$a_{i1}=\cdots=a_{i(i-1)}=a_{i(i+1)}=\cdots=a_{in}=0$

又由 i 任意性 得 $\forall i,j$，$i\neq j$，有 $a_{ij}=0$

∴ $A=\begin{bmatrix}a_{11}&&O\\&a_{22}&\\O&&\ddots\\&&&a_{nn}\end{bmatrix}$，证毕

4. 证明：(1) 不妨设 $A=[a_{ij}]_{n\times n}$，$B=[b_{ij}]_{n\times n}$

则 $tr(AB)=\sum_{i=1}^n\left(\sum_{j=1}^n a_{ij}b_{ji}\right)$

$$=\sum_{i=1}^n\sum_{j=1}^n a_{ij}b_{ji}$$

$$=\sum_{j=1}^n\sum_{i=1}^n b_{ji}a_{ij}$$

$$=\sum_{j=1}^n\left(\sum_{i=1}^n b_{ji}a_{ij}\right)$$

$$=tr(BA)$$

(2) $tr(TAT^{-1}) = tr[T(AT^{-1})] \xlongequal{(1)已证} tr[(AT^{-1})T]$

$= tr[A(T^{-1}T)] = tr(A)$

5.解：(1) 令 $A=\begin{pmatrix}1&2&3\\2&3&1\\3&1&2\end{pmatrix}$，$B=\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix}$

令 $\hat{A}=(A\vdots B)=\left(\begin{array}{ccc|cc}1&2&3&0&0\\2&3&1&1&0\\3&1&2&0&1\end{array}\right)$，对其进行初等行变换

$\hat{A}\to\left(\begin{array}{ccc|cc}1&2&3&0&0\\0&-1&-5&1&0\\0&-5&-7&0&1\end{array}\right)$

$\to\left(\begin{array}{ccc|cc}1&2&3&0&0\\0&1&5&-1&0\\0&-5&-7&0&1\end{array}\right)\to\left(\begin{array}{ccc|cc}1&2&3&0&0\\0&1&5&-1&0\\0&0&18&-5&1\end{array}\right)$

$\to\left(\begin{array}{ccc|cc}1&0&-7&2&0\\0&1&5&-1&0\\0&0&1&-\frac{5}{18}&\frac{1}{18}\end{array}\right)\to\left(\begin{array}{ccc|cc}1&0&0&\frac{1}{18}&\frac{7}{18}\\0&1&0&\frac{7}{18}&-\frac{5}{18}\\0&0&1&-\frac{5}{18}&\frac{1}{18}\end{array}\right)$

故 $X=\frac{1}{18}\begin{pmatrix}1&7\\7&-5\\-5&1\end{pmatrix}$

(2) 令 $A=\begin{pmatrix}1&1&0\\0&1&2\\0&1&1\end{pmatrix}$，$B=\begin{pmatrix}5\\-6\\-2\end{pmatrix}$，$\hat{A}=(A\vdots B)$

对$\hat{A}$作初等行变换，$\hat{A}=\left(\begin{array}{ccc|c}1&1&0&5\\0&1&2&-6\\0&1&1&-2\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&0&5\\0&1&2&-6\\0&0&-1&4\end{array}\right)$

$\to\left(\begin{array}{ccc|c}1&1&0&5\\0&1&0&2\\0&0&-1&4\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&0&3\\0&1&0&2\\0&0&1&-4\end{array}\right)$，故 $X=\begin{pmatrix}3\\2\\-4\end{pmatrix}$

(3) 令 $A_1=\begin{pmatrix}3&1\\2&1\end{pmatrix}$，$A_2=\begin{pmatrix}1&3\\1&2\end{pmatrix}$，$B=\begin{pmatrix}3&3\\2&2\end{pmatrix}$

则 $|A_1|=3-2=1$，$|A_2|=2-3=-1$

故 A_1，A_2 均可逆 且 $A_1^{-1}=\frac{1}{|A_1|}A_1^{*}=\begin{bmatrix}1&-1\\-2&3\end{bmatrix}$

$A_2^{-1}=\frac{1}{|A_2|}A_2^{*}=-\begin{bmatrix}2&-3\\-1&1\end{bmatrix}=\begin{bmatrix}-2&3\\1&-1\end{bmatrix}$

$\because A_1XA_2=B \Rightarrow X=A_1^{-1}BA_2^{-1}$

$=\begin{bmatrix}1&-1\\-2&3\end{bmatrix}\begin{bmatrix}3&3\\2&2\end{bmatrix}\begin{bmatrix}-2&3\\1&-1\end{bmatrix}$

$=\begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}-2&3\\1&-1\end{bmatrix}$

$=\begin{bmatrix}-1&2\\0&0\end{bmatrix}$

6.解：(1)对$[A\vdots I]$作初等行变换：

$[A\vdots I]=\left(\begin{array}{cccc|cccc}1&2&0&0&1&0&0&0\\2&3&0&0&0&1&0&0\\1&-1&1&3&0&0&1&0\\0&1&0&2&0&0&0&1\end{array}\right)\to\left(\begin{array}{cccc|cccc}1&2&0&0&1&0&0&0\\0&-1&0&0&-2&1&0&0\\0&-3&1&3&-1&0&1&0\\0&1&0&2&0&0&0&1\end{array}\right)$

$\to\left(\begin{array}{cccc|cccc}1&0&0&0&-3&2&0&0\\0&-1&0&0&-2&1&0&0\\0&0&1&3&5&-3&1&0\\0&0&0&2&-2&1&0&1\end{array}\right)$

$\to\left(\begin{array}{cccc|cccc}1&0&0&0&-3&2&0&0\\0&1&0&0&2&-1&0&0\\0&0&1&3&5&-3&1&0\\0&0&0&1&-1&\frac{1}{2}&0&\frac{1}{2}\end{array}\right)$

$\to\left(\begin{array}{cccc|cccc}1&0&0&0&-3&2&0&0\\0&1&0&0&2&-1&0&0\\0&0&1&0&8&-\frac{9}{2}&1&-\frac{3}{2}\\0&0&0&1&-1&\frac{1}{2}&0&\frac{1}{2}\end{array}\right)$

~~由~~故 $A^{-1} = \begin{pmatrix} -3 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 8 & -\frac{9}{2} & 1 & -\frac{3}{2} \\ -1 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$

且由~~初等~~变换过程知 ~~[illegible]~~

$$A = \begin{pmatrix} 1&0&0&0 \\ 2&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \end{pmatrix} \times \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 1&0&1&0 \\ 0&0&0&1 \end{pmatrix} \times \begin{pmatrix} 1&-2&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} \times \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&3&1&0 \\ 0&0&0&1 \end{pmatrix} \times \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&-1&0&1 \end{pmatrix} \times \begin{pmatrix} 1&0&0&0 \\ 0&-1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&2 \end{pmatrix} \times \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&3 \\ 0&0&0&1 \end{pmatrix}$$

(2) 对 $[A \vdots I]$ 作初等行变换:

$$[A \vdots I_n] = \left[\begin{array}{cccc|cccc} 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 1 & -1 & \cdots & 0 \\ 0 & 1 & \cdots & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 \end{array}\right]$$

$$\longrightarrow \cdots\cdots \to \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 1 & -1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 \end{array}\right]$$

$\therefore A^{-1} = \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & \ddots & -1 \\ & & & 1 \end{bmatrix}$, 又由变换过程有:

$$A = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

7. 证明: (1) $\because (\lambda A)(\lambda^{-1}A^{-1}) = (\lambda\lambda^{-1})(AA^{-1}) = I$

$(\lambda^{-1}A^{-1})(\lambda A) = (\lambda^{-1}\lambda)(A^{-1}A) = I$

$\therefore (\lambda A)^{-1} = \lambda^{-1}A^{-1}$, 证毕

(2) 记 $B = {}^t(A^{-1})$, 则 ${}^tB = A^{-1}$

则 ${}^tAB = {}^t({}^tBA) = {}^t(A^{-1}A) = {}^tI = I$

$B\,{}^tA = {}^t(A\,{}^tB) = {}^t(AA^{-1}) = {}^tI = I$

$\therefore ({}^tA)^{-1} = B = {}^t(A^{-1})$, 证毕

8. 证明: 设 $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, 则 $tr(A) = a_1 + a_4$, $|A| = a_1a_4 - a_2a_3$

$\because f(A) = A^2 - [tr(A)]A + |A| I_2$

$= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} - (a_1+a_4)\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + (a_1a_4 - a_2a_3)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} a_1^2 + a_2a_3 & a_1a_2 + a_2a_4 \\ a_1a_3 + a_3a_4 & a_2a_3 + a_4^2 \end{pmatrix} - \begin{pmatrix} a_1^2 + a_1a_4 & a_1a_2 + a_2a_4 \\ a_1a_3 + a_3a_4 & a_1a_4 + a_4^2 \end{pmatrix} + \begin{pmatrix} a_1a_4 - a_2a_3 & 0 \\ 0 & a_1a_4 - a_2a_3 \end{pmatrix}$

$= \begin{pmatrix} a_2a_3 - a_1a_4 & 0 \\ 0 & a_2a_3 - a_1a_4 \end{pmatrix} + \begin{pmatrix} a_1a_4 - a_2a_3 & 0 \\ 0 & a_1a_4 - a_2a_3 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$

证毕

9. 证明：不妨设 $\exists m_1, m_2$ 使 $A^{m_1}=0$, $B^{m_2}=0$

记 $m=m_1+m_2$ ~~[crossed out]~~

则 由于 $AB=BA$，故：

$$(A+B)^m = \sum_{k=0}^{m} C_m^k A^k B^{m-k} = \sum_{k=0}^{m_1} C_m^k A^k B^{m-k} + \sum_{k=m_1+1}^{m} C_m^k A^k B^{m-k}$$

而 $k \leq m_1$ 时 $m-k = m_1+m_2-k \geq m_2$，故 $B^{m-k} = B^{m_2+(m_1-k)} = 0$

$k>m_1$ 时 $A^k=0$

$\therefore (A+B)^m = 0+0 = 0$

$\therefore A+B$ 是幂零阵

10. 证明：设 A 对称，B 反对称，$A_1=A^{-1}$，$B_1=B^{-1}$

由于已证：${}^tA_1 = {}^t(A^{-1}) = ({}^tA)^{-1}$

${}^tB_1 = {}^t(B^{-1}) = ({}^tB)^{-1}$

$\because {}^tA=A$，${}^tB = B$

$\therefore$ 由于已证：${}^tA_1 = A^{-1} = A_1$

${}^tB_1 = (-B)^{-1} = -B^{-1} = -B_1$

$\therefore A_1$ 对称，B_1 反对称

证毕

11. 记 $\varepsilon_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_i$ 特别规定 $\varepsilon_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

则 $A = (\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_{n-1})$

且计算易得 $A^k = (\varepsilon_0, \cdots, \varepsilon_0, \overset{k+1}{\varepsilon_1}, \cdots, \varepsilon_{n-k})$

$\therefore A^n=0$，$A^{n-1} = (\varepsilon_0, \varepsilon_0, \cdots, \varepsilon_1) \neq 0$

$\therefore f(x)=x^n$ 为 A 的化零多项式，不妨设 A 的极小多项式为 $\mu_A(x)$，则 $\mu_A(x) \mid f(x)$，故 $\mu_A(x)=x^m$，$m \leq n$

$\because \mu_A(A)=0 \Rightarrow A^m=0$，但 $\forall k<n$，$A^k \neq 0$

$\therefore m \geq n$，综上 $m=n$

$\therefore \mu_A(x) = f(x) = x^n$

12. 证明：不妨设 $A^n=0$ 且 $A^{n-1} \neq 0$ $(n>0)$

则同11 可记 A 的极小多项式 $\mu_A(x)=x^n$

$\therefore \mu_A(x)-1 = x^n-1 = (x-1)(x^{n-1}+\cdots+x+1)$ 并 $=(x^{n-1}+\cdots+x+1)(x-1)$

将 x 以 A 代入，注意 $\mu_A(A)=A^n=0$.

$$-I_n = (A-I_n)(A^{n-1}+\cdots+A+I_n) = (A^{n-1}+\cdots+A+I_n)(A-I_n)$$

记 $B = A^{n-1}+\cdots+A+I_n$

则 $(I_n-A)B = B(I_n-A) = I_n$

$\therefore I_n-A$ 可逆且 $(I_n-A)^{-1}=B$

证明：由题意 $A^2-A=0$，故 $f(x)=x^2-x$ 为A的化零多项式

令 $B=I_n+A$，则 $A=B-I_n$

$\therefore A^2-A=(B-I_n)^2-(B-I_n)=B^2-3B+2I_n=0$

$\Rightarrow B(B-3I_n)=(B-3I_n)B=2I_n$

$\Rightarrow B\left(\frac{B-3I_n}{2}\right)=\left(\frac{B-3I_n}{2}\right)B=I_n$

代回 $B=I_n+A$：

$(I_n+A)(\frac{1}{2}A-I_n)=(\frac{1}{2}A-I_n)(I_n+A)=I_n$

$\therefore (I_n+A)$ 可逆且 $(I_n+A)^{-1}=\frac{1}{2}A-I_n$

14. 证明：设 $X=[X^{(1)},X^{(2)},\cdots,X^{(k)}]$

$B=[B^{(1)},B^{(2)},\cdots,B^{(k)}]$

则 $AX=B \Leftrightarrow [AX^{(1)},AX^{(2)},\cdots AX^{(k)}]=[B^{(1)},B^{(2)},\cdots,B^{(k)}]$

故 $AX=B$ 有解 $\Leftrightarrow AX^{(i)}=B^{(i)}$ 有解，$\forall i\in\{1,2,\cdots k\}$

设 $A=[a_{st}]_{m\times n}$，$B=[b_{st}]_{m\times k}$

对 $[A|B^{(1)}]$ 进行初等行变换，将其化为：

$$[A|B^{(1)}]\longrightarrow\left[\begin{array}{cccccc|c}\hat{a}_{11} & \hat{a}_{12} & \cdots & \hat{a}_{1t} & \cdots & \hat{a}_{1n} & \hat{b}_{11}\\ \vdots & \vdots & & & & & \vdots\\ 0 & 0 & \cdots & \hat{a}_{s't} & \cdots & \hat{a}_{s'n} & \hat{b}_{s'1}\\ 0 & 0 & \cdots & 0 & \cdots & 0 & \hat{b}_{(s'+1)1}\\ \vdots & \vdots & & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & \cdots & 0 & \hat{b}_{m1}\end{array}\right]$$

记为 $[A|B^{(1)}]\xrightarrow{P}[\hat{A}\,\vdots\,\hat{B}^{(1)}]$

对 $[A|B^{(2)}]$、$\cdots$ $[A|B^{(k)}]$ 进行同样的初等行变换，

$[A|B^{(h)}]\xrightarrow{P}[\hat{A}\,\vdots\,\hat{B}^{(h)}]$，$2\le h\le k$

设 $\hat{B}^{(h)}=\begin{pmatrix}\hat{b}_{1h}\\ \vdots\\ \hat{b}_{s'h}\\ \hat{b}_{(s'+1)h}\\ \vdots\\ \hat{b}_{mh}\end{pmatrix}$，则 $AX^{(h)}=B^{(h)}$ 有解 $\Leftrightarrow r(A)=r([A\,\vdots\,B^{(h)}])$

$\Leftrightarrow \hat{b}_{(s'+1)h}=\cdots=\hat{b}_{mh}=0$

对 $[A\,\vdots\,B]$ 进行相同的初等行变换：

$[A\,\vdots\,B]\xrightarrow{P}[\hat{A}\,\vdots\,\hat{B}]$，则 $\hat{B}=[\hat{B}^{(1)},\cdots\hat{B}^{(k)}]$

$\therefore AX=B$ 有解 $\Leftrightarrow \underbrace{AX^{(i)}=B^{(i)}\text{ 有解}}_{\forall 1\le i\le k} \overset{\text{已证}}{\Longleftrightarrow} \hat{b}_{(s'+1)i}=\cdots=\hat{b}_{mi}=0$ $\forall 1\le i\le k$

$\Leftrightarrow \hat{B}=\begin{pmatrix}\hat{b}_{11} & \cdots & \hat{b}_{1k}\\ \vdots & & \vdots\\ \hat{b}_{s'1} & \cdots & \hat{b}_{s'k}\\ 0 & \cdots & 0\\ \vdots & & \vdots\\ 0 & \cdots & 0\end{pmatrix} \Leftrightarrow r(A)=r([A\,\vdots\,B])$

证毕

15. 证明：不妨设 $C=(I_n+AB)^{-1}$

令 $D=I_n-BCA$

则 $D(I_n+BA)=I_n+BA-BCA-BCABA$

$=(I_n+BA)-BC(I_n+AB)A$

而 $C(I_n+AB)=I_n$

∴ $D(I_n+BA)=I_n+BA-BI_nA=I_n$

同理 $(I_n+BA)D=I_n$

∴ I_n+BA 可逆且 $(I_n+BA)^{-1}=D=I_n-BCA=I_n-B(I_n+AB)^{-1}A$

16. 解：$A=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}(y_1,y_2,\cdots y_n)$，记为 $A=\alpha\beta^T$

∴ $r(A)\le\max\{r(\alpha),r(\beta)\}\le 1$

①若 $\alpha=0$ 或 $\beta=0$，则显然 $A=0$，$r(A)=0$

②若 $\alpha\ne0$ 且 $\beta\ne0$，则 $\exists\, x_i,y_j\ne0$，故 $x_iy_j\ne0$

∴ $A\ne O$，故 $r(A)\ge1$，又 $r(A)\le1$，故 $r(A)=1$

17. 解：$f(x)=x^5+x^4+3x^3+2x^2+5x+2$

∴ $f(A)=A^5+A^4+3A^3+2A^2+5A+2I_n$

18. 证明：∵ $i\ne j$

∴ $E_{ij}^2=E_{ij}E_{ij}=O$

∴ $(I_n+E_{ij})(I_n-E_{ij})=I_n^2-E_{ij}^2=I_n-O=I_n$

同理 $(I_n-E_{ij})(I_n+E_{ij})=I_n$

∴ I_n+E_{ij} 可逆 且 $(I_n+E_{ij})^{-1}=\cancel{I_n+E_{ij}}\ I_n-E_{ij}$

19. 证明："⇒" 显然

"⇐"：取 $B_1=\begin{pmatrix}1&&\\&2&\\&&\ddots\\&&&n\end{pmatrix}=diag(1,2,\cdots n)$，显然

设 $A=[a_{ij}]_{n\times n}$

则 $AB_1=\begin{bmatrix}a_{11}&2a_{12}&\cdots&na_{1n}\\a_{21}&2a_{22}&\cdots&na_{2n}\\\vdots&\vdots&&\vdots\\a_{n1}&2a_{n2}&\cdots&na_{nn}\end{bmatrix}$

$B_1A=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\2a_{21}&2a_{22}&\cdots&2a_{2n}\\\vdots&\vdots&&\vdots\\na_{n1}&na_{n2}&\cdots&na_{nn}\end{bmatrix}$

由 $AB_1=B_1A$ 有 $ia_{ij}=ja_{ij}\Rightarrow(i-j)a_{ij}=0$

∴ $i\ne j$ 时 $a_{ij}=0$

∴ $A=\begin{bmatrix}a_{11}&&&\\&a_{22}&&\\&&\ddots&\\&&&a_{nn}\end{bmatrix}=diag(a_{11},a_{22},\cdots a_{nn})$

取 $B_2=\begin{array}{c|c} & \begin{matrix}\ \ &\ \ & i & j & \ \ \end{matrix}\\ \begin{matrix}\\ \\ i\\ j\\ \\ \end{matrix} & \begin{bmatrix}1&&&&\\&1&&&\\&&0&1&\\&&1&0&\\&&&&\ddots&\\&&&&&1\end{bmatrix}\end{array}$ 显然 B_2 可逆，且 $B_2^{-1}=B_2$

∴ $AB_2=B_2A\Rightarrow A=B_2AB_2^{-1}=B_2AB_2$

∵ $B_2AB_2=\begin{bmatrix}a_{11}&&&\\&\ddots a_{jj}&&\\&&a_{ii}\ddots&\\&&&a_{nn}\end{bmatrix}=diag(a_{11}\cdots a_{jj}\cdots a_{ii}\cdots a_{nn})$

∴ $A = B_2 A B_2 \Rightarrow a_{ii} = a_{jj}$

又由 i,j 任意性知 $a_{11}=a_{22}=\cdots=a_{nn}$，设 $\lambda=a_{11}=\cdots=a_{nn}$

∴ $A=\begin{bmatrix}\lambda&&\\&\lambda&\\&&\ddots\\&&&\lambda\end{bmatrix}=\lambda I_n$，证毕

20. 解：断定 $A=0$，下证之：

取 $M_n(K)$ 标准基 $E_{11},\cdots E_{1n},\cdots,E_{n1},\cdots,E_{nn}$

其中 E_{ij} 满足：$\begin{cases}\mathrm{ent}_{ij}E_{ij}=1\\ \mathrm{ent}_{st}E_{ij}=0\ (s\neq i \text{ 或 } t\neq j)\end{cases}$

则对 $\forall i,j$，有 $tr(A\cdot E_{ij})=0$

不妨设 $A=[a_{mk}]_{n\times n}$

则 $A\cdot E_{ij}=\begin{array}{c}\begin{array}{ccccc}&&j&&\end{array}\\ \begin{bmatrix}0&\cdots&a_{1i}&\cdots&0\\0&\cdots&a_{2i}&\cdots&0\\\vdots&&\vdots&&\vdots\\0&\cdots&a_{ni}&\cdots&0\end{bmatrix}\end{array}$

∴ $tr(A\cdot E_{ij})=a_{ji}=0$

由 i,j 任意性知 $\forall m,k$，$a_{mk}=tr(AE_{km})=0$

∴ $A=0$，证毕

21. 证明：对 A 的阶采用归纳法：

① $n=1$ 时显然成立

② 假设 $n=m-1$ 时成立，即 $\begin{cases}A\text{为上三角阵}\\A\in M_{m-1}(\mathbb{R})\\A\,{}^tA={}^tAA\end{cases}\Rightarrow A$ 为对角阵

则 $n=m$ 时：

不妨设 $A=\begin{bmatrix}a_{11}&{}^t\alpha\\0&A_1\end{bmatrix}$，$\alpha=\begin{pmatrix}a_{12}\\a_{13}\\\vdots\\a_{1m}\end{pmatrix}$，显然 A_1 是上三角阵

则 $A\,{}^tA={}^tAA\Rightarrow\begin{pmatrix}a_{11}&{}^t\alpha\\0&A_1\end{pmatrix}\begin{pmatrix}a_{11}&0\\\alpha&{}^tA_1\end{pmatrix}=\begin{pmatrix}a_{11}&0\\\alpha&{}^tA_1\end{pmatrix}\begin{pmatrix}a_{11}&{}^t\alpha\\0&A_1\end{pmatrix}$

$\Rightarrow\begin{pmatrix}a_{11}^2+{}^t\alpha\alpha&{}^t\alpha\,{}^tA_1\\A_1\alpha&A_1{}^tA_1\end{pmatrix}=\begin{pmatrix}a_{11}^2&a_{11}{}^t\alpha\\a_{11}\alpha&{}^tA_1A_1\end{pmatrix}$

∴ $a_{11}^2+{}^t\alpha\alpha=a_{11}^2\Rightarrow{}^t\alpha\alpha=0\Rightarrow a_{12}^2+a_{13}^2+\cdots+a_{1m}^2=0$

$\Rightarrow a_{12}=a_{13}=\cdots=a_{1m}=0\Rightarrow\alpha=0$

∴ ${}^t\alpha\,{}^tA_1=a_{11}{}^t\alpha=0$，$A_1\alpha=a_{11}\alpha=0$，故 $A=\begin{pmatrix}a_{11}&0\\0&A_1\end{pmatrix}$

而可看出 $A_1{}^tA_1={}^tA_1A_1$

由归纳假设，A_1 为对角阵

∴ $A=\begin{pmatrix}a_{11}&0\\0&A_1\end{pmatrix}$ 为对角阵，即 $n=m$ 时命题成立

由(1)(2)，命题得证

$=tr(BA)$

第19周作业（高代A）

第一次（20211220）

习题 4.3

8. 证明：设 $A=(a_{ij})_{n\times n}=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

则 $|A|=\begin{vmatrix} a_{11} & \cdots & a_{1(n-1)} & a_{1n} \\ \vdots & & \vdots & \vdots \\ a_{(n-1)1} & \cdots & a_{(n-1)(n-1)} & a_{(n-1)n} \\ a_{n1} & \cdots & a_{n(n-1)} & a_{nn} \end{vmatrix}$

$=\begin{vmatrix} a_{11} & \cdots & a_{1(n-1)} & a_{1n} \\ \vdots & & \vdots & \vdots \\ a_{(n-1)1} & \cdots & a_{(n-1)(n-1)} & a_{(n-1)n} \\ a_{n1} & & a_{n(n-1)} & 0 \end{vmatrix}+\begin{vmatrix} a_{11} & \cdots & a_{1(n-1)} & 0 \\ \vdots & & \vdots & \vdots \\ a_{(n-1)1} & \cdots & a_{(n-1)(n-1)} & 0 \\ a_{n1} & \cdots & a_{n(n-1)} & a_{nn} \end{vmatrix}$

记 $A_1=\begin{pmatrix} a_{11} & \cdots & a_{1(n-1)} \\ \vdots & & \vdots \\ a_{(n-1)1} & \cdots & a_{(n-1)(n-1)} \end{pmatrix}$，$f(x_1,\cdots x_{n-1})=\begin{vmatrix} A_1 & X \\ {}^tX & 0 \end{vmatrix}$，$X=\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$

$\because$ A_1的各阶主子式对应等于A的$1\sim(n-1)$阶主子式，设为$\Delta_1,\cdots,\Delta_{n-1}$

而A正定；故$\Delta_1>0,\cdots;\Delta_{n-1}>0$

$\therefore$ A_1正定，故由7已证，$f(x_1,\cdots x_{n-1})$为负定二次型

$\because A$正定 $\Rightarrow A$实对称 $\Rightarrow a_{in}=a_{ni}$

$\therefore |A|=f(a_{1n}\cdots a_{(n-1)n})+a_{nn}\Delta_{n-1}$

$\because f$负定 $\Rightarrow f(a_{1n},\cdots a_{(n-1)n})\le 0$

$\therefore |A|\le a_{nn}\Delta_{n-1}$

9. 证明：设 $A=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$，其各阶主子式为$\Delta_1,\cdots\Delta_n$

又设 $A_1=\begin{pmatrix} a_{11} & \cdots & a_{1(n-1)} \\ \vdots & & \vdots \\ a_{(n-1)1} & & a_{(n-1)(n-1)} \end{pmatrix}$，则由8已证，$A_1$正定

由8已证：$|A|\le a_{nn}\Delta_{n-1}=a_{nn}|A_1|$

故对A_1继续使用8已证结论：$|A_1|\le a_{(n-1)(n-1)}\Delta_{n-2}$

故同理，一直进行下去得：$|A|\le a_{nn}\cdot a_{(n-1)(n-1)}\cdot\cdots\cdot a_{11}$

$=a_{11}a_{22}\cdots a_{nn}$，证毕

10. 证明：$\because A$为实对称阵

$\therefore A$正交相似于对角阵

即$\exists Q$为正交阵，使${}^tQAQ=\mathrm{diag}(\lambda_1,\cdots\lambda_n)$，其中$\lambda_1,\cdots\lambda_n$为$A$特征值

记 $\Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_n)$，$X=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$，$Y=\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}={}^tQX$

则 ${}^tYY={}^tXQ{}^tQX={}^tXX\Rightarrow y_1^2+\cdots+y_n^2=x_1^2+\cdots+x_n^2$

记 $\lambda=\max\{1,\lambda_1,\cdots\lambda_n\}$，则$\lambda>0$

且 $A=Q\Lambda{}^tQ\Rightarrow {}^tXAX={}^tXQ\Lambda{}^tQX={}^tY\Lambda Y$

$=\lambda_1y_1^2+\cdots+\lambda_ny_n^2$

$\le\lambda(y_1^2+\cdots+y_n^2)$

$=\lambda(x_1^2+\cdots+x_n^2)$

即 ${}^tXAX\le\lambda(x_1^2+\cdots+x_n^2)$，证毕

11. 证明：充分性：设$\forall X\in\mathbb{R}^n$，则${}^tXAX={}^tX{}^tBBX={}^t(BX)(BX)\ge 0$

$\therefore A$为半正定阵

必要性：∵A为实对称阵

∴∃正交阵Q，使 ${}^tQAQ=\mathrm{diag}(\lambda_1,\cdots,\lambda_n)$

∵A半正定

∴ $\lambda_1\cdots\lambda_n\ge 0$

①若A≠0：不妨选择适当的Q，使 $\lambda_1\cdots\lambda_s>0$，$\lambda_{s+1},\cdots,\lambda_n=0$

令 $\Lambda=\mathrm{diag}(\lambda_1,\cdots\lambda_n)=\mathrm{diag}(\lambda_1,\cdots\lambda_s,0,\cdots0)$

又令 $M=\mathrm{diag}(\sqrt{\lambda_1},\cdots,\sqrt{\lambda_s},0,\cdots,0)$，则 $r(M)=r(\Lambda)=r(A)$

则 ${}^tMM=M^2=\Lambda$

∴ $A=Q\Lambda{}^tQ=Q{}^tMM{}^tQ={}^t(M{}^tQ)(M{}^tQ)$

令 $B=M{}^tQ$，则由Q为正交阵，有 $r(B)=r(M)=r(A)$

且 $A={}^tBB$，证毕

②若A=0，则结论显然

综合①②结论证毕

12. 证明：设 $\forall A,B\in M_n(\mathbb{R})$，$\forall k\in\mathbb{R}$

则 $f_T(A+B)=T^{-1}(A+B)T=T^{-1}AT+T^{-1}BT$

$=f_T(A)+f_T(B)$

$f_T(kA)=T^{-1}(kA)T=kT^{-1}AT=kf_T(A)$

$f_T(AB)=T^{-1}(AB)T=(T^{-1}AT)(T^{-1}BT)=f_T(A)f_T(B)$

证毕

13. 证明：设 $T=T_1+iT_2$，其中 $T_1,T_2\in M_n(\mathbb{R})$

则 $B=T^{-1}AT\Rightarrow TB=AT\Rightarrow T_1B+iT_2B=AT_1+iAT_2$

∵ $A,B\in M_n(\mathbb{R})$

∴ $\begin{cases}T_1B=AT_1\\T_2B=AT_2\end{cases}$

令 $f(t)=|T_1+tT_2|$，$t\in\mathbb{R}$

则 $f(t)$ 为一实系数多项式，且由 $f(i)\ne0$（因T可逆）知 f不是零多项式

∴ f至多有有限个实零点

∴ t充分大时 $f(t)\ne0$，即 $\exists r>0,\forall t>r$，$f(t)\ne0$

取 $C=T_1+(r+1)T_2$，则 $|C|=f(r+1)\ne0$，故C可逆

∴ $CB=T_1B+(r+1)T_2B=AT_1+A(r+1)T_2=AC$

$\Rightarrow B=C^{-1}AC$，证毕

14. 证明：由题意 $f(I_n)\ne0$，否则 $\forall A\in M_n(K)$

有 $f(A)=f(AI_n)=f(A)f(I_n)=0$

这是与f非零映射矛盾的

设 E_{ij} 定义为：$ent_{kl}(E_{ij})=\begin{cases}1 & k=i,l=j\\0 & k\ne i或l\ne j\end{cases}$

则 $I_n=E_{11}+\cdots+E_{nn}$

∴ $f(I_n) = f(E_{11}) + \cdots + f(E_{nn})$

$\Rightarrow f(I_n)V = f(E_{11})V + \cdots + f(E_{nn})V$

其中 $V = k^n$（列向量空间），$f(X)V = \{f(X)\alpha \mid \alpha \in V\}$

记 $W = f(I_n)V$，$W_i = f(E_{ii})V$

则断言 $W_i \neq \{0\}$，以反证法证之：

假设 $W_i = \{0\}$，则 $f(E_{ii}) = 0$

∴ $f(E_{jj}) = f(E_{ji}E_{ii}E_{ij}) = f(E_{ji})f(E_{ii})f(E_{ij}) = 0, \forall j$

∴ $f(I_n) = 0 + 0 + \cdots + 0 = 0$，与已证 $f(I_n) \neq 0$ 矛盾

∴ $W_i \neq \{0\}$，故 $\dim_k W_i \geq 1$

下证 $W_1 + \cdots + W_n$ 为直和：

设 $U_i = W_1 + \cdots + W_{i-1} + W_{i+1} + \cdots + W_n$，则 $W_i + U_i =$

设 $\beta \in W_i \cap U_i$，则 $\beta = f(E_{ii})\alpha_1 = \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} f(E_{jj})\alpha_{2j}$

∴ $f(E_{ii})\beta = f(E_{ii})f(E_{ii})\alpha_1 = f(E_{ii}E_{ii})\alpha_1 = f(E_{ii})\alpha_1 = \beta$

但 $\forall j \neq i$，$f(E_{ii})f(E_{jj}) = f(E_{ii}E_{jj}) = f(0) = 0$

∴ $f(E_{ii})\beta = f(E_{ii})\sum\limits_{\substack{j=1 \\ j \neq i}}^{n} f(E_{jj})\alpha_{2j} = \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} [f(E_{ii})f(E_{jj})]\alpha_{2j}$

$= \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} 0 \cdot \alpha_{2j} = 0$

∴ $\beta = f(E_{ii})\beta = 0$，即 $W_i \cap U_i = \{0\}$

∴ $W_1 + \cdots + W_n$ 为直和，故 $\dim_k W = \dim_k W_1 + \cdots + \dim_k W_n \geq n$

但 $W \subset V \Rightarrow \dim_k W \leq \dim_k V = n$

∴ 综合即得：$\begin{cases} \dim_k W = n \\ \dim_k W_i = 1, \forall i \end{cases}$ 故此也知 $f(I_n)$ 可逆

∴ $f(I_n) = f(I_n \cdot I_n) = f(I_n) \cdot f(I_n) \xrightarrow{\text{且 } f(I_n) \text{ 可逆}} I_n = f(I_n)$

∴ $I_n = f(E_{11}) + \cdots + f(E_{nn})$，$V = W = W_1 \oplus \cdots \oplus W_n$

∴ $\eta = f(E_{11})\eta + \cdots + f(E_{nn})\eta$

设 $\eta \in V$ 使 $f(E_{ii})\eta \neq 0$ $\forall 1 \leq i \leq n$（因 $\dim_k W_i = 1 \neq 0$，故 η 必存在），记 $\eta_j = f(E_{j1})\eta$ $1 \leq j \leq n$

则断言 $\eta_1 \cdots \eta_n$ 线性无关，下证之：

(1) 先证 $f(E_{st})\eta \neq 0$，$\forall s, t \in \{1, 2, \cdots n\}$：

采用反证法，假设 $f(E_{st})\eta = 0$，则由 $E_{tt} = E_{ts}E_{st}$：

$f(E_{tt})\eta = f(E_{ts}E_{st})\eta = f(E_{ts})[f(E_{st})\eta] = 0$，与 $f(E_{tt})\eta \neq 0$ $\forall 1 \leq i \leq n$ 矛盾

∴ $f(E_{st})\eta \neq 0$

② 再证 $\eta_1 \cdots \eta_n$ 线性无关：

设 $k_1\eta_1 + \cdots + k_n\eta_n = 0 \in V$，则 $f(E_{1j})(k_1\eta_1 + \cdots + k_n\eta_n) = f(E_{1j})0 = 0$

$\Rightarrow k_1[f(E_{ij})\eta_1]+\cdots+k_n[f(E_{ij})\eta_n]=0$

$\because f(E_{ij})\eta_p = f(E_{ij})f(E_{p1})\eta$

$= f(E_{ij}E_{p1})\eta$

而 $E_{ij}E_{k1}=\begin{cases}0 & j\neq p\\ E_{i1} & j=p\end{cases}$

$\therefore \sum_{p=1}^{n} k_p[f(E_{ij})\eta_p] = k_j f(E_{i1})\eta = 0$

$\because f(E_{i1})\eta\neq 0 \quad \therefore k_j=0,\ \forall 1\le j\le n$

$\therefore \eta_1\cdots\eta_n$ 线性无关

记 $S=[\eta_1,\cdots,\eta_n]$，则 S 可逆

且 $f(E_{ij})S=[f(E_{ij})\eta_1\cdots f(E_{ij})\eta_n]$

同上分析有：$f(E_{ij})\eta_p=\begin{cases}f(E_{i1})\eta, & j=p\\ 0, & j\neq p\end{cases}$

$\therefore f(E_{ij})S=[0,\cdots,0,\overset{\text{第}j\text{列}}{f(E_{i1})\eta},0,\cdots 0]=[0,\cdots 0,\overset{\text{第}j\text{列}}{\eta_i},0\cdots 0]$

而 $SE_{ij}=[\eta_1,\cdots,\eta_n]\begin{array}{c}\text{第}j\text{列}\\ \begin{pmatrix}0 & \cdots & 0 & \cdots & 0\\ \vdots & & \vdots & & \vdots\\ 0 & \cdots & 1 & \cdots & 0\\ \vdots & & \vdots & & \vdots\\ 0 & \cdots & 0 & \cdots & 0\end{pmatrix}\end{array}\text{第}i\text{行}$

$=[0,\cdots 0,\overset{\text{第}j\text{列}}{\eta_i},0,\cdots 0]$

$\therefore f(E_{ij})S=SE_{ij}\Rightarrow f(E_{ij})=SE_{ij}S^{-1}$

$\therefore \forall X\in M_n(k)$，设 $X=\begin{pmatrix}x_{11}&\cdots&x_{1n}\\ \vdots & & \vdots\\ x_{n1}&\cdots&x_{nn}\end{pmatrix}$，则 $X=\sum_{i,j}x_{ij}E_{ij}$

$\therefore f(X)=f(\sum_{i,j}x_{ij}E_{ij})=\sum_{i,j}x_{ij}f(E_{ij})=\sum_{i,j}x_{ij}SE_{ij}S^{-1}$

$=S(\sum_{i,j}x_{ij}E_{ij})S^{-1}=SXS^{-1}$

记 $T=S^{-1}$，则 $f(X)=T^{-1}XT$，证毕

15. 证明：设 $A=\begin{pmatrix}a_{11}&\cdots&a_{1n}\\ \vdots & & \vdots\\ a_{n1}&\cdots&a_{nn}\end{pmatrix}$，$E_{ij}$ 定义为：$\mathrm{ent}_{st}E_{ij}=\begin{cases}1 & s=i,t=j\\ 0 & s\neq i \text{或} t\neq j\end{cases}$

定义 $a_{ij}=f(E_{ji})$，则断言 $f(X)=tr(A\cdot X)$

证之：设 $X=\begin{pmatrix}x_{11}&\cdots&x_{1n}\\ \vdots & & \vdots\\ x_{n1}&\cdots&x_{nn}\end{pmatrix}$，则 $X=\sum_{i,j}x_{ij}E_{ij}$

$\therefore f(X)=f(\sum_{i,j}x_{ij}E_{ij})\overset{f\text{为线性函数}}{=\!=}\sum_{i,j}x_{ij}f(E_{ij})=\sum_{i=1}^{n}\sum_{j=1}^{n}x_{ij}a_{ji}$

$=\sum_{j=1}^{n}\sum_{i=1}^{n}a_{ji}x_{ij}=\sum_{j=1}^{n}(\sum_{i=1}^{n}a_{ji}x_{ij})=tr(AX)$

证毕

习题4.4.

1. 证明：记 $M=\begin{pmatrix}A&-B\\ B&A\end{pmatrix}$，$M_1=\begin{pmatrix}A&B\\ -B&A\end{pmatrix}$

则 $M_1 = \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix}\begin{pmatrix} A & -B \\ B & A \end{pmatrix}\begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix}$ 记为 $M_1 = PMP$

$\therefore MM_1 = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \overset{AB=BA}{=\!=} \begin{pmatrix} A^2+B^2 & O \\ O & A^2+B^2 \end{pmatrix}$

$\therefore |M||M_1| = \begin{vmatrix} A^2+B^2 & O \\ O & A^2+B^2 \end{vmatrix} = |A^2+B^2|^2$

$\because |M_1| = |P||M||P| = |P|^2|M|$

而 $|P| = (-1)^{n^2}$，故 $|P|^2 = 1$

$\therefore |M||M_1| = |M|^2 = |A^2+B^2|^2 \Rightarrow |M| = \pm|A^2+B^2| \quad (*)$

取 $B=0$，则 $M = \begin{pmatrix} A & O \\ O & A \end{pmatrix} \Rightarrow |M| = |A^2|$，故$(*)$中应取"+"

$\therefore |M| = |A^2+B^2|$

2. 解：令 $M = \begin{pmatrix} A & B & C & D \\ B & A & D & C \\ C & D & A & B \\ D & C & B & A \end{pmatrix}$，则 $M\begin{pmatrix} I_n & O & O & O \\ I_n & I_n & O & O \\ I_n & O & I_n & O \\ I_n & O & O & I_n \end{pmatrix}$

$= \begin{pmatrix} A+B+C+D & B & C & D \\ A+B+C+D & A & D & C \\ A+B+C+D & D & A & B \\ A+B+C+D & C & B & A \end{pmatrix}$

$\therefore |M| \begin{vmatrix} I_n & O & O & O \\ I_n & I_n & O & O \\ I_n & O & I_n & O \\ I_n & O & O & I_n \end{vmatrix} = \begin{vmatrix} A+B+C+D & B & C & D \\ A+B+C+D & A & D & C \\ A+B+C+D & D & A & B \\ A+B+C+D & C & B & A \end{vmatrix}$

$= \begin{vmatrix} A+B+C+D & O & O & O \\ O & I_n & O & O \\ O & O & I_n & O \\ O & O & O & I_n \end{vmatrix} \begin{vmatrix} I_n & B & C & D \\ I_n & A & D & C \\ I_n & D & A & B \\ I_n & C & B & A \end{vmatrix}$

$\Rightarrow |M| = |A+B+C+D| \begin{vmatrix} I_n & B & C & D \\ I_n & A & D & C \\ I_n & D & A & B \\ I_n & C & B & A \end{vmatrix}$，记 $M_1 = \begin{pmatrix} I_n & B & C & D \\ I_n & A & D & C \\ I_n & D & A & B \\ I_n & C & B & A \end{pmatrix}$

则 $M_1 \begin{pmatrix} I_n & -B & -C & -D \\ O & I_n & O & O \\ O & O & I_n & O \\ O & O & O & I_n \end{pmatrix} = \begin{pmatrix} I_n & O & O & O \\ I_n & A-B & D-C & C-D \\ I_n & D-B & A-C & B-D \\ I_n & C-B & B-C & A-D \end{pmatrix}$

记 $M_2 = \begin{pmatrix} A-B & D-C & C-D \\ D-B & A-C & B-D \\ C-B & B-C & A-D \end{pmatrix}$，则 $\begin{pmatrix} I_n & -I_n & O \\ O & I_n & O \\ O & O & I_n \end{pmatrix} M_2 \begin{pmatrix} I_n & O & O \\ I_n & I_n & O \\ O & O & I_n \end{pmatrix}$

$= \begin{pmatrix} I_n & -I_n & O \\ O & I_n & O \\ O & O & I_n \end{pmatrix}\begin{pmatrix} A-B-C+D & D-C & C-D \\ A-B-C+D & A-C & B-D \\ O & B-C & A-D \end{pmatrix} = \begin{pmatrix} A-B-C+D & D-C & C-D \\ O & A-D & B-C \\ O & B-C & A-D \end{pmatrix}$

$= \begin{pmatrix} I_n & D-C & C-D \\ O & A-D & B-C \\ O & B-C & A-D \end{pmatrix}\begin{pmatrix} A-B-C+D & O & O \\ O & I_n & O \\ O & O & I_n \end{pmatrix}$

$\therefore \begin{vmatrix} I_n & -I_n & O \\ O & I_n & O \\ O & O & I_n \end{vmatrix} |M_2| \begin{vmatrix} I_n & O & O \\ I_n & I_n & O \\ O & O & I_n \end{vmatrix} = \begin{vmatrix} A-B-C+D & O & O \\ O & I_n & O \\ O & O & I_n \end{vmatrix} \begin{vmatrix} I_n & D-C & C-D \\ O & A-D & B-C \\ O & B-C & A-D \end{vmatrix}$

$\Rightarrow |M_2| = |A-B-C+D| \begin{vmatrix} A-D & B-C \\ B-C & A-D \end{vmatrix}$

5. 证明: $\begin{pmatrix} A & B \\ -2A & -5B \end{pmatrix} = \begin{pmatrix} I_m & I_m \end{pmatrix} \begin{pmatrix} A & O \end{pmatrix} \cdots$

$$\therefore |M_1| \begin{vmatrix} I_n & -B & -C & -D \\ 0 & I_n & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \end{vmatrix} = \begin{vmatrix} I_n & 0 & 0 & 0 \\ I_n & A-B & D-C & C-D \\ I_n & D-B & A-C & B-D \\ I_n & C-B & B-C & A-D \end{vmatrix} = |M_2|$$

$$\Rightarrow |M_1| = |M_2| = |A-B-C+D| \begin{vmatrix} A-D & B-C \\ B-C & A-D \end{vmatrix}$$

$$\therefore |M| = |A+B+C+D|\,|M_1| = |A+B+C+D|\,|A-B-C+D| \begin{vmatrix} A-D & B-C \\ B-C & A-D \end{vmatrix}$$

记 $M_3 = \begin{pmatrix} A-D & B-C \\ B-C & A-D \end{pmatrix}$, 则 $M_3 \begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix} = \begin{pmatrix} A+B-C-D & B-C \\ A+B-C-D & A-D \end{pmatrix}$

$$\therefore |M_3| \begin{vmatrix} I_n & 0 \\ I_n & I_n \end{vmatrix} = \left|\begin{pmatrix} I_n & B-C \\ I_n & A-D \end{pmatrix}\right| \left|\begin{pmatrix} A+B-C-D & 0 \\ 0 & I_n \end{pmatrix}\right|$$

$$= |A+B-C-D| \begin{vmatrix} I_n & B-C \\ I_n & A-D \end{vmatrix}$$

记 $M_4 = \begin{pmatrix} I_n & B-C \\ I_n & A-D \end{pmatrix}$, 则 $\begin{pmatrix} I_n & 0 \\ -I_n & I_n \end{pmatrix} M_4 = \begin{pmatrix} I_n & B-C \\ 0 & A-B+C-D \end{pmatrix}$

$$\therefore \left|\begin{pmatrix} I_n & 0 \\ -I_n & I_n \end{pmatrix}\right| |M_4| = \begin{vmatrix} I_n & B-C \\ 0 & A-B+C-D \end{vmatrix}$$

$\Rightarrow |M_4| = |A-B+C-D|$, 故 $|M_3| = |A+B-C-D|\,|A-B+C-D|$

$\therefore |M| = |A+B+C+D|\,|A-B-C+D|\,|A+B-C-D|\,|A-B+C-D|$

3. 解: 设 $M = \begin{pmatrix} A & O \\ O & B \end{pmatrix}$, 则断言 $M^* = \begin{pmatrix} |B|A^* & O \\ O & |A|B^* \end{pmatrix}$

7. 证明: …

证明之: (1) 先考虑 A, B 均可逆的情形:

$\because A, B$ 可逆 $\therefore |M| = |A||B| \neq 0$, 故 M 可逆

$$\therefore M^* = |M| M^{-1} = |A||B| \begin{pmatrix} A^{-1} & O \\ O & B^{-1} \end{pmatrix} = \begin{pmatrix} |B||A|A^{-1} & O \\ O & |A||B|B^{-1} \end{pmatrix}$$

$= \begin{pmatrix} |B|A^* & O \\ O & |A|B^* \end{pmatrix}$, 证毕

(2) 考虑一般情形: 设 $M(t) = M + tI_{2n} = \begin{pmatrix} A+tI_n & O \\ O & B+tI_n \end{pmatrix}$

记 $f(t) = |A+tI_n|$, $g(t) = |B+tI_n|$

则 $f(t)$、$g(t)$ 均为次数有限的多项式

而由 $f(t)$, $g(t)$ 最高次首一可知 $f(t)$, $g(t)$ 均不为0多项式

$\therefore f(t)$, $g(t)$ 均至多有有限个零点, 即 $\exists a>0$, $\forall t>a$, $f(t), g(t) \neq 0$

对 $t>a$ 讨论:

$\because$ $\cancel{M+tI_n}$ $f(t), g(t) \neq 0$

$\therefore A+tI_n$、$B+tI_n$ 均可逆

$\therefore$ 由(1)已证: $(M(t))^* = \begin{pmatrix} |B+tI_n|(A+tI_n)^* & O \\ O & |A+tI_n|(B+tI_n)^* \end{pmatrix}$

由定义知 $(M(t))^*$、$(A+tI_n)^*$、$(B+tI_n)^*$ 均应为多项式矩阵

设 $P(t) = |B+tI_n|(A+tI_n)^*$, $Q(t) = |A+tI_n|(B+tI_n)^*$

则 $P(t)$、$Q(t)$ 亦为多项式矩阵

设 $(M(t))^* = \begin{pmatrix} m_{11}(t) & \cdots & m_{1(2n)}(t) \\ \vdots & & \vdots \\ m_{(2n)1}(t) & \cdots & m_{(2n)(2n)}(t) \end{pmatrix}$

$P(t) = \begin{pmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{pmatrix}$，$Q(t) = \begin{pmatrix} q_{11}(t) & \cdots & q_{1n}(t) \\ \vdots & & \vdots \\ q_{n1}(t) & \cdots & q_{nn}(t) \end{pmatrix}$

$$\therefore \begin{pmatrix} m_{11}(t) & \cdots & m_{1(2n)}(t) \\ \vdots & & \vdots \\ m_{(2n)1}(t) & \cdots & m_{(2n)(2n)}(t) \end{pmatrix} = \begin{pmatrix} p_{11}(t) & \cdots & p_{1n}(t) & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & q_{11}(t) & \cdots & q_{1n}(t) \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & q_{n1}(t) & \cdots & q_{nn}(t) \end{pmatrix}, \forall t>a$$

分析 $m_{11}(t) = a_{11}(t)$：∵ $m_{11}(t)$、$a_{11}(t)$ 均为多项式且次数有限

又而 $\forall t>a$，$m_{11}(t) = a_{11}(t)$

∴ $m_{11}(t) \equiv a_{11}(t)$，$\forall t$

∴ $m_{11}(0) = a_{11}(0)$

同理可得 $m_{ij}(0) = s_{ij}(0)$，$s_{ij}(t)$ 为分块多项式矩阵中对应元

$$\therefore (M(0))^* = \begin{pmatrix} |B+0I_n|(A+0I_n)^* & O \\ O & |A+0I_n|(B+0I_n)^* \end{pmatrix}$$

$$\Rightarrow (M(0))^* = \begin{pmatrix} |B|A^* & O \\ O & |A|B^* \end{pmatrix}$$

∵ $M(0) = M$

∴ $M^* = \begin{pmatrix} |B|A^* & O \\ O & |A|B^* \end{pmatrix}$，证毕

综合(1)(2)结论成立

4. 证明：(1) 当A可逆时：

$$\begin{pmatrix} I_n & O \\ -CA^{-1} & I_n \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ O & D-CA^{-1}B \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} I_n & O \\ -CA^{-1} & I_n \end{vmatrix}\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} A & B \\ O & D-CA^{-1}B \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A||D-CA^{-1}B| = |AD-ACA^{-1}B| \xlongequal{AC=CA} |AD-CB|$$

(2) 一般情况时：令 $A(t) = A+tI_n$，且 $f(t) = |A(t)| = |A+tI_n|$，$t\in\mathbb{R}$

则 $f(t)$ 为一首一且次数有限多项式，故其至多有有限实根

又∴ $\exists a>0$，$\forall t>a$，$f(t)\neq 0$，故 $t>a$ 时 $A(t)$ 可逆

∴由(1)已证：$\begin{vmatrix} A(t) & B \\ C & D \end{vmatrix} = |A(t)D-CB|$，$\forall t>a$

记 $g(t) = \begin{vmatrix} A(t) & B \\ C & D \end{vmatrix}$，$h(t) = |A(t)D-CB|$

则 $g(t)$、$h(t)$ 均为次数有限的多项式

且 $g(t) = h(t)$，$\forall t>a$

∴可知 $g(t) \equiv h(t)$，故 $g(0) = h(0)$，即 $\begin{vmatrix} A(0) & B \\ C & D \end{vmatrix} = |A(0)D-CB|$

∵ $A(0) = A$ ∴ $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD-CB|$

综上结论证毕

5. 证明：$\begin{pmatrix} A & B \\ 2A & -5B \end{pmatrix} = \begin{pmatrix} I_m & I_m \\ 2I_m & -5I_m \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$，记为 $M_1 = CM_2$

由4已证：

$|C| = \begin{vmatrix} I_m & I_m \\ 2I_m & -5I_m \end{vmatrix} = |-5I_m - 2I_m| = |-7I_m| = (-7)^n \neq 0$

∴ C 可逆，故 $r(M_1) = r(M_2)$

设 $r(A) = r_1$，$r(B) = r_2$

则 ∃ P_1, Q_1, P_2, Q_2 均可逆，使 $P_1AQ_1 = \begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix}$，$P_2BQ_2 = \begin{pmatrix} I_{r_2} & 0 \\ 0 & 0 \end{pmatrix}$

（其中 $P_1, P_2 \in M_m(\mathbb{R})$，$Q_1 \in M_s(\mathbb{R})$，$Q_2 \in M_t(\mathbb{R})$）

记 $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$，$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$，则 P、Q 均可逆

且 $PM_2Q = \begin{pmatrix} P_1AQ_1 & 0 \\ 0 & P_2BQ_2 \end{pmatrix} = \begin{pmatrix} I_{r_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{r_2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

∴ $r(M_2) = r(I_{r_1}) + r(I_{r_2}) = r_1 + r_2$

∴ $r(M_1) = r(M_2) = r_1 + r_2 = r(A) + r(B)$

6. 证明：$\begin{pmatrix} A & AB \\ B & B+B^2 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix} \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix}$

记为 $M = M_1C_1C_2$，则显然 $|C_1| = |C_2| = 1$，故 C_1, C_2 可逆

∴ $r(M) = r(M_1)$，而5已证 $r\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = r(A) + r(B)$

∴ $r(M) = r(A) + r(B)$，证毕

7. 证明：设 $A^{(i)} = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix}$，$T = \begin{pmatrix} t \\ t \\ \vdots \\ t \end{pmatrix}$

则 $A(t) = [A^{(1)}+T, \cdots, A^{(n)}+T]$

∴ $|A(t)| = |A^{(1)}+T, \cdots, A^{(n)}+T|$

设 $D(i_1 \cdots i_s)$ 为将 $|A(0)|$ 中第 $i_1 \cdots i_s$ 列换为 T 所得的行列式，并规定 $D(0) = |A(0)|$

则由行列式列的线性性质得：

$$|A(t)| = \sum_{s=0}^{n} \sum_{0<i_1<\cdots<i_s\le n} D(i_1, \cdots, i_s)$$

∵ $s \geq 2$ 时，$D(i_1, \cdots, i_s)$ 中有相同列，故 $D(i_1, \cdots, i_s) = 0$

∴ $|A(t)| = D(0) + \sum_{k=1}^{n} D(k) = |A(0)| + \sum_{i=1}^{n} |A^{(1)} \cdots A^{(i-1)} T A^{(i+1)} \cdots A^{(n)}|$

将 $|A^{(1)} \cdots A^{(i-1)} T A^{(i+1)} \cdots A^{(n)}|$ 按第 i 列展开：

$$|A^{(1)} \cdots A^{(i-1)} T A^{(i+1)} \cdots A^{(n)}| = \sum_{j=1}^{n} t A_{ij}$$

∴ $|A(t)| = |A(0)| + t \sum_{i,j} A_{ij}$，证毕

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1. 证明：(1) 若$A=O$，则结论显然

(2) 若$A\neq O$，则A中元素必有一绝对值最小者

①：以第I类初等行变换将其换至a_{11}位

得$A\sim A_1$，$A_1=\begin{pmatrix} a_{11}^{(1)} & \cdots & a_{1n}^{(1)} \\ \vdots & & \vdots \\ a_{m1}^{(1)} & \cdots & a_{mn}^{(1)} \end{pmatrix}$

若$a_{12}^{(1)}, \cdots a_{1n}^{(1)}$中有不能被$a_{11}^{(1)}$整除者，(不妨设为$a_{1i}^{(1)}$) 则作带余除法：

$a_{1i}^{(1)}=k\cdot a_{11}^{(1)}+d$，则 $0<|d|<|a_{11}^{(1)}|$

以第III类初等行变换：

$A_1 \xrightarrow{c_i-kc_1} \begin{pmatrix} a_{11}^{(1)} & \cdots & d & \cdots & a_{1n}^{(1)} \\ \vdots & & \vdots & & \vdots \\ a_{m1}^{(1)} & \cdots & a_{mi}^{(1)}-ka_{m1}^{(1)} & \cdots & a_{mn}^{(1)} \end{pmatrix}$

$\xrightarrow{c_i\leftrightarrow c_1} \begin{pmatrix} d & \cdots & a_{11}^{(1)} & \cdots & a_{1n}^{(1)} \\ \vdots & & \vdots & & \vdots \\ a_{mi}^{(1)}-ka_{m1}^{(1)} & \cdots & a_{m1}^{(1)} & \cdots & a_{mn}^{(1)} \end{pmatrix}$

若第一行其余元素中有不整除d者，则继续重复上述步骤

由于每完成一次上述步骤，(1,1)位元素绝对值均变小

但$|a_{11}|<+\infty$，故有限次后过程终止，此时(1,1)位元素整除第一行所有元素

设此时得矩阵$A_2=\begin{pmatrix} a_{11}^{(2)} & \cdots & a_{1n}^{(2)} \\ \vdots & & \vdots \\ a_{m1}^{(2)} & \cdots & a_{mn}^{(2)} \end{pmatrix}$，$a_{11}^{(2)} \mid a_{1i}^{(2)}$，$i=1,\cdots n$

② 若$a_{21}^{(2)}\cdots a_{m1}^{(2)}$中有不整除$a_{11}^{(2)}$者(不妨设为$a_{i1}^{(2)}$)

则作带余除法：$a_{i1}^{(2)}=k^{(2)}\cdot a_{11}^{(2)}+d^{(2)}$，$|d^{(2)}|<|a_{11}^{(2)}|$

对A_2作第I类、III类初等变换：

$A_2=\begin{pmatrix} a_{11}^{(2)} & \cdots & a_{1n}^{(2)} \\ \vdots & & \vdots \\ a_{m1}^{(2)} & \cdots & a_{mn}^{(2)} \end{pmatrix} \xrightarrow{r_i-k^{(2)}r_1} \begin{pmatrix} a_{11}^{(2)} & \cdots & a_{1n}^{(2)} \\ \vdots & & \vdots \\ d^{(2)} & \cdots & a_{in}^{(2)}-k^{(2)}a_{1n}^{(2)} \\ \vdots & & \vdots \\ a_{m1}^{(2)} & \cdots & a_{mn}^{(2)} \end{pmatrix}$

$\xrightarrow{r_i\leftrightarrow r_1} \begin{pmatrix} d^{(2)} & \cdots & a_{in}^{(2)}-k^{(2)}a_{1n}^{(2)} \\ \vdots & & \vdots \\ a_{11}^{(2)} & \cdots & a_{1n}^{(2)} \\ \vdots & & \vdots \\ a_{m1}^{(2)} & \cdots & a_{mn}^{(2)} \end{pmatrix}$

将所得矩阵重复过程①后继续重复上述步骤

而每次重复后(1,1)元绝对值均减小，故有限步后即终止

此时(1,1)元整除第一行、第一列所有元素。

设此时得矩阵$A_3=\begin{pmatrix} a_{11}^{(3)} & \cdots & a_{1n}^{(3)} \\ \vdots & & \vdots \\ a_{m1}^{(3)} & \cdots & a_{mn}^{(3)} \end{pmatrix}$，$a_{11}^{(3)} \mid a_{i1}^{(3)}$，$i=1,2,\cdots m$
$a_{11}^{(3)} \mid a_{1j}^{(3)}$，$j=1,2,\cdots n$

③ 若$a_{ij}^{(3)}$ $(2\le i\le m, 2\le j\le n)$中有不整除$a_{11}^{(3)}$者，不妨设为$a_{st}^{(3)}$

设$a_{i1}^{(3)}=p_i a_{11}^{(3)}$，$a_{1j}^{(3)}=q_j a_{11}^{(3)}$，以$a_{st}^{(3)}$为被除数，$a_{11}^{(3)}$为除数作带余除法。

$a_{st}^{(3)} = k^{(3)} a_{11}^{(3)} + \underbrace{d^{(3)}}_{|d^{(3)}|<|a_{11}^{(3)}|}$，对$A_3$作第I类初等变换

$$A_3 \xrightarrow{r_s-(p_s-1)r_1} \begin{pmatrix} a_{11}^{(3)} & \cdots & a_{1t}^{(3)} & \cdots & a_{1n}^{(3)} \\ \vdots & & \vdots & & \vdots \\ a_{11}^{(3)} & \cdots & a_{st}^{(3)}-(p_s-1)q_t a_{11}^{(3)} & \cdots & a_{sn}^{(3)}-(p_s-1)a_{1n}^{(3)} \\ \vdots & & \vdots & & \vdots \\ a_{m1}^{(3)} & \cdots & a_{mt}^{(3)} & \cdots & a_{mn}^{(3)} \end{pmatrix}$$

$$\xrightarrow{c_t-[k^{(3)}-(p_s-1)q_t]c_1} \begin{pmatrix} a_{11}^{(3)} & \cdots & a_{1t}^{(3)}-[k^{(3)}-(p_s-1)q_t]a_{11}^{(3)} & \cdots & a_{1n}^{(3)} \\ \vdots & & \vdots & & \vdots \\ a_{11}^{(3)} & \cdots & d^{(3)} & \cdots & a_{sn}^{(3)}-(p_s-1)a_{1n}^{(3)} \\ \vdots & & \vdots & & \vdots \\ a_{m1}^{(3)} & \cdots & a_{mt}^{(3)}-[k^{(3)}-(p_s-1)q_t] & \cdots & a_{mn}^{(3)} \end{pmatrix}$$

再由第II类初等变换将$d^{(3)}$换至(1,1)元的位置处，将所得矩阵再进行过程①、②，并继续重复上述过程，由于每次进行后(1,1)元绝对值减小，故至多有限步后即终止，则此时(1,1)元整除矩阵中所有元素。设此时得矩阵$B=(b_{ij})$，则$b_{11}|b_{ij}$且$A\sim B$，证毕

2. 证明：设$A=(a_{ij})_{m\times n}\in M_{m\times n}(\mathbb{Z})$为非零矩阵

由1已证：A初等等价于$B=(b_{ij}^{(0)})$，且$b_{11}^{(0)}|b_{ij}^{(0)}$，不妨$b_{11}^{(0)}>0$

∴可将A经初等变换化为：

$$B_1=\begin{pmatrix} b_{11}^{(1)} & 0 & \cdots & 0 \\ 0 & b_{22}^{(1)} & \cdots & b_{2n}^{(1)} \\ \vdots & \vdots & & \vdots \\ 0 & b_{m2}^{(1)} & \cdots & b_{mn}^{(1)} \end{pmatrix}$$ 且 $b_{11}^{(1)}=|b_{11}^{(0)}|\,|\,b_{ij}^{(1)}$，$2\le i\le m, 2\le j\le n$

记 $$B_2=\begin{pmatrix} b_{22}^{(1)} & \cdots & b_{2n}^{(1)} \\ \vdots & & \vdots \\ b_{m2}^{(1)} & \cdots & b_{mn}^{(1)} \end{pmatrix}$$

故可采用归纳法：

由1结论可知本题

当B只有一行或一列时结论显然成立

假设 $s=m-1$，$t=n-1$ 时结论对 $A\in M_{s\times t}(\mathbb{Z})$ 成立

则B_2初等等价于 $$\begin{pmatrix} b_{22}^{(2)} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & b_{33}^{(2)} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots & & \vdots \\ 0 & 0 & \cdots & b_{rr}^{(2)} & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} := B_3$$

且 $b_{ii}^{(2)}\,|\,b_{(i+1)(i+1)}^{(2)}$，$b_{ii}^{(2)}>0$

P, Q可逆

由初等等价定义：$B_3=PB_2Q$，设 $P=(p_{ij})_{(m-1)\times(m-1)}$，$Q=(q_{ij})_{(n-1)\times(n-1)}$

$$\therefore b_{22}^{(2)}=\sum_{j=1}^{n-1}\left(\sum_{i=1}^{m-1}p_{1i}b_{(i+1)(j+1)}^{(1)}\right)q_{j1} \xlongequal{\mathbb{Z}为交换环} \sum_{i=1}^{m-1}\sum_{j=1}^{n-1}b_{(i+1)(j+1)}^{(1)}p_{1i}q_{j1}$$

$\because b_{11}^{(1)}=b_{11}^{(0)}$ ∴令 $b_{11}^{(2)}=b_{11}^{(1)}=|b_{11}^{(0)}|$，则由 $b_{11}^{(1)}|b_{ij}^{(1)}$，$\forall\, 2\le i\le m$，$2\le j\le n$

知 $b_{11}^{(2)}|b_{22}^{(2)}$ 且 $b_{11}^{(2)}>0$

由P、Q可逆知

$$\because \begin{pmatrix}1&0\\0&P\end{pmatrix}B_1\begin{pmatrix}1&0\\0&Q\end{pmatrix}=\begin{pmatrix}b_{11}^{(2)}&0\\0&B_3\end{pmatrix}$$ 则 $\begin{pmatrix}1&0\\0&P\end{pmatrix}$、$\begin{pmatrix}1&0\\0&Q\end{pmatrix}$ 均可逆

$\therefore B_1\sim\begin{pmatrix}b_{11}^{(2)}&0\\0&B_3\end{pmatrix}$，记 $B=\begin{pmatrix}b_{11}&0\\0&B_3\end{pmatrix}$，$b_{ii}=b_{ii}^{(2)}$

则 $A\sim B_1\sim B$

且 $B=\begin{pmatrix} b_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_{rr} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$ 且 $b_{ii} \mid b_{(i+1)(i+1)}$

$\therefore s=m, t=n$ 时结论亦成立

综上，本题得证

3. 证明：由于可逆阵必非零，故不妨设为 A

由 2 已证：$A \sim B = \begin{pmatrix} b_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_{rr} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$ 且 $b_{ii} \mid b_{(i+1)(i+1)}$，$b_{ii}>0$

$\therefore A = A_1^{(l)} \cdots A_s^{(l)} B A_1^{(r)} \cdots A_t^{(r)}$

其中 $\{A_i^{(l)}\}$，$\{A_j^{(r)}\}$ 为一系列初等矩阵

$\because A$ 可逆，$A_i^{(l)}$ 可逆，$A_j^{(r)}$ 可逆

$\therefore B$ 可逆，故 $B=\begin{pmatrix} b_{11} & & \\ & \ddots & \\ & & b_{nn} \end{pmatrix}$，$b_{ii} \mid b_{(i+1)(i+1)}$，$b_{ii}>0$

而 $|B| = b_{11}\cdots b_{nn}$，故由 B 可逆知 $\left(\prod_{i=1}^{n} b_{ii}\right) \in \mathbb{Z}$ 可逆

又 $b_{ii}>0$，故 $\prod_{i=1}^{n} b_{ii} = 1 \Rightarrow b_{ii}=1$，$\forall i \in 1,2,\cdots n$

$\therefore B=I$，故 $A = A_1^{(l)} \cdots A_s^{(l)} A_1^{(r)} \cdots A_t^{(r)}$

$\therefore A$ 可写成初等矩阵的乘积

4. 解：记 $A(x) = x \cdot I_3 - A = \begin{pmatrix} x-3 & -2 & 3 \\ -4 & x-10 & 12 \\ -3 & -6 & x+7 \end{pmatrix}$

则：$A(x) \xrightarrow{r_3-r_2} \begin{pmatrix} x-3 & -2 & 3 \\ -4 & x-10 & 12 \\ 1 & -x+4 & x-5 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{pmatrix} 1 & -x+4 & x-5 \\ -4 & x-10 & 12 \\ x-3 & -2 & 3 \end{pmatrix}$

$\xrightarrow[r_3-(x-3)r_1]{r_2+4r_1} \begin{pmatrix} 1 & -x+4 & x-5 \\ 0 & -3x+6 & 4x-8 \\ 0 & (x-2)(x-5) & -(x-2)(x-6) \end{pmatrix} \xrightarrow[c_3-(x-5)c_1]{c_2+(x-4)c_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3(x-2) & 4(x-2) \\ 0 & (x-2)(x-5) & -(x-2)(x-6) \end{pmatrix}$

$\xrightarrow{c_2+c_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & x-2 & 4(x-2) \\ 0 & (x-2) & -(x-2)(x-6) \end{pmatrix} \xrightarrow{r_3-r_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & x-2 & 4(x-2) \\ 0 & 0 & -(x-2)^2 \end{pmatrix}$

$\xrightarrow[-1\times r_3]{c_3-4c_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & x-2 & 0 \\ 0 & 0 & (x-2)^2 \end{pmatrix}$

5. 解：$|xI_3-A| = \begin{vmatrix} x-1 & -1 & 1 \\ -3 & x+2 & 0 \\ 1 & -1 & x+1 \end{vmatrix} = \begin{vmatrix} x-1 & -1 & 1 \\ -3 & x+2 & 0 \\ 2-x^2 & x & 0 \end{vmatrix}$

$= \begin{vmatrix} -3 & x+2 \\ 2-x^2 & x \end{vmatrix} = -3x-(2-x^2)(x+2)$

$= (x^2-2)(x+2)-3x = x^3+2x^2-5x-4$

6. 证明：设 $A=(a_{ij})$，$A(x)=xI_n-A$，记 $B(x)=(b_{ij}(x))=A(x)$

(1) $A(x) = \begin{pmatrix} x-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & x-a_{22} & \cdots & -a_{2n} \\ \vdots & & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & x-a_{nn} \end{pmatrix}$

$\therefore |A(x)| = |B(x)| = \sum\limits_{\pi\in S_n} \varepsilon_\pi\, b_{1\pi(1)}(x)\cdots b_{n\pi(n)}(x)$

$= [b_{11}(x)\cdots b_{nn}(x)] + \sum\limits_{\substack{\pi\in S_n\\ \pi\neq id}} \varepsilon_\pi\, b_{1\pi(1)}(x)\cdots b_{n\pi(n)}(x)$

若 $\pi\neq id$，则 $\exists\, i,j$，使 $i\neq j$ 且 $\pi(i)\neq i$，$\pi(j)\neq j$ （因为置换不可能只移动1个元素）

$\therefore$ 或有 $b_{1\pi(1)}(x)\cdots b_{n\pi(n)}(x)=0$

或有 $\deg[b_{1\pi(1)}(x)\cdots b_{n\pi(n)}(x)] = \sum\limits_{k=1}^{n}\deg b_{k\pi(k)}(x) \leq n-2$

$\therefore$ 设 $|A(x)| = b_{11}(x)\cdots b_{nn}(x) + g(x)$

则 $g(x)=0$ 或 $\deg g(x)\leq n-2$

$\because b_{11}(x)\cdots b_{nn}(x) = \prod\limits_{k=1}^{n}(x-a_{kk})$

$= x^n - \left(\sum\limits_{k=1}^{n}a_{kk}\right)x^{n-1} + p(x)$

其中 $\deg p(x)\leq n-2$ 或 $p(x)=0$

$\therefore$ 记 $h(x)=p(x)+g(x)$，则 $\deg h(x)\leq n-2$ 或 $h(x)=0$

$\therefore |A(x)| = x^n - \sum\limits_{k=1}^{n}a_{kk}x^{n-1} + h(x)$

即 $|A(x)|$ 为首一 n 次多项式

(2) 由(1)知 $|A(x)| = x^n - \sum\limits_{k=1}^{n}a_{kk}x^{n-1} + h(x)$

$h(x)=0$ 或 $\deg h(x)\leq n-2$

$\therefore$ 对比系数即得 $a_1 = -\sum\limits_{k=1}^{n}a_{kk} = -\mathrm{tr}A$，证毕

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第6周作业 (高代A)　　　　汪翊平

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第一次 (2021 09 22)

1,2

1. 证明：设 $p(n)$ 为关于自然数 $n\ (n\geq k_0)$ 的命题

其满足：(1) $p(k_0)$ 正确

(2) $\forall\, m>k_0$，$p(k)$ 对 $\forall k_0\leq k<m$ 正确 $\Rightarrow p(m)$ 正确

设 $F=\{n \mid n\geq k_0,\ p(n)$ 不正确$\}$

假设 $F\neq\varnothing$，则 F 中有最小数 n_0。

由于 $p(k_0)$ 正确，故 $k_0\notin F$

$\because n_0=\min F$

$\therefore \forall k<n_0$ 且 $k\geq k_0$，$k\notin F$，$n_0\in F$

$\therefore \forall k_0\leq k<n_0$，有 $p(k)$ 正确

由(2)有 $p(n_0)$ 正确，故 $n_0\notin F$，与 $n_0\in F$ 矛盾

$\therefore$ 假设错误，故 $F=\varnothing$

$\therefore \forall n\geq k_0$ 且 $n\in \mathbb{N}_+$，$p(n)$ 正确

2. 证明：设 $p(n)$ 为关于自然数 $n\ (n\geq k_0)$ 的命题，则：

(I) (数学归纳原理)：$\left\{\begin{array}{l}(1)\ p(k_0)\text{ 正确}\\ (2)\ \forall m\geq k_0,\ \text{有：} p(m)\text{正确} \Rightarrow p(m+1)\text{正确}\end{array}\right. \Rightarrow \forall n\geq k_0,\ p(n)$ 正确

(II) (完全归纳原理)：$\left\{\begin{array}{l}(1)\ p(k_0)\text{ 正确}\\ (2)\ \forall m>k_0,\ \forall k_0\leq k<m,\ p(k)\text{正确} \Rightarrow p(m)\text{正确}\end{array}\right. \Rightarrow \forall n\geq k_0,\ p(n)$ 正确

题意即求证 (I)$\Leftrightarrow$(II)

先证 $\Rightarrow$)：由 (1')，$p(k_0)$ 正确

取 $\forall m\geq k_0$，且 $m\in\mathbb{N}_+$，设 $p(m)$ 正确

① 若 $m=k_0$，显然由(2')：$p(m+1)=p(k_0+1)$ 正确

② 若 $m>k_0$，则同①有 $p(k_0+1)$ 正确

由 $p(k_0)$，$p(k_0+1)$ 正确，由(2')可知 $p(k_0+2)$ 正确

由 $p(k_0)$，$p(k_0+1)$，$p(k_0+2)$ 正确，由(2')可知 $p(k_0+3)$ 正确

……

由 $p(k_0)$，$p(k_0+1)$，…$p(m-1)$ 正确，及 $p(m)$ 正确，由(2')可知 $p(m+1)$ 正确

即 $p(m)$ 正确 $\Rightarrow p(m+1)$ 正确

由(I)可知 $\forall n\geq k_0$，$p(n)$ 正确

(I)$\Rightarrow$(II) 证毕

证 $\Leftarrow$)：由(1) $p(k_0)$ 正确

取 $\forall m>k_0$ 且 $m\in\mathbb{N}_+$，假设 $\forall k_0\leq k<m$，$p(k)$ 正确

则显然 $p(m-1)$ 正确，故由(2)，$p(m)$ 正确

$\therefore \forall k_0\leq k<m$，$p(k)$ 正确 $\Rightarrow p(m)$ 正确

由(II)可知 $\forall n\geq k_0$，$p(n)$ 正确

故 (I)$\Leftarrow$(II) 证毕

3. 证明：设命题 $P_L(m)$ 为："$\overbrace{\forall n\geq b,\ p(m,n)}^{\text{取定 } m\geq a}$ 正确"

显然 $P_L(m)$ 正确 $\Leftrightarrow \forall n\geq b,\ p(m,n)$ 正确

故 $P_L(a)$ 也正确

下采用完全归纳原理证明 $\forall m\geq a$，$P_L(m)$ 正确：

设 $\forall m>a$，对 $\forall a\leq k<m$，$P_L(k)$ 正确

由于 $P_L(m)$ 正确 $\Leftrightarrow \forall n\ge b$, $P(m',n)$ 正确

故可采用完全归纳原理证之.

证 ① 由题意, $P(m',b)$ 正确

② 假设 $\forall n'>b$, 对于 $\forall b\le k<n'$, $P(m',k)$ 正确

∵ $\forall a\le k<m'$, $P_L(k)$ 正确

∴ $\forall a\le k<m'$, $\forall n\ge b$, $P(k,n)$ 正确

∴ $P(k,n')$ 正确, $\forall a\le k<m'$

$P(m',k')$ 正确, $\forall b\le k'<n'$

故由条件(2): $P(m',n')$ 正确

∴ $\forall n\ge b$, $P(m',n)$ 正确, 故 $P_L(m')$ 正确

∴ $\forall m\ge a$, $P_L(m)$ 正确

即: $\forall m\ge a, n\ge b$, $P(m,n)$ 正确

4.5.6: 由于4需用5, 5需用6结论, 故下做题顺序为 6→5→4.

6. 证明: 记 $P(x)=(1+x)^{n+1}$, $Q(x)=(1+x)^n$

则 $P(x)=(1+x)Q(x)$

由二项式定理: $P(x)=\sum_{k=0}^{n+1} C_{n+1}^k x^k$

$Q(x)=\sum_{k=0}^{n} C_n^k x^k$

∴ $(1+x)Q(x)=\sum_{k=0}^{n} C_n^k x^k+\sum_{k=0}^{n} C_n^k x^{k+1}$

$=\sum_{k=0}^{n} C_n^k x^k+\sum_{k=1}^{n+1} C_n^{k-1} x^k$

$=1+\sum_{k=1}^{n} (C_n^k+C_n^{k-1}) x^k+x^{n+1}$

对比系数可得: $C_{n+1}^0=1$

$C_{n+1}^k=C_n^k+C_n^{k-1}$

$C_{n+1}^{n+1}=1$

∴ $\forall n,k\in N_+$, $n\ge k$, 有 $C_{n+1}^k=C_n^k+C_n^{k-1}$, 证毕

5. 证明: 记 $T(m)=\sum_{k=0}^{m} C_{n-2+k}^k$

① 若 $m=0$, 则 $T(0)=C_{n-2}^0=1=C_{n+0-1}^0$, 等式成立

② 若 $m>0$, 则: $T(m)=C_{n-2}^0+\sum_{k=1}^{m} C_{n-2+k}^k$

$=1+\sum_{k=1}^{m} C_{n-2+k}^k$

∴ $T(m)+\sum_{k=1}^{m} C_{n-2+k}^{k-1}=1+\sum_{k=1}^{m} (C_{n-2+k}^k+C_{n-2+k}^{k-1})$

$=1+\sum_{k=1}^{m} C_{n-1+k}^k=\sum_{k=0}^{m} C_{n-1+k}^k$

⇒ $T(m)=\sum_{k=0}^{m} C_{n-1+k}^k-\sum_{k=1}^{m} C_{n-2+k}^{k-1}$

$=\sum_{k=0}^{m} C_{n-1+k}^k-\sum_{k=0}^{m-1} C_{n-2+k+1}^k=C_{n+m-1}^m$, 证毕

综合①②: $T(m)=\sum_{k=0}^{m} C_{n-2+k}^{k} = C_{n+m-1}^{m}$

$\forall m\ge 0,\ n\ge 2$ 且 $m,n\in N_+$

4. 证明: 由于 $x_1,x_2,\cdots x_n\in N$, $x_1+x_2+\cdots+x_n=m$

∴ $\forall i\in\{1,2,\cdots n\},\ 0\le x_i\le m$

记 $A_{m,n}=\{(x_1,x_2,\cdots x_n)\in N^n \mid x_1+x_2+\cdots+x_n=m\}$

$B_{k,n}=\{(x_1,x_2,\cdots x_n)\in N^n\mid x_1+x_2+\cdots+x_{n-1}=m-k,\ x_n=k\}$

则显然: $A_{m,n}=\bigcup_{k=0}^{m} B_{k,n}$

$\forall k_1\ne k_2$, $B_{k_1,n}\cap B_{k_2,n}=\phi$

∴ $A_{m,n}=\sum_{k=0}^{m} B_{k,n} \Rightarrow \# A_{m,n}=\sum_{k=0}^{m} \# B_{k,n}$

而 $\#B_{k,n}=\#\{(x_1,x_2\cdots x_{n-1})\in N^{n-1}\mid x_1+x_2+\cdots+x_{n-1}=m-k\}=\#A_{m-k,n-1}$

∴ $P(m,n)=\sum_{k=0}^{m} P(m-k,n-1)=\sum_{k=0}^{m} P(k,n-1)$

显然 $m\ge0,\ n\ge1$

而 $P(0,n)=\#\{(x_1,x_2,\cdots x_n)\in N^n\mid x_1+x_2+\cdots+x_n=0\}=\frac{1}{0!}=C_{n+0-1}^{0}$

$P(m,1)=\#\{x_1\in N\mid x_1=m\}=\frac{(1+m-1)\times\cdots\times 1}{m!}=C_{1+m-1}^{m}$

设命题 $T(n)$: $P(m,n)=C_{n+m-1}^{m}$, $\forall m\ge0$

显然命题 $T(1)$ 正确

假设 $\forall n>1$, 命题 $T(n-1)$ 正确

则 $\forall k\in N$, 有: $P(k,n-1)=C_{n-2+k}^{k}$

∴ $\forall m\in N$, 有: $P(m,n)=\sum_{k=0}^{m} P(k,n-1)=\sum_{k=0}^{m} C_{n-2+k}^{k}$

由5式已证: $P(m,n)=C_{n+m-1}^{m}$, 故命题 $T(n)$ 正确

∴由数学归纳原理, $\forall n\ge1$, $T(n)$正确

即: $\forall n\ge1,\ m\ge0$, 有 $P(m,n)=C_{n+m-1}^{m}=\frac{(n+m-1)(n+m-2)\cdots n}{m!}$

1.3.

1.解: (a) 原式 $=\frac{8-i+24i+3}{4+4i-1}=\frac{11+23i}{3+4i}=\frac{(11+23i)(3-4i)}{(3+4i)(3-4i)}=\frac{125-25i}{25}=5-i$

(b) 设 $\omega_1=-\frac12+\frac{\sqrt3}{2}i$, $\omega_2=-\frac12-\frac{\sqrt3}{2}i$, 则原式 $=\omega_1^3$ 或 ω_2^3

∵ $\omega_1^2+\omega_1+1=\frac14-\frac{\sqrt3}{2}i-\frac34-\frac12+\frac{\sqrt3}{2}i+1=0$

$\omega_2^2+\omega_2+1=\frac14+\frac{\sqrt3}{2}i-\frac34-\frac12-\frac{\sqrt3}{2}i+1=0$

∴ $\begin{cases}\omega_1^3-1=(\omega_1-1)(\omega_1^2+\omega_1+1)=0\\ \omega_2^3-1=(\omega_2-1)(\omega_2^2+\omega_2+1)=0\end{cases} \Rightarrow \begin{cases}\omega_1^3=1\\ \omega_2^3=1\end{cases}$

∴原式 $=1$

(c). 记 $\omega=\frac{\sqrt3+i}{1-i}=\frac{(\sqrt3+i)(1+i)}{(1-i)(1+i)}=\frac{\sqrt3-1+(\sqrt3+1)i}{2}$

则 $\omega^2=\left[\frac{(\sqrt3-1)+(\sqrt3+1)i}{2}\right]^2=\frac{(\sqrt3-1)^2-(\sqrt3+1)^2+2(\sqrt3-1)(\sqrt3+1)i}{4}=-\sqrt3+i$

记 $\omega' = \omega^2 = -\sqrt{3}+i$,

则 $\left(\frac{\omega'}{2}\right)^2 + \left(\frac{\omega'}{2}\right) + 1 = \frac{3-1-2\sqrt{3}i}{4} + \frac{-\sqrt{3}+i}{2} + 1 = 0$

$\Rightarrow \therefore \left(\frac{\omega'}{2}\right)^3 - 1 = \left(\frac{\omega'}{2} - 1\right)\left[\left(\frac{\omega'}{2}\right)^2 + \frac{\omega'}{2} + 1\right] = 0$

$\Rightarrow \left(\frac{\omega'}{2}\right)^3 = 1 \Rightarrow \omega'^3 = 8$

$\therefore$ 原式 $= \omega^{30} = (\omega')^{15} = (\omega'^3)^5 = 8^5 = 2^{15}$

2. 解：(a) 设 $z = |z|(\cos\theta + i\sin\theta)$

则 $|z| + z = |z|(1+\cos\theta + i\sin\theta)$

$= |z|\left(2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$

$= 2|z|\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$

而 $8+4i = 4\sqrt{5}\left(\frac{2}{\sqrt{5}} + i\cdot\frac{1}{\sqrt{5}}\right)$

$\therefore \begin{cases} \cos\frac{\theta}{2} = \frac{2}{\sqrt{5}} \\ \sin\frac{\theta}{2} = \frac{1}{\sqrt{5}} \\ 2|z|\cos\frac{\theta}{2} = 4\sqrt{5} \end{cases} \Rightarrow \begin{cases} \cos\theta = 2\cos^2\frac{\theta}{2} - 1 = \frac{3}{5} \\ \sin\theta = 2\cos\frac{\theta}{2}\sin\frac{\theta}{2} = \frac{4}{5} \\ |z| = 5 \end{cases}$

$\therefore z = 5\left(\frac{3}{5} + \frac{4}{5}i\right) = 3+4i$

(b). $z^2 = 3-4i = 5(\cos\theta_0 + i\sin\theta_0)$，其中 $\cos\theta_0 = \frac{3}{5}$, $\sin\theta_0 = -\frac{4}{5}$

故方程的解为：$\begin{cases} z_1 = \sqrt{5}\left(\cos\frac{\theta_0}{2} + i\sin\frac{\theta_0}{2}\right) \\ z_2 = \sqrt{5}\left(\cos\frac{\theta_0+2\pi}{2} + i\sin\frac{\theta_0+2\pi}{2}\right) = \sqrt{5}\left(-\cos\frac{\theta_0}{2} - i\sin\frac{\theta_0}{2}\right) \end{cases}$

而 $\begin{cases} \cos^2\frac{\theta_0}{2} = \frac{1+\cos\theta_0}{2} = \frac{4}{5} \\ \sin\frac{\theta_0}{2}\cos\frac{\theta_0}{2} = \frac{\sin\theta_0}{2} = -\frac{2}{5} \end{cases}$ 解得 $\begin{cases} \cos\frac{\theta_0}{2} = \frac{2}{\sqrt{5}} \\ \sin\frac{\theta_0}{2} = -\frac{1}{\sqrt{5}} \end{cases}$ 或 $\begin{cases} \cos\frac{\theta_0}{2} = -\frac{2}{\sqrt{5}} \\ \sin\frac{\theta_0}{2} = \frac{1}{\sqrt{5}} \end{cases}$

代入均有：$z_{1,2} = \pm\left[\sqrt{5}\left(\frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}}i\right)\right]$

即 $z_{1,2} = \pm(2-i)$

(c) 设 $x, y \in \mathbb{R}$，则 $(2x+y) + (x+2y)i = 1-4i$

$\therefore \begin{cases} 2x+y = 1 \\ x+2y = -4 \end{cases}$ 解之有：$\begin{cases} x = 2 \\ y = -3 \end{cases}$

(d). $\begin{cases} (1+i)z_1 + (1-i)z_2 = 1+i & ① \\ (1-i)z_1 + (1+i)z_2 = 1+3i & ② \end{cases}$

①$\times(1+i)$ − ②$\times(1-i)$ 得：$[(1+i)^2 - (1-i)^2]z_1 = (1+i)^2 - (1+3i)(1-i)$ ③

①$\times(1-i)$ − ②$\times(1+i)$ 得：$[(1-i)^2 - (1+i)^2]z_2 = (1+i)(1-i) - (1+3i)(1+i)$ ④

③ $\Leftrightarrow 4i z_1 = 2i - (1+3+3i-i) = -4$

$\Leftrightarrow z_1 = -\frac{4}{4i} = i$

④ $\Leftrightarrow (-4i)z_2 = 2 - (1-3+3i+i) = 4-4i$

$\Leftrightarrow z_2 = \frac{4-4i}{-4i} = 1+i$

故综上，解为 $\begin{cases} z_1 = i \\ z_2 = 1+i \end{cases}$

3. 证明：分两步证之：

(1) 首先证 $R[z]$ 是一个子环：

$\because R\subset\mathbb{C}$

$\therefore \forall a_i\in R$，必有 $a_i\in\mathbb{C}$

又 $z\in\mathbb{C}$，故 $\forall n\geq 0$，$z^n\in\mathbb{C}$

$\therefore \forall a_i\in R,\ n\geq 0$，$a_0+a_1z+\cdots+a_nz^n\in\mathbb{C}$，即 $R[z]\subset\mathbb{C}$

$\because 1=1+0\cdot z+0\cdot z^2+\cdots+0\cdot z^n\in R$，即R中至少有1个非零数

记 $f(z)=a_0+a_1z+\cdots+a_nz^n\in R[z]$　其中，a_i,b_i,m,n 均任意

$g(z)=b_0+b_1z+\cdots+b_mz^m\in R[z]$

则：$f(z)\pm g(z)=(a_0^*\pm b_0^*)+(a_1^*\pm b_1^*)z+\cdots+(a_k^*\pm b_k^*)z^k$

其中：$k=\max\{m,n\}$

$a_i^*\pm b_i^*$ 定义为：$a_i^*\pm b_i^*=\begin{cases}a_i\pm b_i, & i\leq\min\{m,n\}\\ a_i, & m<i\leq n\\ \pm b_i, & n<i\leq m\end{cases}$

显然，由于 $a_i,b_i\in R$，故 $a_i^*\pm b_i^*\in R$

$\therefore f(z)\pm g(z)\in R[z]$

而 $f(z)\cdot g(z)=\sum\limits_{k=0}^{m+n}\left(\sum\limits_{i+j=k}a_ib_j\right)z^k$

由于 $a_i,b_j\in R$，故 $\sum\limits_{i+j=k}a_ib_j\in R$，故 $f(z)\cdot g(z)\in R[z]$

$\therefore R[z]$ 是 $\mathbb{C}$ 的一个子环

(2) 再证 $R[z]$ 最小，即 $\mathbb{C}$ 的任意子环 R'，若 $R\cup\{z\}\in R'$，则 $R'\supset R[z]$

设 $R\cup\{z\}\subset R'\subset\mathbb{C}$

由 R' 是子环，对任：$\forall a_i\in R$，$a_i\in R'$

而 $z\in R'$，故 $\forall n\geq 0$，$z^n=z\cdot z\cdot\cdots\cdot z\in R'$

$\therefore \forall i\geq 0$，$a_iz^i\in R'$

$\therefore a_0+a_1z+\cdots+a_nz^n\in R'$，$\forall n\geq 0$，$a_i\in R$

$\therefore R[z]\subset R'$，证毕

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1. 证明: 记 $\bigcup_{i\in I} A_i = C$, $\bigcap_{i\in I} A_i = D$

(1) ①

取 $\forall a\in C\cap B$, 则 $a\in C$, $a\in B$

$\therefore \exists k\in I, a\in A_k$, 故 $a\in A_k\cap B$

$\therefore a\in \bigcup_{i\in I}(A_i\cap B)$, 即 $(C\cap B)\subset \bigcup_{i\in I}(A_i\cap B)$

② 取 $\forall a'\in \bigcup_{i\in I}(A_i\cap B)$, 则 $\exists j\in I$, $a'\in A_j\cap B$

$\therefore a'\in A_j$, $a'\in B$, 故 $a'\in \bigcup_{i\in I} A_i = C$

$\therefore a'\in C\cap B$, 即 $(C\cap B)\supset \bigcup_{i\in I}(A_i\cap B)$

综合①②: $(C\cap B) = (\bigcup_{i\in I} A_i)\cap B = \bigcup_{i\in I}(A_i\cap B)$

(2) ① 取 $\forall b\in D\cup B$, 则 $b\in D$或$b\in B$

若$b\in D$, 则 $\forall i\in I$, $b\in A_i$, 故 $b\in A_i\cup B$

若$b\in B$, 则 $\forall i\in I$, $b\in B\subset A_i\cup B$, 即 $b\in A_i\cup B$

$\therefore b\in \bigcap_{i\in I}(A_i\cup B)$, 即 $(D\cup B)\subset \bigcap_{i\in I}(A_i\cup B)$

② 取 $\forall b'\in \bigcap_{i\in I}(A_i\cup B)$, 则 $\forall i\in I$, $b'\in A_i\cup B$

即 $b'\in A_i$或$b'\in B$

若$b'\in B$, 则 $b'\in D\cup B$

若$b'\notin B$, 则 $\forall i\in I$, $b'\in A_i$, 故 $b'\in \bigcap_{i\in I} A_i = D$, 故 $b'\in D\cup B$

$\therefore b'\in D\cup B$, 即 $(D\cup B)\supset \bigcap_{i\in I}(A_i\cup B)$

综合①②: $D\cup B = (\bigcap_{i\in I} A_i)\cup B = \bigcap_{i\in I}(A_i\cup B)$

2. 证明: 记 $\bigcup_{i\in I} A_i = C_1$, $\bigcap_{i\in I} A_i = C_2$

$\bigcup_{i\in I} \bar{A_i} = D_1$, $\bigcap_{i\in I} \bar{A_i} = D_2$

(1) ① 取 $\forall a\in \bar{C_1}$, 则 $\forall i\in I$, $a\notin A_i$ (否则 $a\in \bigcup_{i\in I} A_i = C_1$)

$\therefore \forall i\in I$, $a\in \bar{A_i}$, 故 $a\in \bigcap_{i\in I}\bar{A_i} = D_2$, 即 $\bar{C_1}\subset D_2$

② 取 $\forall a'\in D_2$, 则 $\forall i\in I$, $a\in \bar{A_i}$, 故 $a\notin A_i$, 故 $a\notin \bigcup_{i\in I} A_i = C_1$

$\therefore a\in \bar{C_1}$, 即 $\bar{C_1}\supset D_2$

综合①②: $\bar{C_1} = D_2$, 即 $\overline{\bigcup_{i\in I} A_i} = \bigcap_{i\in I}\bar{A_i}$

(2) ① 取 $\forall b\in \bar{C_2}$, 则 $b\notin C_2$, 即 $\exists k\in I$, $b\notin A_k$, 故 $b\in \bar{A_k}$

$\therefore b\in \bigcup_{i\in I}\bar{A_i} = D_1$, 即 $\bar{C_2}\subset D_1$

② 取 $\forall b'\in D_1$, 则 $\exists k\in I$, $b'\in \bar{A_k}$, 故 $b'\notin A_k$, 故 $b'\notin \bigcap_{i\in I} A_i = C_2$

$\therefore b'\in \bar{C_2}$, 即 $\bar{C_2}\supset D_1$

综合①②: $\bar{C_2} = D_1$, 即 $\overline{\bigcap_{i\in I} A_i} = \bigcap_{i\in I}\bar{A_i}$

3. 证明：先证 $|A\cup B|=|A|+|B|-|A\cap B|$：

设 $C=A\cap B$，由于 $|A|,|B|<+\infty$，故 $|C|<+\infty$

设 $|A|=m$，$|B|=n$，$|C|=r$

则不妨设 $C=\{c_1,c_2,\cdots c_r\}$，而 $C=A\cap B$，故 $c_i\in A$，$c_i\in B$ $(i=1,\cdots,r)$

∴不妨设 $A=\{a_1,a_2,\cdots a_{m-r},c_1,c_2,\cdots c_r\}$

$B=\{b_1,b_2,\cdots b_{n-r},c_1,c_2,\cdots c_r\}$

则 $a_i\notin B$，$b_j\notin A$ $(i=1,2,\cdots m-r,\ j\in 1,2,\cdots n-r)$

∴ $A\cup B=\{a_1,a_2,\cdots a_{m-r},b_1,b_2,\cdots b_{n-r},c_1,c_2,\cdots c_r\}$

∴ $|A\cup B|=m-r+n-r+r=m+n-r=|A|+|B|-|C|$

$=|A|+|B|-|A\cap B|$

再证待证式：采用数学归纳法：

① $n=1$时显然成立，$n=2$时由上已证

② 设 $n=p$ 时其成立 $(p\geq 2)$

则 $n=p+1$ 时：

记：$B_1=A_1$，$B_2=A_2$，$\cdots$ $B_p=A_p\cup A_{p+1}$

则 $\bigcup_{i=1}^{p+1}A_i=\bigcup_{j=1}^{p}B_i$

∴由归纳假设：$\left|\bigcup_{i=1}^{p}B_i\right|=\sum_{k=1}^{p}(-1)^{k-1}\sum_{1\leq i_1<\cdots<i_k\leq p}|B_{i_1}\cap B_{i_2}\cap\cdots\cap B_{i_k}|$

下面进行讨论：(I)：若 $p\notin\{i_1,i_2,\cdots i_k\}$

则 $B_{i_1}\cap B_{i_2}\cap\cdots\cap B_{i_k}=A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_k}$

(II)：若 $p\in\{i_1,i_2,\cdots i_k\}$，由 $i_1<i_2<\cdots<i_k\leq p$

知必有 $i_k=p$

∴ $\bigcap_{j=1}^{k}B_{i_j}=\left(\bigcap_{j=1}^{k-1}B_{i_j}\right)\cap B_p$

$=\left(\bigcap_{j=1}^{k-1}B_{i_j}\right)\cap(A_p\cup A_{p+1})$

$=\left(\bigcap_{j=1}^{k-1}A_{i_j}\right)\cap(A_p\cup A_{p+1})$

（当$k=1$时规定 $\bigcap_{j=1}^{k-1}A_{i_j}=\phi$）

由第1题已证：

$\bigcap_{j=1}^{k}B_{i_j}=\left(\bigcap_{j=1}^{k-1}A_{i_j}\cap A_p\right)\cup\left(\bigcap_{j=1}^{k-1}A_{i_j}\cap A_{p+1}\right)$

由已证：$\left|\bigcap_{j=1}^{k}B_{i_j}\right|=\left|\bigcap_{j=1}^{k-1}A_{i_j}\cap A_p\right|+\left|\bigcap_{j=1}^{k-1}A_{i_j}\cap A_{p+1}\right|-\left|\bigcap_{j=1}^{k-1}A_{i_j}\cap A_p\cap A_{p+1}\right|$

∴ $(-1)^{k-1}\sum_{1\leq i_1<\cdots<i_k\leq p}\left|\bigcap_{j=1}^{k}B_{i_j}\right|$ $\cancel{=(-1)^{k-1}\sum_{1\leq i_1<\cdots<i_k\leq p-1}\left|\bigcap_{j=1}^{k}A_{i_j}\right|+(-1)^{k-1}}$

$=(-1)^{k-1}\sum_{1\leq i_1<\cdots<i_k\leq p-1}\left|\bigcap_{j=1}^{k}A_{i_j}\right|+(-1)^{k-1}\sum_{1\leq i_1<\cdots<i_{k-1}\leq p-1}\left(\left|\bigcap_{j=1}^{k-1}A_{i_j}\cap A_p\right|+\left|\bigcap_{j=1}^{k-1}A_{i_j}\cap A_{p+1}\right|-\left|\bigcap_{j=1}^{k-1}A_{i_j}\cap A_p\cap A_{p+1}\right|\right)$

$=(-1)^{k-1}\sum_{1\leq i_1<\cdots<i_k\leq p-1}\left|\bigcap_{j=1}^{k}A_{i_j}\right|+(-1)^{k-1}\sum_{\substack{1\leq i_1<\cdots<i_{k-1}\leq p-1\\ i_k=p\text{或}p+1}}\left|\bigcap_{j=1}^{k}A_{i_j}\right|+(-1)^{k}\sum_{1\leq i_1<\cdots<i_{k-1}\leq p-1}\left|\bigcap_{j=1}^{k-1}A_{i_j}\cap A_p\cap A_{p+1}\right|$

$t<1$ 时，若 $\bigcap_{j=1}^{t} A_{i_j} = \phi$

则 $\sum_{k=1}^{p}(-1)^{k-1}\sum_{1\le i_1<\cdots<i_k\le p}\left|\bigcap_{j=1}^{k}B_{i_j}\right|$

$$= \sum_{k=1}^{p}\left[(-1)^{k-1}\sum_{1\le i_1<\cdots<i_k\le p-1}\left|\bigcap_{j=1}^{k} A_{i_j}\right| + (-1)^{k-1}\sum_{\substack{1\le i_1<\cdots<i_{k-1}\le p-1\\ i_k=p\text{或}p+1}}\left|\bigcap_{j=1}^{k} A_{i_j}\right|\right] + \sum_{k=1}^{p}(-1)^{k}\sum_{1\le i_1<\cdots<i_{k-1}\le p-1}\left|\bigcap_{j=1}^{k-1}A_{i_j}\cap A_p\cap A_{p+1}\right|$$

$$=\sum_{i=1}^{p+1}|A_i| + \sum_{k=2}^{p}\left[(-1)^{k-1}\sum_{1\le i_1<\cdots<i_k\le p-1}\left|\bigcap_{j=1}^{k} A_{i_j}\right| + (-1)^{k-1}\sum_{\substack{1\le i_1<\cdots<i_{k-1}\le p-1\\ i_k=p\text{或}p+1}}\left|\bigcap_{j=1}^{k} A_{i_j}\right|\right] + \sum_{k=2}^{p}(-1)^{k-1}\sum_{1\le i_1<\cdots<i_{k-2}\le p-1}\left|\bigcap_{j=1}^{k-2}A_{i_j}\cap A_p\cap A_{p+1}\right|$$

$$+(-1)^{p}\left|\bigcap_{j=1}^{p+1}A_j\right|$$

$$=\sum_{i=1}^{p+1}(-1)^{1-1}|A_i| + \sum_{k=2}^{p}(-1)^{k-1}\sum_{j\in I}\left|\bigcap_{j=1}^{k} A_{i_j}\right| + (-1)^{p+1-1}\left|\bigcap_{j=1}^{p+1}A_j\right|$$

其中：$I=\{i_1,i_2,\cdots i_k \mid 1\le i_1<i_2<\cdots<i_k\le p-1$ 或 $1\le i_1<\cdots<i_{k-2}\le p-1,\ i_{k-1}=p,\ i_k=p+1$ 或 $1\le i_1<\cdots<i_{k-1}\le p-1,\ i_k=p$ 或 $i_k=p+1\}$

显然 $I=\{i_1,i_2,\cdots i_k\mid 1\le i_1<i_2<\cdots<i_k\le p+1\}$

$$\therefore \left|\bigcup_{i=1}^{p}B_i\right| = \sum_{k=1}^{p}(-1)^{k-1}\sum_{1\le i_1<\cdots<i_k\le p}\left|\bigcap_{j=1}^{k}B_{i_j}\right| = \sum_{k=1}^{p+1}(-1)^{k-1}\sum_{1\le i_1<i_2<\cdots<i_k\le p+1}|A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_k}|$$

$\therefore \left|\bigcup_{i=1}^{p+1}A_i\right| = \sum_{k=1}^{p+1}(-1)^{k+1}\sum_{1\le i_1<i_2<\cdots<i_k\le p+1}\left|\bigcap_{j=1}^{k}A_{i_j}\right|$，即 $n=p+1$ 时成立

综合(1)(2)，命题得证

4. 解：(1) $x^2=1 \Leftrightarrow x=\pm1$

$\therefore f^{-1}(1)=\{1,-1\}$

(2) $0<x^2<1 \Leftrightarrow -1<x<0$ 或 $0<x<1$

$\therefore f^{-1}(\{x|0<x<1\})=(-1,0)\cup(0,1)$

(3) $x^2>4 \Leftrightarrow x>2$ 或 $x<-2$

$\therefore f^{-1}(\{x|x>4\})=(-\infty,-2)\cup(2,+\infty)$

5. (1) 证明：① 取 $\forall x\in f^{-1}(A\cup B)$，则 $f(x)\in A\cup B$

故 $f(x)\in A$ 或 $f(x)\in B$，进而 $x\in f^{-1}(A)$ 或 $x\in f^{-1}(B)$

$\therefore x\in f^{-1}(A)\cup f^{-1}(B)$，即 $f^{-1}(A\cup B)\subset[f^{-1}(A)\cup f^{-1}(B)]$

② 取 $\forall x'\in f^{-1}(A)\cup f^{-1}(B)$，则 $x'\in f^{-1}(A)$ 或 $x'\in f^{-1}(B)$

$\therefore f(x')\in A$ 或 $f(x')\in B$，故 $f(x')\in A\cup B$

$\therefore x'\in f^{-1}(A\cup B)$，即 $f^{-1}(A\cup B)\supset[f^{-1}(A)\cup f^{-1}(B)]$

综合①②，$f^{-1}(A\cup B)=f^{-1}(A)\cup f^{-1}(B)$

(2) 证明：① 取 $\forall x''\in f^{-1}(A\cap B)$，则 $f(x'')\in A\cap B$

$\therefore f(x'')\in A$，$f(x'')\in B$，故 $x''\in f^{-1}(A)$ 且 $x''\in f^{-1}(B)$

$\therefore x''\in f^{-1}(A)\cap f^{-1}(B)$，即 $f^{-1}(A\cap B)\in f^{-1}(A)\cap f^{-1}(B)$

② 取 $\forall x'''\in f^{-1}(A)\cap f^{-1}(B)$，则 $x'''\in f^{-1}(A)$，$x'''\in f^{-1}(B)$

$\therefore f(x''')\in A$ 且 $f(x''')\in B$，故 $f(x''')\in A\cap B$

$\therefore x''' \in f^{-1}(A\cap B)$，即 $f^{-1}(A\cap B) \supset f^{-1}(A)\cap f^{-1}(B)$

综合①②：$f^{-1}(A\cap B) = f^{-1}(A)\cap f^{-1}(B)$

6. 证明：(a). 设 $x_1, x_2 \in X$ 且 $g\cdot f(x_1) = g\cdot f(x_2)$

则：$f(x_1), f(x_2) \in Y$，且 $g(f(x_1)) = g(f(x_2))$

由 g 是单射：$f(x_1) = f(x_2)$

又由 f 是单射：$x_1 = x_2$

$\therefore g\cdot f$ 是单射

(b) 设 $z\in Z$，由 g 为满射：$\exists y\in Y$，$g(y)=z$

又由 f 是满射：$\exists x\in X$，$f(x)=y$

$\therefore g(y) = g(f(x)) = z \Rightarrow g\cdot f(x) = z$

$\therefore (g\cdot f)^{-1}(z)$ 非空，故 $g\cdot f$ 是满射

(c) f 是单射，下证之：

假设 f 不是单射，则 $\exists x_1, x_2\in X$ 且 $x_1\neq x_2$，使 $f(x_1)=f(x_2)$

由于 g 是映射，故 $g(f(x_1)) = g(f(x_2)) \Rightarrow g\cdot f(x_1) = g\cdot f(x_2)$

这与 $g\cdot f$ 是单射矛盾，故假设错误，即 f 是单射

(d) 不一定，下举反例：

设 $X = Z = [0,1]$，$Y=[-1,1]$

$f: X\to Y$，$g: Y\to Z$

$x \to y=\sqrt{x}$　　$y\to z = y^2$

则 $\forall x\in X$，有 $g\cdot f(x) = g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 = x$

而 $X = Z$，故 $g\cdot f = 1_X = 1_Z$，显然其是满射

而 $\forall y\in Y$，$g^{-1}(y) = \{-\sqrt{y}, \sqrt{y}\} \neq \phi$

故 g 也是满射

但若取 $-1\in Y$，无 $x\in X$ 使得 $f(x) = \sqrt{x} = -1$

故 f 不是满射

7. 解：由题意 $f(A) = \{2x \mid x\in A\}$　$(A\subset \mathbb{R})$

$\therefore$ (1) $f(\mathbb{Z}) = \{2k \mid k\in\mathbb{Z}\}$

(2) $f(\mathbb{N}) = \{2n \mid n=0,1,2,\dots\}$

(3) $f(\mathbb{R}) = \{2x \mid x\in\mathbb{R}\}$，显然 $f(\mathbb{R})\subset\mathbb{R}$

但 $\forall x\in\mathbb{R}$，$\frac{x}{2}\in\mathbb{R}$，故 $x = 2\cdot\frac{x}{2}\in f(\mathbb{R})$

$\therefore f(\mathbb{R})\supset\mathbb{R}$

故 $f(\mathbb{R}) = \mathbb{R}$

8. 证明：(1) ① 取 $\forall y\in f(A\cup B)$，则 $(A\cup B)\cap f^{-1}(y)\neq\phi$

故 $[A\cap f^{-1}(y)]\cup[B\cap f^{-1}(y)]\neq\phi$

不妨设 $A\cap f^{-1}(y)\neq\phi$，则 $\exists x\in A\cap f^{-1}(y)$

故 $x\in A$ 且 $x\in f^{-1}(y)$

$f(x)=y$，故 $y\in f(A)$，进而 $y\in f(A)\cup f(B)$

即 $f(A\cup B)\subset f(A)\cup f(B)$

② 取 $\forall y'\in f(A)\cup f(B)$，则 $y'\in f(A)$ 或 $y'\in f(B)$，不妨 $y'\in f(A)$

则：$\exists x'\in A$，$f(x')=y$，而 $x'\in A$，故 $x'\in A\cup B$

$\therefore \exists x'\in A\cup B$，$f(x')=y$，故 $y\in f(A\cup B)$

即 $f(A\cup B)\supset f(A)\cup f(B)$

综上：$f(A\cup B)=f(A)\cup f(B)$

(2) 取 $\forall y''\in f(A\cap B)$，则 $\exists x''\in A\cap B$ 使 $f(x'')=y''$

而 $x''\in A\cap B\Rightarrow x''\in A$，$x''\in B$，故 $\exists x''\in A$，$f(x'')=y''$

$\exists x''\in B$，$f(x'')=y''$

$\therefore y''\in f(A)$，$y''\in f(B)$，故 $y''\in f(A)\cap f(B)$

$\therefore f(A\cap B)\subseteq f(A)\cap f(B)$

A+

逻辑都没出错，
但易读性很差…

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（转专业补修）

A

习题1.3.

4. 证明：(1) 使用归纳法即可：

① $\varepsilon_0 = 1 = \varepsilon_1^0$，命题正确

② 设 $\varepsilon_k = \varepsilon_1^k$，$k \geq 0$

则：$\varepsilon_k \varepsilon_1 = \varepsilon_1^{k+1}$

且 $\varepsilon_k \varepsilon_1 = \left(\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}\right)\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)$

$= \left(\cos\frac{2\pi k}{n}\cos\frac{2\pi}{n} - \sin\frac{2\pi k}{n}\sin\frac{2\pi}{n}\right) + i\left(\sin\frac{2\pi k}{n}\cos\frac{2\pi}{n} + \cos\frac{2\pi k}{n}\sin\frac{2\pi}{n}\right)$

$= \cos\left(\frac{2\pi k}{n} + \frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi k}{n} + \frac{2\pi}{n}\right)$

$= \cos\left[\frac{2\pi(k+1)}{n}\right] + i\sin\left[\frac{2\pi(k+1)}{n}\right]$

$= \varepsilon_{k+1}$

$\therefore \varepsilon_{k+1} = \varepsilon_1^{k+1}$

由①②即知 $\forall\, 0 \leq k < n$，$\varepsilon_k = \varepsilon_1^k$

(2) 同(1)可证 $\forall\, p \in \mathbb{N}$，$\cos\frac{2\pi p}{n} + i\sin\frac{2\pi p}{n} = \varepsilon_1^p$

故 $\varepsilon_1^n = \cos 2\pi + i\sin 2\pi = 1$

$\therefore \varepsilon_k \cdot \varepsilon_l = \varepsilon_1^k \cdot \varepsilon_1^l = \varepsilon_1^{k+l}$

由于 $k<n$，$l<n$，故 $k+l<2n$

$\therefore$ $k+l<n$ 时，$\varepsilon_k \cdot \varepsilon_l = \varepsilon_1^{k+l} = \varepsilon_{k+l}$

$n \leq k+l < 2n$ 时，$0 \leq k+l-n < n$，$\varepsilon_k \cdot \varepsilon_l = \varepsilon_1^{k+l} = \varepsilon_1^{k+l-n} \cdot \varepsilon_1^n = \varepsilon_1^{k+l-n} = \varepsilon_{k+l-n}$

(3) 由上已知：$\forall\, 0 \leq k < n-1$，$(\sqrt[n]{a}\,\varepsilon_k)^n - a = a\varepsilon_k^n - a = a\varepsilon_1^{kn} - a = a(\varepsilon_1^n)^k - a = a \cdot 1^k - a = 0$

$\therefore \sqrt[n]{a}\,\varepsilon_k = \sqrt[n]{a}\,\varepsilon_1^k$ 是 $x^n - a = 0$ 的一个根

又 $\forall\, 0 < k < n-1$，显 $\varepsilon_1^k \neq 1$

$\therefore \forall\, i \neq j$，且 $0 \leq i, j < n-1$，$\varepsilon_i \neq \varepsilon_j$，故 $\sqrt[n]{a}$、$\sqrt[n]{a}\,\varepsilon_1$、……$\sqrt[n]{a}\,\varepsilon_1^{n-1}$ 互不相等

又 n 次方程至多有 n 个互异根

$\therefore \sqrt[n]{a}$、$\sqrt[n]{a}\,\varepsilon_1$、……$\sqrt[n]{a}\,\varepsilon_1^{n-1}$ 为 $x^n - a = 0$ 的全部根

5. 解：$\mathbb{Q}[\alpha] = \{a_0 + a_1\alpha + \cdots + a_n\alpha^n \mid a_i \in \mathbb{Q}, n \geq 0\}$

$\because \alpha$ 满足：$\alpha^3 - 2 = 0$ 即 $\alpha^3 = 2$

$\therefore \mathbb{Q}[\alpha] = S := \{a_0 + a_1\alpha + a_2\alpha^2 \mid a_1, a_2, a_3 \in \mathbb{Q}\}$

下证之：显然 $\mathbb{Q}[\alpha] \supset S$

取 $\forall\, a = a_0 + a_1\alpha + \cdots + a_n\alpha^n$

则 $\exists\, p \in \mathbb{N}$，$3p < n < 3p+3$

记 $a = \sum_{k=0}^{p}(a_{3k} + a_{3k+1}\alpha + a_{3k+2}\alpha^2)\alpha^{3k}$

规定：当 $i > n$ 时，$a_i = 0$

$\because \alpha^3 = 2$

$\therefore \alpha^{3k} = 2^k$

$\therefore a = \sum_{k=0}^{p}(a_{3k} + a_{3k+1}\alpha + a_{3k+2}\alpha^2)2^k = \sum_{k=0}^{p}2^k a_{3k} + \sum_{k=0}^{p}(2^k a_{3k+1})\alpha + \sum_{k=0}^{p}(2^k a_{3k+2})\alpha^2$

$\because a_i \in \mathbb{Q}$ $\therefore \sum_{k=0}^{p}2^k a_{3k}$、$\sum_{k=0}^{p}2^k a_{3k+1}$、$\sum_{k=0}^{p}2^k a_{3k+2} \in \mathbb{Q}$

$\therefore a\in S$，即 $Q[\alpha]\subset S$，故 $Q[\alpha]=S=\{a_0+a_1\alpha+a_2\alpha^2 \mid a_0,a_1,a_2\in Q\}$

附：已下证 $Q[\alpha]\neq\{a_0+a_1\alpha \mid a_0,a_1\in Q\}$、$\{a_0+a_2\alpha^2 \mid a_0,a_2\in Q\}$、$\{a_1\alpha+a_2\alpha^2 \mid a_1,a_2\in Q\}$

只需证 $Q[\alpha]\neq\{a_0+a_1\alpha \mid a_0,a_1\in Q\}$ 即可，~~其~~其他同理

假设 $Q[\alpha]=\{a_0+a_1\alpha \mid a_0,a_1\in Q\}$

则由 $\alpha^2\in Q[\alpha]$，$\exists a,b\in Q$，使 $\alpha^2=a+b\alpha$

$\Rightarrow \alpha^3=a\alpha+b\alpha^2=a\alpha+ab+b^2\alpha=2$

$\Rightarrow (a+b^2)\alpha=2-ab$ (*)

①若 $a+b^2\neq 0$，则 $\alpha=\frac{2-ab}{a+b^2}\in Q$，由(4)已证：$x^3-2=0$ 的根仅有 $\sqrt[3]{2}$，$\sqrt[3]{2}\varepsilon_1$，$\sqrt[3]{2}\varepsilon_2$ 均非有理数，矛盾

②若 $a+b^2=0$，则 $2-ab=0 \Rightarrow ab=2$

$\therefore a+b^2=0 \Rightarrow ab+b^3=0 \Rightarrow 2+b^3=0 \Rightarrow (-b)^3=2$，同①知 $b\notin Q$，矛盾

$\therefore$ 假设错误，故 $Q[\alpha]\neq\{a_0+a_1\alpha \mid a_0,a_1\in Q\}$

综上：$Q[\alpha]=\{a_0+a_1\alpha+a_2\alpha^2 \mid a_0,a_1,a_2\in Q\}$

实际上定义 $Q[\alpha]=\{$系数在 Q 上的多项式$\}$ 证明后面这些属实没必要 (1,3,4)

6. 证明：显然 $\alpha\in\mathbb{C}$，故 $\mathbb{R}\cup\{\alpha\}\subset\mathbb{C}$，由定义有 $\mathbb{R}[\alpha]\subset\mathbb{C}$

故只需证 $\mathbb{C}\subset\mathbb{R}[\alpha]$

设 $\forall z=x+yi\in\mathbb{C}$，其中 $x,y\in\mathbb{R}$，$i^2=-1$

设 $\beta=\alpha+\frac{b}{2}$，则 $\beta^2=\alpha^2+b\alpha+\frac{b^2}{4}=-c+\frac{b^2}{4}=\frac{b^2-4c}{4}<0$

$\therefore \frac{4c-b}{4}>0$，故可令 $\delta=\sqrt{\frac{4c-b}{4}}=\frac{\sqrt{4c-b}}{2}>0$，令 $\gamma=\frac{1}{\delta}\beta$，则 $\gamma^2=\frac{1}{\delta^2}\beta^2=-1$

又由于 $\beta\in\mathbb{R}[\alpha]$，$\delta\in\mathbb{R}\subset\mathbb{R}[\alpha]$，故 $\gamma\in\mathbb{R}[\alpha]$，同理 $-\gamma\in\mathbb{R}[\alpha]$

而 $\gamma^2=-1 \Leftrightarrow \gamma=\pm i$，故 $\gamma,-\gamma$ 之一为 i，故 $i\in\mathbb{R}[\alpha]$

$\therefore x+yi\in\mathbb{R}[\alpha]$，故 $\mathbb{C}\subset\mathbb{R}[\alpha]$，

$\therefore \mathbb{R}[\alpha]=\mathbb{C}$，证毕

7. 解：(1) $x^6-i=0 \Leftrightarrow x^6=i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$

$\therefore$ 方程解为 $x=\cos\frac{\frac{\pi}{2}+2k\pi}{6}+i\sin\frac{\frac{\pi}{2}+2k\pi}{6}$

$=\cos\frac{(4k+1)\pi}{12}+i\sin\frac{(4k+1)\pi}{12}$

$k=0,1,2,3,4,5$

(2) $x^6-64=0 \Leftrightarrow x^6=64=2^6\cdot 1=2^6(\cos 0+i\sin 0)$

$\therefore$ 方程解为 $x=2(\cos\frac{2k\pi}{6}+i\sin\frac{2k\pi}{6})$

$=2(\cos\frac{k\pi}{3}+i\sin\frac{k\pi}{3})$

$k=0,1,2,3,4,5$

(3) $x^{10}-512(1-i\sqrt{3})=0 \Leftrightarrow x^{10}=2^{10}(\frac{1}{2}-i\frac{\sqrt{3}}{2})$

$=2^{10}[\cos(-\frac{\pi}{3})+i\sin(-\frac{\pi}{3})]$

$\therefore$ 方程解为 $x=2(\cos\frac{-\frac{\pi}{3}+2k\pi}{10}+i\sin\frac{-\frac{\pi}{3}+2k\pi}{10})$

$=2[\cos\frac{(6k-1)\pi}{30}+i\sin\frac{(6k-1)\pi}{30}]$

$k=0,1,2,\cdots,9$

习题1.4

1. 解：$\pi(1)=3$，$\pi^2(1)=\pi(3)=6$，$\pi^3(1)=\pi(6)=2$，$\pi^4(1)=\pi(2)=7$，$\pi^5(1)=\pi(7)=4$，$\pi^6(1)=\pi(4)=5$，$\pi^7(1)=\pi(5)=1$

$\therefore \pi=(1362745)$

$\varepsilon_\pi=(-1)^{7-1}=(-1)^6=1$

$\therefore \alpha \in S$，即 $\mathbb{Q}[\alpha] \subset S$，故 $\mathbb{Q}[\alpha] = S = \{a_0 + a_1\alpha + a_2\alpha^2 \mid a_0, a_1, a_2 \in \mathbb{Q}\}$

附：下证 $\mathbb{Q}[\alpha] \neq \{a_0 + a_1\alpha \mid a_0, a_1 \in \mathbb{Q}\}$、$\{a_0 + a_2\alpha^2 \mid a_0, a_2 \in \mathbb{Q}\}$、$\{a_1\alpha + a_2\alpha^2 \mid a_1, a_2 \in \mathbb{Q}\}$

只需证 $\mathbb{Q}[\alpha] \neq \{a_0 + a_1\alpha \mid a_0, a_1 \in \mathbb{Q}\}$ 即可，其它同理

假设 $\mathbb{Q}[\alpha] = \{a_0 + a_1\alpha \mid a_0, a_1 \in \mathbb{Q}\}$

则由 $\alpha^2 \in \mathbb{Q}[\alpha]$，$\exists a, b \in \mathbb{Q}$，使 $\alpha^2 = a + b\alpha$

$\Rightarrow \alpha^3 = a\alpha + b\alpha^2 = a\alpha + ab + b^2\alpha = 2$

$\Rightarrow (a+b^2)\alpha = 2 - ab \quad (*)$

实际上定义 $\mathbb{Q}[\alpha]$={系数在 $\mathbb{Q}$ 上的多项式}证明 后面这些属实没必要. (1,3,4)

①若 $a+b^2 \neq 0$，则 $\alpha = \frac{2-ab}{a+b^2} \in \mathbb{Q}$，由(4)已证：$x^3 - 2 = 0$ 的根仅有 $\sqrt[3]{2}$、$\sqrt[3]{2}\varepsilon_1$、$\sqrt[3]{2}\varepsilon_2$ 均非有理数，矛盾

②若 $a+b^2 = 0$，则 $2 - ab = 0 \Rightarrow ab = 2$

$\therefore a + b^2 = 0 \Rightarrow ab + b^3 = 0 \Rightarrow 2 + b^3 = 0 \Rightarrow (-b)^3 = 2$，同①知 $b \notin \mathbb{Q}$，矛盾

$\therefore$ 假设错误，故 $\mathbb{Q}[\alpha] \neq \{a_0 + a_1\alpha \mid a_0, a_1 \in \mathbb{Q}\}$

综上：$\mathbb{Q}[\alpha] = \{a_0 + a_1\alpha + a_2\alpha^2 \mid a_0, a_1, a_2 \in \mathbb{Q}\}$

6. 证明：显然 $\alpha \in \mathbb{C}$，故 $\mathbb{R} \cup \{\alpha\} \subset \mathbb{C}$，由定义有 $\mathbb{R}[\alpha] \subset \mathbb{C}$

故只需证 $\mathbb{C} \subset \mathbb{R}[\alpha]$

设 $\forall z = x + yi \in \mathbb{C}$，其中 $\forall x, y \in \mathbb{R}$，$i^2 = -1$

设 $\beta = \alpha + \frac{b}{2}$，则 $\beta^2 = \alpha^2 + b\alpha + \frac{b^2}{4} = -c + \frac{b^2}{4} = \frac{b^2 - 4c}{4} < 0$

$\therefore \frac{4c-b}{4} > 0$，故可令 $\delta = \sqrt{\frac{4c-b}{4}} = \frac{\sqrt{4c-b}}{2} > 0$，令 $\gamma = \frac{1}{\delta}\beta$，则 $\gamma^2 = \frac{1}{\delta^2}\beta^2 = -1$

由于 $\beta \in \mathbb{R}[\alpha]$，$\delta \in \mathbb{R} \subset \mathbb{R}[\alpha]$，故 $\gamma \in \mathbb{R}[\alpha]$，同理 $-\gamma \in \mathbb{R}[\alpha]$

而 $\gamma^2 = -1 \Leftrightarrow \gamma = \pm i$，故 $\gamma, -\gamma$ 一为 i，故 $i \in \mathbb{R}[\alpha]$

$\therefore x + yi \in \mathbb{R}[\alpha]$，故 $\mathbb{C} \subset \mathbb{R}[\alpha]$，

$\therefore \mathbb{R}[\alpha] = \mathbb{C}$，证毕

7. 解：(1) $x^6 - i = 0 \Leftrightarrow x^6 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$\therefore$ 方程解为 $x = \cos\frac{\frac{\pi}{2} + 2k\pi}{6} + i\sin\frac{\frac{\pi}{2} + 2k\pi}{6}$

$= \cos\frac{(4k+1)\pi}{12} + i\sin\frac{(4k+1)\pi}{12}$

$k = 0, 1, 2, 3, 4, 5$

(2) $x^6 - 64 = 0 \Leftrightarrow x^6 = 64 = 2^6 \cdot 1 = 2^6(\cos 0 + i\sin 0)$

$\therefore$ 方程解为 $x = 2(\cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6})$

$= 2(\cos\frac{k\pi}{3} + i\sin\frac{k\pi}{3})$

$k = 0, 1, 2, 3, 4, 5$

(3) $x^{10} - 512(1 - i\sqrt{3}) = 0 \Leftrightarrow x^{10} = 2^{10}(\frac{1}{2} - i\frac{\sqrt{3}}{2})$

$= 2^{10}[\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})]$

$\therefore$ 方程解为 $x = 2(\cos\frac{-\frac{\pi}{3} + 2k\pi}{10} + i\sin\frac{-\frac{\pi}{3} + 2k\pi}{10})$

$= 2[\cos\frac{(6k-1)\pi}{30} + i\sin\frac{(6k-1)\pi}{30}]$

$k = 0, 1, 2, \cdots, 9$

习题1.4

1. 解：$\pi(1) = 3$，$\pi^2(1) = \pi(3) = 6$，$\pi^3(1) = \pi(6) = 2$，$\pi^4(1) = \pi(2) = 7$，$\pi^5(1) = \pi(7) = 4$，

$\pi^6(1) = \pi(4) = 5$，$\pi^7(1) = \pi(5) = 1$

$\therefore \pi = (1362745)$

$\varepsilon_\pi = (-1)^{7-1} = (-1)^6 = 1$

解：$\sigma\cdot\tau=\begin{pmatrix}1&2&3&4&5&6\\3&1&2&6&4&5\end{pmatrix}$

$\tau\cdot\sigma=\begin{pmatrix}1&2&3&4&5&6\\5&1&6&3&2&4\end{pmatrix}$

$\sigma^{-1}=\begin{pmatrix}5&6&3&1&4&2\\1&2&3&4&5&6\end{pmatrix}=\begin{pmatrix}1&2&3&4&5&6\\4&6&3&5&1&2\end{pmatrix}$

$\sigma^{2}=\begin{pmatrix}1&2&3&4&5&6\\4&2&3&5&1&6\end{pmatrix}$

$\sigma^{3}=\sigma\cdot\sigma^{2}=\begin{pmatrix}1&2&3&4&5&6\\1&6&3&4&5&2\end{pmatrix}$

$\tau^{-1}\cdot\tau=1\in S_n$

$\because \sigma(1)=5,\ \sigma^{2}(1)=\sigma(5)=4,\ \sigma^{3}(1)=\sigma(4)=1$

$\tau(1)=3,\ \tau^{2}(1)=\tau(3)=6,\ \tau^{3}(1)=\tau(6)=1$

$\therefore$ ~~$\sigma=\sigma_1$~~ $\sigma=(154)\sigma_1,\ \sigma_1=\begin{pmatrix}2&3&6\\6&3&2\end{pmatrix}$

$\tau=(136)\tau_1,\ \tau_1=\begin{pmatrix}2&4&5\\4&2&5\end{pmatrix}$

同理有：$\sigma_1=(26)1,\ 1\in S_n$

$\tau_1=(24)1,\ 1\in S_n$

$\therefore \sigma=(154)(26),\ \tau=(136)(24)$

$\therefore \sigma^{m}=(154)^{m}\cdot(26)^{m},\ \tau^{m}=(136)^{m}(24)^{m}$

且 $\sigma^{m}=1\Leftrightarrow(154)^{m}=(26)^{m}=1$，故 $m=3p=2k,\ p,k\in\mathbb{N}$

$\tau^{m}=1\Leftrightarrow(136)^{m}=(24)^{m}=1$，故 $n=3p'=2k',\ p',k'\in\mathbb{N}$

又 $m,n>0$，故 $m=n=[3,2]=6$

即 σ、τ 的阶均为6

3.(1)：$\sigma=\begin{pmatrix}1&2&3&4&5&6&7&8\\2&5&7&3&8&6&4&1\end{pmatrix}=(1258)\begin{pmatrix}3&4&6&7\\7&3&6&4\end{pmatrix}=(1258)(374)(6)$

$=(1258)(374)$

$\therefore \sigma=(12)(25)(58)(37)(74)$

(2)：$\because \sigma=\begin{pmatrix}1&2&3&4&5\\2&3&1&5&4\end{pmatrix}=(123)\begin{pmatrix}4&5\\5&4\end{pmatrix}=(123)(45)$

$\therefore \sigma=(12)(23)(45)$

4.证明：由于 $\forall\sigma\in S_n$，σ 可写成一列不交的循环 $\sigma_1,\sigma_2\ldots\sigma_s$ 的乘积

每个循环 σ_i 能写成 $(l(\sigma_i)-1)$ 个对换的乘积

$\therefore \forall\sigma\in S_n$，$\sigma$ 可写成一系列对换 (i_1j_1)、$(i_2j_2)\ldots,(i_tj_t)$ 的乘积

故只需证明 $\forall(ij)\in S_n$，(ij) 可写成形如 (12)、$\cdots$、$(1n)$ 的乘积即可

方便起见：记 $\tau_1=1\in S_n$，$\tau_k=(1k)\in S_n,k=2,\cdots,n$

$\because$ ①若 $i=j$，则 $(ij)=1\in S_n$，即 $(ij)=\tau_2^{2}$

②若 $i\neq j$，则 i,j 中至少有1个不为1，不妨设 $j\neq1$

则：$i\neq1$ 时：

$(1i)(ij)=(1ij)=(j1i)=(j1)(1i)=(1j)(1i)$

$\Rightarrow[(1i)(1i)](ij)=(1i)(1j)(1i)$

$\Rightarrow (ij)=(1i)(1j)(1i)=\tau_i\tau_j\tau_i$

$i=1$ 时 $(ij)=(1j)=\tau_j=1\cdot\tau_j\cdot1$，上式亦成立

$\therefore$ 综合①②，$\forall(ij)\in S_n$，(ij) 可写成形如 (12)、$(13)\cdots$、$(1n)$ 的对换的乘积

证毕

5.证明：由4已证，任意置换 $\sigma\in S_n$ 都可写成形如 (12)、(13)、$\cdots$、$(1n)$ 的对换的乘积，若 σ 为偶置换，则 σ 必可写成偶数个上述形式的对换乘积

$\therefore$ 只需证：$\forall k, l \in \{2,\cdots,n\}$ 且 $k \neq l$，$(1k)(1l)$ 可写成形如 $(123),(124),\cdots(12n)$ 的3-轮换乘积即可

方便起见：同4设 $\tau_k=(1k)$，$k=2,\cdots n$ 且
（实际上还不如 $(1k)$ 方便）~~(12k)~~ 设 $\sigma_k=(12k)$，$k=3,\cdots,n$ 且

$\therefore \tau_k\tau_l=(1k)(1l)=(1lk)$ （$k,l\in\{2,\cdots n\}$ 且 $k\neq l$）

(1) 当 $l,k\neq 2$ 时，

$\because (12l)(12k)^2=\begin{pmatrix}1&2&l&k&i_1&\cdots&i_{n-4}\\2&l&1&k&i_1&\cdots&i_{n-4}\end{pmatrix}\begin{pmatrix}1&2&l&k&i_1&\cdots&i_{n-4}\\2&k&l&1&i_1&\cdots&i_{n-4}\end{pmatrix}\begin{pmatrix}1&2&l&k&i_1&\cdots&i_{n-4}\\2&k&l&1&i_1&\cdots&i_{n-4}\end{pmatrix}$

$=\begin{pmatrix}1&2&l&k&i_1&\cdots&i_{n-4}\\2&l&1&k&i_1&\cdots&i_{n-4}\end{pmatrix}\begin{pmatrix}1&2&l&k&i_1&\cdots&i_{n-4}\\k&1&l&2&i_1&\cdots&i_{n-4}\end{pmatrix}$

$=\begin{pmatrix}1&2&l&k&i_1&\cdots&i_{n-4}\\k&2&1&l&i_1&\cdots&i_{n-4}\end{pmatrix}$

$=(1kl)$

即 $(1kl)=\sigma_l\sigma_k^2$　故同理 $(1lk)=\sigma_k\sigma_l^2$　$\therefore \tau_k\tau_l=\sigma_k\sigma_l^2$

(2) 当 l,k 中有一为2时，不妨 $l=2$，则 $\tau_2\tau_k=(12)(1k)=(1k2)=(12k)^2$，$\tau_k\tau_2=(1k)(12)=(12k)$

综合(1)(2)，$\forall k\neq l$ 且 $k,l\in\{2,\cdots,n\}$，$\tau_k\tau_l$ 可写成形如 $\sigma_3,\cdots,\sigma_n$ 的~~乘积~~3-轮换的乘积

证毕

6. 解：$\because$ $\sigma(1)=2$, $\sigma(2)=3$, $\sigma(3)=1$, $\sigma(4)=5$, $\sigma(5)=4$

$\therefore f(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)},x_{\sigma(4)},x_{\sigma(5)})=f(x_2,x_3,x_1,x_5,x_4)$

$=x_2+2x_3^2+3x_1^3+4x_5^4+5x_4^5$

习题1.5

1. 解：(1) $f(x_1,x_2,x_3)+g(x_1,x_2,x_3)$

$=5+3x_1-2x_2+2x_1x_2+2x_2^2-x_2x_3+3x_3^2+x_1^3-2x_3^3+x_2^4$

$f(x_1,x_2,x_3)\cdot g(x_1,x_2,x_3)=[(x_1^3-2x_3^3)+(2x_2^2-x_2x_3)-x_1+2][x_2^4+(2x_1x_2+3x_3^2)+(4x_1-2x_2)+3]$

$=(x_1^3-2x_3^3)x_2^4+(2x_2^2-x_2x_3)x_2^4+[-x_1x_2^4+(x_1^3-2x_3^3)(2x_1x_2+3x_3^2)]$
$+[(x_1^3-2x_3^3)(4x_1-2x_2)+(2x_2^2-x_2x_3)(2x_1x_2+3x_3^2)]+[3(x_1^3-2x_3^3)+(2x_2^2-x_2x_3)(4x_1-2x_2)-x_1(2x_1x_2+3x_3^2)]+[3(2x_2^2-x_2x_3)+2(2x_1x_2+3x_3^2)-x_1(4x_1-2x_2)]+[2(4x_1-2x_2)-3x_1]+6$

$=(x_1^3x_2^4-2x_2^4x_3^3)+(2x_2^6-x_2^5x_3)+(2x_1^4x_2+3x_1^3x_3^2-x_1x_2^4-4x_1x_2x_3^3-2x_3^5)$
$+(4x_1^4-2x_1^3x_2+4x_1x_2^3-2x_1x_2^2x_3-8x_1x_3^3+6x_2^2x_3^2+x_2x_3^3)$
$+(3x_1^3-2x_1^2x_2+8x_1x_2^2-4x_1x_2x_3-3x_1x_3^2-4x_2^3+2x_2^2x_3-6x_3^3)$
$+(-4x_1^2+6x_1x_2+6x_2^2-3x_2x_3+6x_3^2)+(5x_1-4x_2)+6$

(2) $f_0=2$, $f_1=-x_1$, $f_2=2x_2^2-x_2x_3$, $f_3=x_1^3-2x_3^3$

$g_0=3$, $g_1=4x_1-2x_2$, $g_2=2x_1x_2+3x_3^2$, $g_3=0$, $g_4=x_2^4$

(3) $\deg(f)=3$, $\deg(g)=4$, $\deg(f+g)=4$

$\deg(f\cdot g)=7$

解：(1) ∵

$$
\begin{array}{r|rrrrr|l}
1\ -3\ 1 & 2 & -3 & 4 & -5 & 6 & 2\ \ 3\ \ 11 \\
 & 2 & -6 & 2 & & & \\
\hline
 & & 3 & 2 & -5 & 6 & \\
 & & 3 & -9 & 3 & & \\
\hline
 & & & 11 & -8 & 6 & \\
 & & & 11 & -33 & 11 & \\
\hline
 & & & & 25 & -5 & \\
\end{array}
$$

∴ $2x^4-3x^3+4x^2-5x+6=(x^2-3x+1)(2x^2+3x+11)+(25x-5)$

即：$q(x)=2x^2+3x+11$，$r(x)=25x-5$

(2) ∵

$$
\begin{array}{r|rrrr|l}
3\ -2\ 1 & 1 & -3 & -1 & -1 & \frac{1}{3}\ \ -\frac{7}{9} \\
 & 1 & -\frac{2}{3} & \frac{1}{3} & & \\
\hline
 & & -\frac{7}{3} & -\frac{4}{3} & -1 & \\
 & & -\frac{7}{3} & \frac{14}{9} & -\frac{7}{9} & \\
\hline
 & & & -\frac{26}{9} & -\frac{2}{9} & \\
\end{array}
$$

∴ $x^3-3x^2-x-1=(3x^2-2x+1)(\frac{1}{3}x-\frac{7}{9})+(-\frac{26}{9}x-\frac{2}{9})$

即：$q(x)=\frac{1}{3}x-\frac{7}{9}$，$r(x)=-\frac{26}{9}x-\frac{2}{9}$

3. 证明：统设 $f(x)=a_mx^m+\cdots+a_0$ 则 $a_m\neq0, b_n\neq0, c_s\neq0$

$g(x)=b_nx^n+\cdots+b_0$

$h(x)=c_sx^s+\cdots+c_0$

(1) ∵ $h(x)=f(x)g(x)=\sum_{k=0}^{s}(\sum_{i+j=k}a_ib_j)x^k$

∴ $c_k=\sum_{i+j=k}a_ib_j$，$k=0,\cdots,s$

∴ $\forall\alpha\in\mathbb{R}^n$，$f(\varphi)g(\varphi)(\alpha)=f(\varphi)[g(\varphi)(\alpha)]$

$=f(\varphi)(b_n\varphi^{(n)}(\alpha)+\cdots+b_1\varphi(\alpha)+b_0\alpha)$ (*)

(2) 题意即证：(I) φ是线性映射 $\Leftrightarrow$ φ满足：$\begin{cases}① \ \varphi\cdot\lambda=\lambda\cdot\varphi \\ ② \ \varphi\cdot(\varphi_1+\varphi_2)=\varphi\cdot\varphi_1+\varphi\cdot\varphi_2 \\ \quad \forall\varphi_1,\varphi_2\in M(\mathbb{R}^n)\end{cases}$

(II) 在满足(I)中某一条件下 $h(\varphi)=f(\varphi)g(\varphi)$

证明(I)：$\Rightarrow$：∵ φ为线性映射

∴ $\forall\alpha\in\mathbb{R}^n$，有：$\varphi(\lambda\alpha)=\lambda\varphi(\alpha)$，即 $\varphi\cdot\lambda(\alpha)=\lambda\cdot\varphi(\alpha)$

$\forall\varphi_1,\varphi_2\in M(\mathbb{R}^n)$ $\varphi(\varphi_1(\alpha)+\varphi_2(\alpha))=\varphi(\varphi_1(\alpha))+\varphi(\varphi_2(\alpha))$

$=\varphi\cdot\varphi_1(\alpha)+\varphi\cdot\varphi_2(\alpha)$

$\Rightarrow\varphi(\varphi_1+\varphi_2)(\alpha)=\varphi\cdot\varphi_1(\alpha)+\varphi\cdot\varphi_2(\alpha)$

∴ $\varphi\cdot\lambda=\lambda\cdot\varphi$

$\varphi\cdot(\varphi_1+\varphi_2)=\varphi\cdot\varphi_1+\varphi\cdot\varphi_2$

$\Leftarrow$：设 $id\in M(\mathbb{R}^n)$ 为：$id:\mathbb{R}^n\to\mathbb{R}^n$，$\alpha\mapsto\alpha$

则 $\forall\lambda\in\mathbb{R}$，$\forall\alpha,\beta\in\mathbb{R}^n$，

有：$\varphi\cdot\lambda(\alpha)=\lambda\cdot\varphi(\alpha)\Rightarrow\varphi(\lambda\alpha)=\lambda\varphi(\alpha)$

$\varphi(\alpha+\beta)=\varphi(id(\alpha)+\beta)$

而$\alpha,\beta\in\mathbb{R}^n$，故 $\exists\varphi_0$ 使 $\varphi_0(\alpha)=\beta$

∴ $\varphi(\alpha+\beta)=\varphi(id(\alpha)+\varphi_0(\alpha))=\varphi\cdot(id+\varphi_0)(\alpha)=\varphi\cdot id(\alpha)+\varphi\cdot\varphi_0(\alpha)$

$= \varphi(id(\alpha)) + \varphi(\varphi_0(\alpha)) = \varphi(\alpha) + \varphi(\beta)$

$\therefore \varphi$ 是线性映射

综上(I)得证

证明(II)：运用"φ为线性映射"的条件

则 $(*) \Leftrightarrow f(\varphi)\, g(\varphi)(\alpha) = f(\varphi)\left(\sum_{j=0}^{n} b_j \varphi^{(j)}(\alpha)\right) = \sum_{i=0}^{m} a_i \varphi^{(i)}\left(\sum_{j=0}^{n} b_j \varphi^{(j)}(\alpha)\right)$

$\because \forall \alpha, \beta \in \mathbb{R}^n, a, b \in \mathbb{R}, t \in \mathbb{N}_+$，有：

$\varphi^{(t)}(a\alpha + b\beta) = \varphi^{(t-1)}(\varphi(a\alpha + b\beta)) = \varphi^{(t-1)}(a\varphi(\alpha) + b\varphi(\beta))$

$= \cdots\cdots = a\varphi^{(t)}(\alpha) + b\varphi^{(t)}(\beta)$

而 $t=0$ 时 $\varphi^{(0)}(a\alpha + b\beta) = a\alpha + b\beta = a\varphi^{(0)}(\alpha) + b\varphi^{(0)}(\beta)$

$\therefore (*) \Leftrightarrow f(\varphi)\, g(\varphi)(\alpha) = \sum_{i=1}^{m} a_i \sum_{j=0}^{n} b_j\, \varphi^{(i)}(\varphi^{(j)}(\alpha))$

$= \sum_{i=0}^{m} \sum_{j=0}^{n} a_i b_j\, \varphi^{(i+j)}(\alpha)$

$= \sum_{k=0}^{m+n} \left(\sum_{i+j=k} a_i b_j\right) \varphi^{(k)}(\alpha)$

$\because f(x)g(x) = h(x) \Rightarrow m+n = s$

$\therefore (*) \Leftrightarrow f(\varphi)\, g(\varphi)(\alpha) = \sum_{k=0}^{s} \left(\sum_{i+j=k} a_i b_j\right) \varphi^{(k)}(\alpha) = h(\varphi)(\alpha)$

$\therefore h(\varphi) = f(\varphi)\, g(\varphi)$

证毕

(2) 证明：$\forall \alpha \in \mathbb{R}^n$，有 $h(\varphi)(\alpha) = c_s \varphi^{(s)}(\alpha) + \cdots + c_1 \varphi^{(1)}(\alpha) + c_0 \alpha$

$\because h(x) = f(x) + g(x)$

$\therefore c_k = a_k' + b_k'$，其中 a_k', b_k' 定义为：

$$a_k' = \begin{cases} a_k & (m \geq n \text{ 或 } m < n, k \leq m) \\ 0 & (m < k \leq n) \end{cases}$$

$$b_k' = \begin{cases} b_k & (m \leq n \text{ 或 } m > n, k \leq n) \\ 0 & (n < k \leq m) \end{cases}$$

且：$s = \max\{k \mid a_k' + b_k' \neq 0\}$

故 $f(\varphi)(\alpha) + g(\varphi)(\alpha) = \sum_{k=0}^{m} a_k \varphi^{(k)}(\alpha) + \sum_{k=0}^{n} b_k \varphi^{(k)}(\alpha)$

$= \sum_{k=0}^{\max\{m,n\}} (a_k' + b_k')\varphi^{(k)}(\alpha)$

$= \sum_{k=0}^{s} (a_k' + b_k')\varphi^{(k)}(\alpha)$

$= h(\varphi)(\alpha)$

即 $\Rightarrow (f(\varphi) + g(\varphi))(\alpha) = h(\varphi)(\alpha)$

$\Rightarrow h(\varphi) = f(\varphi) + g(\varphi)$ （$\forall\varphi$）

证毕

第9周作业（高代A）

第一次（20211011）

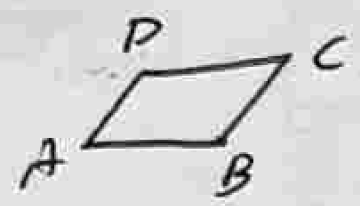
汪树晖

3019201080

转专业补修

A

1.1. 解：如图：



$\vec{AC}=\vec{AB}+\vec{BC}$

$\vec{BD}=\vec{BC}+\vec{CD}$

而 $\vec{CD}=-\vec{AB}$

∴ $\vec{BC}+\vec{AB}=\alpha$

$\vec{BC}-\vec{AB}=\beta$

$\Rightarrow \begin{cases}\vec{AB}=\dfrac{\alpha-\beta}{2}\\ \vec{BC}=\dfrac{\alpha+\beta}{2}\end{cases}$

1.2. 证明：$\lambda\vec{OA}+\mu\vec{OB}+\nu\vec{OC}+\omega\vec{OD}$

$=\lambda\vec{OA}+\mu\vec{OA}+\mu\vec{AB}+\nu\vec{OA}+\nu\vec{AC}+\omega\vec{OA}+\omega\vec{AD}$

$=(\lambda+\mu+\nu+\omega)\vec{OA}+\mu\vec{AB}+\nu\vec{AC}+\omega\vec{AD}$ (*)

又：A、B、C、D 共面 $\Leftrightarrow$ $\vec{AB}$、$\vec{AC}$、$\vec{AD}$ 共面

$\Leftrightarrow$ $\vec{AB},\vec{AC},\vec{AD}$ 线性相关 $\Leftrightarrow$ $\exists\underbrace{\lambda',\mu',\nu'}_{\text{不全为0的}}$，使 $\lambda'\vec{AB}+\mu'\vec{AC}+\nu'\vec{AD}=\vec{0}$

故：(1) 充分性：∵ λ,μ,ν,ω 不全为0 而 $\lambda+\mu+\nu+\omega=0$

∴ μ,ν,ω 不全为0，否则 $\lambda=0$，矛盾

由(*)：$\lambda\vec{OA}+\mu\vec{OB}+\nu\vec{OC}+\omega\vec{OD}=\vec{0}$

$\Leftrightarrow (\lambda+\mu+\nu+\omega)\vec{OA}+\mu\vec{AB}+\nu\vec{AC}+\omega\vec{AD}=\vec{0}$

$\xLeftrightarrow{\lambda+\mu+\nu+\omega=0}$ $\mu\vec{AB}+\nu\vec{AC}+\omega\vec{AD}=\vec{0}$，$\mu,\nu,\omega$ 不全为0

∴ A,B,C,D 共面

(2) 必要性：∵ A、B、C、D 共面

∴ ∃不全为0的 λ',μ',ν'，$\lambda'\vec{AB}+\mu'\vec{AC}+\nu'\vec{AD}=\vec{0}$

$\Leftrightarrow -(\lambda'+\mu'+\nu')\vec{OA}+\lambda'\vec{OB}+\mu'\vec{OC}+\nu'\vec{OD}=\vec{0}$

令 $\lambda''=-(\lambda'+\mu'+\nu')$

$\mu''=\lambda'$

$\nu''=\mu'$

$\omega''=\nu'$

由于 λ',μ',ν' 不全为0，故 $\lambda'',\mu'',\nu'',\omega''$ 不全为0

且 $\lambda''+\mu''+\nu''+\omega''=-(\lambda'+\mu'+\nu')+\lambda'+\mu'+\nu'=0$

∴ ∃不全为0的 $\lambda'',\mu'',\nu'',\omega''$ 且 $\lambda''+\mu''+\nu''+\omega''=0$

使 $\lambda''\vec{OA}+\mu''\vec{OB}+\nu''\vec{OC}+\omega''\vec{OD}=\vec{0}$

证毕

1.3. 证明：∵ O、A、B 不共线

∴ $\vec{AB}\neq\vec{0}$

∴ C、A、B共线 $\Leftrightarrow$ $\vec{AC}$、$\vec{AB}$ 线性相关 $\Leftrightarrow$ $\exists\lambda\in\mathbb{R}$，$\vec{AC}=\lambda\vec{AB}$

其中λ由C、A、B唯一确定 (*)

又：$\vec{OC}-(1-s)\vec{OA}-s\vec{OB}=\vec{OC}-(1-s)\vec{OC}-(1-s)\vec{CA}-s\vec{OC}-s\vec{CB}$

$=-(1-s)\vec{CA}-s\vec{CA}-s\vec{AB}$

$=\vec{AC}-s\vec{AB}$

设"$\exists s\in\mathbb{R}$ 使 $\vec{OC}=(1-s)\vec{OA}+s\vec{OB}$ 且 s 由A,B,C唯一确定，与O无关"为条件(**)

~~[illegible]~~

则(**) $\Leftrightarrow \begin{cases} \vec{OC}-(1-s)\vec{OA}-s\vec{OB}=\vec{0} \\ s由A,B,C唯一确定 \end{cases} \Leftrightarrow \begin{cases} \vec{AC}=s\vec{AB} \\ s由A,B,C唯一确定 \end{cases} \Leftrightarrow$ (*) $\Leftrightarrow$ A,B,C 共线

证毕

1.4. 证明：$\vec{OM}=\lambda\vec{OA}+\mu\vec{OB}+\nu\vec{OC} \Leftrightarrow \vec{OM}=\lambda\vec{OM}+\mu\vec{OM}+\nu\vec{OM}+\lambda\vec{MA}+\mu\vec{MB}+\nu\vec{MC}$

$\xLeftrightarrow{\lambda+\mu+\nu=1}$ $\lambda\vec{MA}+\mu\vec{MB}+\nu\vec{MC}=\vec{0}$ $\Leftrightarrow$ $\vec{MA}$、$\vec{MB}$、$\vec{MC}$ 线性相关

λ,μ,ν不全为0

$\Leftrightarrow$ $\vec{MA}$、$\vec{MB}$、$\vec{MC}$ ~~线性相关~~ 共面 $\Leftrightarrow$ M,A,B,C 共面

1.5 证明：点M在△ABC内部(包括边界，即三角形的闭包)

$\Leftrightarrow$ $\vec{CM}=\lambda'\vec{CA}+\mu'\vec{CB}$，其中$\begin{cases}\lambda'\ge0,\mu'\ge0\\ \lambda'+\mu'\le1\end{cases}$ 且$\lambda',\mu'\in\mathbb{R}$

$\xLeftrightarrow{\forall O}$ $\vec{CO}+\vec{OM}=\lambda'\vec{CO}+\lambda'\vec{OA}+\mu'\vec{CO}+\mu'\vec{OB}$，$\lambda',\mu'\in\mathbb{R}$，$\lambda'\ge0,\mu'\ge0,\lambda'+\mu'\le1$

$\xLeftrightarrow{\forall O}$ $\vec{OM}=\lambda'\vec{OA}+\mu'\vec{OB}+(1-\lambda'-\mu')\vec{OC}$，$\lambda',\mu'\in\mathbb{R}$，$\lambda'\ge0,\mu'\ge0,\lambda'+\mu'\le1$

$\xLeftrightarrow[1-\lambda'-\mu'=\nu]{\lambda'=\lambda,\ \mu'=\mu}$ $\forall O$ $\vec{OM}=\lambda\vec{OA}+\mu\vec{OB}+\nu\vec{OC}$，$\lambda,\mu,\nu\in\mathbb{R}$，$\lambda\ge0,\mu\ge0,\nu=1-(\lambda+\mu)\ge0$

证毕

1.6 (1) 证明：∵ α,β,γ 共面

∴ α,β,γ 线性相关

∴ ∃不全为0的$\lambda,\mu,\nu\in\mathbb{R}$，$\lambda\alpha+\mu\beta+\nu\gamma=0$

不妨$\lambda\ne0$，则$\alpha=-\frac{\mu}{\lambda}\beta-\frac{\nu}{\lambda}\gamma$，即$\alpha$可由$\beta,\gamma$线性表示

(2) 解：不是，下举反例：

α,β线性无关，$\gamma=-\alpha$

则 $\alpha+0\cdot\beta+\gamma=\alpha-\alpha=0$

但β不能由α,γ线性表出

否则设 $\beta=\lambda'\alpha+\mu'\gamma=(\lambda'-\mu')\alpha$

$\Leftrightarrow \beta+(\mu'-\lambda')\alpha=0$

与α,β线性无关矛盾

1.7. 证明：

(1) ①存在性：

设$\alpha=\vec{A_1A_2}+\vec{A_1A_3}+\cdots+\vec{A_1A_n}$

设$\beta=\frac{1}{n}\alpha$

则由β和A_1可确定一点M使 $\vec{A_1M}=\beta$

则 $\vec{MA_1}+\vec{MA_2}+\cdots+\vec{MA_n}$

$=-\beta+(\vec{MA_1}+\vec{A_1A_2})+\cdots+(\vec{MA_1}+\vec{A_1A_n})$

$=-\beta+(n-1)\cdot(-\beta)+(\vec{A_1A_2}+\cdots+\vec{A_1A_n})$

$=-n\beta+\alpha$

$=\vec{0}$

② 唯一性：设M,M'均满足 $\vec{PA_1}+\cdots+\vec{PA_n}=0$ (P=M,M')

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相减：$\sum_{k=1}^{n}(\vec{MA_k}-\vec{M'A_k})=\vec{0}$

$\Rightarrow \sum_{k=1}^{n}\vec{MM'}=\vec{0}$

$\Rightarrow n\vec{MM'}=\vec{0}$

$\Rightarrow \vec{MM'}=\vec{0}$

∴ $M=M'$

(2) $\forall O$，有：$\sum_{k=1}^{n}(\vec{MA_k})=(\sum_{k=1}^{n}\vec{MO}+\vec{OA_k})=\vec{0}$

$\Rightarrow n\vec{MO}+\sum_{k=1}^{n}\vec{OA_k}=\vec{0}$

$\Rightarrow \vec{OA_1}+\vec{OA_2}+\cdots+\vec{OA_n}=n\vec{OM}$

证毕

1.8. 证明：设E为P、Q中点，则 $\vec{PE}=\frac{1}{2}\vec{PQ}$，$\vec{QE}=\frac{1}{2}\vec{QP}$

则 $\alpha:=\vec{EA}+\vec{EB}+\vec{EC}+\vec{ED}$

$=\vec{EP}+\vec{PA}+\vec{EP}+\vec{PB}+\vec{EQ}+\vec{QC}+\vec{EQ}+\vec{QD}$

$=2(\vec{EP}+\vec{EQ})+\vec{PA}+\vec{PB}+\vec{QC}+\vec{QD}$

∵ P为AB中点，Q为CD中点

∴ $\vec{PA}=\frac{1}{2}\vec{BA}$，$\vec{PB}=\frac{1}{2}\vec{AB}$，故 $\vec{PA}+\vec{PB}=\vec{0}$

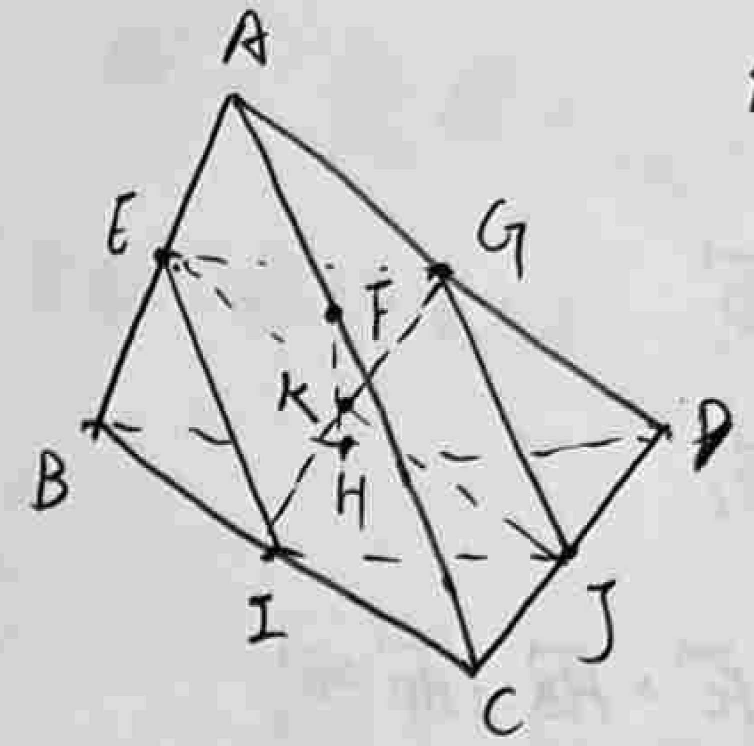
同理 $\vec{QC}+\vec{QD}=\vec{0}$，$\vec{EP}+\vec{EQ}=\vec{0}$

∴ $\alpha=2\cdot\vec{0}+\vec{0}+\vec{0}=\vec{0}$

即 $\vec{EA}+\vec{EB}+\vec{EC}+\vec{ED}=\vec{0}$

∴ E为A,B,C,D重心

1.9. 证明：如图：



设E、F、G、H、I、J分别是线段AB、AC、AD、BD、BC、CD的中点，线段EJ∩线段IG=K

故需证：(1) 线段FH∩线段EJ∩线段IG=K

(2) $\vec{KA}+\vec{KB}+\vec{KC}+\vec{KD}=\vec{0}$

先证一引理：若P_1为线段P_2P_3中点，则当且仅当$\forall Q$，有：$\vec{QP_1}=\frac{1}{2}(\vec{QP_2}+\vec{QP_3})$

证：P_1为P_2P_3中点 $\Leftrightarrow \vec{P_1P_2}=\frac{1}{2}\vec{P_3P_2} \Leftrightarrow \vec{P_1Q}+\vec{QP_2}=\frac{1}{2}\vec{P_3Q}+\frac{1}{2}\vec{QP_2}$

$\Leftrightarrow \vec{QP_1}=\frac{1}{2}\vec{QP_2}+\frac{1}{2}\vec{QP_3}$

故，(1)：由中位线定理：$\begin{cases}|\vec{EI}|=\frac{1}{2}|\vec{AC}| \\ EI // AC\end{cases}$

∴ $\vec{EI}=\frac{1}{2}\vec{AC}$

同理 $\vec{GJ}=\frac{1}{2}\vec{AC}$

∴ $\vec{EI}=\vec{GJ}$，故四边形EIJG为平行四边形

∴ K为线段EJ、IG的中点

要证(1)只需证 $K\in$ 线段FH，即F、K、H共线即可

由所设有：$\vec{AE}=\frac{1}{2}\vec{AB}$，$\vec{AJ}=\frac{1}{2}(\vec{AC}+\vec{AD})$

∴ $\vec{AK}=\frac{1}{2}(\vec{AE}+\vec{AJ})=\frac{1}{4}(\vec{AB}+\vec{AC}+\vec{AD})$

而 $\vec{AF}=\frac{1}{2}\vec{AC}$，$\vec{AH}=\frac{1}{2}(\vec{AB}+\vec{AD})$

∴ $\vec{AK}=\frac{1}{2}(\vec{AF}+\vec{AH})$

∴ K为FH中点

∴ K,F,H共线，证毕

(2) 由(1)已证：K是F、H中点，故 $\vec{KF}=\frac{1}{2}\vec{HF}$，$\vec{KH}=\frac{1}{2}\vec{FH}$，故 $\vec{KF}+\vec{KH}=\vec{0}$

∴ $\vec{KA}+\vec{KB}+\vec{KC}+\vec{KD}$

$=\vec{KF}+\vec{FA}+\vec{KH}+\vec{HB}+\vec{KF}+\vec{FC}+\vec{KH}+\vec{HD}$

$=2(\vec{KF}+\vec{KH})+(\vec{FA}+\vec{FC})+(\vec{HB}+\vec{HD})$

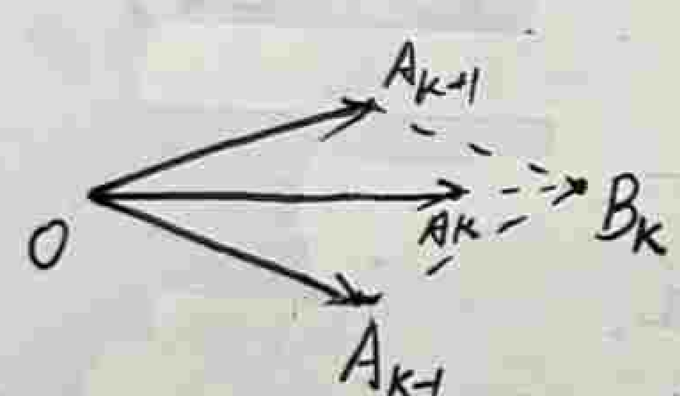
而F为AC中点，H为BD中点，故同理 $\vec{FA}+\vec{FC}=\vec{HB}+\vec{HD}=\vec{0}$

∴ $\vec{KA}+\vec{KB}+\vec{KC}+\vec{KD}=2\cdot\vec{0}+\vec{0}+\vec{0}=\vec{0}$

∴ K为A、B、C、D重心

1.10. 证明：设圆半径为r，由 $A_1, A_2 \cdots A_n$ 为n等分点 可知 $n=2$ 时命题显然成立

则 $n>2$ 时，如图：$\forall\, 2\le k\le n-1$



$\vec{OA_{k-1}}+\vec{OA_{k+1}}=\vec{OB_k}=\lambda\vec{OA_k}$

设 $\angle A_{k+1}OA_{k-1}=\theta_k$

显然 $\theta_k=\frac{2\pi}{n}\cdot 2=\frac{4\pi}{n}$

∴ 由余弦定理：$|\vec{OB_k}|=\sqrt{|\vec{OA_{k+1}}|^2+|\vec{OA_{k-1}}|^2-2|\vec{OA_k}||\vec{OA_{k-1}}|\cos(\pi-\theta_k)}$

$=\sqrt{r^2+r^2+2r^2\cos\frac{4\pi}{n}}=\sqrt{2}\,r\sqrt{1+\cos\frac{4\pi}{n}}$

∴ $|\lambda|=\frac{|\vec{OB_k}|}{|\vec{OA_k}|}=\sqrt{2}\sqrt{1+\cos\frac{4\pi}{n}}$

同样有 $\vec{OA_1}+\vec{OA_{n-1}}=\lambda\vec{OA_n}$

$\vec{OA_n}+\vec{OA_2}=\lambda\vec{OA_1}$

① 当 $n=3$ 时，$\theta_k>\pi$，$\lambda=-|\lambda|=-\sqrt{2}\cdot\sqrt{1+\cos\frac{4\pi}{3}}=-\sqrt{3}$

∴ $\vec{OA_1}+\vec{OA_2}+\vec{OA_3}=\vec{OA_3}+\vec{OA_1}+\vec{OA_2}$

$\Rightarrow 2(\vec{OA_1}+\vec{OA_2}+\vec{OA_3})=(\vec{OA_1}+\vec{OA_3})+(\vec{OA_2}+\vec{OA_1})+(\vec{OA_3}+\vec{OA_2})$

$=-\sqrt{3}(\vec{OA_2}+\vec{OA_3}+\vec{OA_1})$

$\Rightarrow (2+\sqrt{3})(\vec{OA_1}+\vec{OA_2}+\vec{OA_3})=\vec{0}$

$\Rightarrow \vec{OA_1}+\vec{OA_2}+\vec{OA_3}=\vec{0}$

② 当 $n\ge 4$ 时：记 $\alpha=\vec{OA_1}+\cdots+\vec{OA_n}$

则 $\lambda=|\lambda|=\sqrt{2}\sqrt{1+\cos\frac{4\pi}{n}}$

∴ $\alpha=\vec{OA_1}+\vec{OA_2}+\cdots+\vec{OA_n}$

$\alpha=\vec{OA_{n-1}}+\vec{OA_n}+\cdots+\vec{OA_{n-2}}$

$\Rightarrow \alpha+\alpha=\lambda\vec{OA_n}+\lambda\vec{OA_1}+\cdots+\lambda\vec{OA_{n-1}}$

$\Rightarrow 2\alpha=\lambda\alpha$

$\Rightarrow (2-\lambda)\alpha=\vec{0}$

∵ $n\ge 4 \Rightarrow 0<\frac{4\pi}{n}\le\pi$

∴ $\cos\frac{4\pi}{n}<1 \Rightarrow 1+\cos\frac{4\pi}{n}<2$

∴ $\sqrt{2}\sqrt{1+\cos\frac{4\pi}{n}}<2 \Rightarrow \lambda<2 \Rightarrow 2-\lambda>0$

∴ $(2-\lambda)\alpha=\vec{0} \Rightarrow \alpha=\vec{0} \Rightarrow$ 即 $\vec{OA_1}+\cdots+\vec{OA_n}=\vec{0}$

综上，命题得证

1.11. 证明：∵ f为双射

∴ $\{f(A_1), f(A_2), \cdots f(A_n)\}=\{B_1, B_2, \cdots B_n\}$

$\overrightarrow{A_1f(A_1)}+\overrightarrow{A_1f(A_2)}+\cdots+\overrightarrow{A_1f(A_n)}=\overrightarrow{A_1B_1}+\overrightarrow{A_2B_2}+\cdots+\overrightarrow{A_1B_n}$

而 $\overrightarrow{A_1f(A_1)}+\overrightarrow{A_2f(A_2)}+\cdots+\overrightarrow{A_nf(A_n)}$

$=\overrightarrow{A_1f(A_1)}+(\overrightarrow{A_2A_1}+\overrightarrow{A_1f(A_2)})+\cdots+(\overrightarrow{A_nA_1}+\overrightarrow{A_1f(A_n)})$

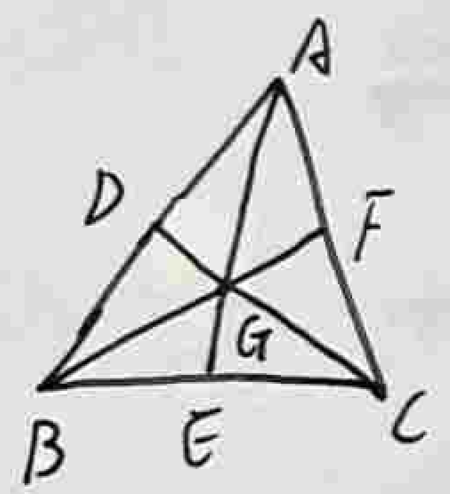
$=\overrightarrow{A_1f(A_1)}+\overrightarrow{A_1f(A_2)}+\cdots+\overrightarrow{A_1f(A_n)}-(\overrightarrow{A_1A_2}+\cdots+\overrightarrow{A_1A_n})$

$=\overrightarrow{A_1B_1}+\overrightarrow{A_2B_2}+\cdots\overrightarrow{A_1B_n}-(\overrightarrow{A_1A_2}+\cdots+\overrightarrow{A_1A_n})$

$=\overrightarrow{A_1B_1}+\overrightarrow{A_2B_2}+\cdots+\overrightarrow{A_nB_n}$

即该式结果与 f 无关

1.12. 证明：如图：



设 $BF\cap CD=G$，由题意 $\lambda,\mu,\nu\neq0$，故 $0<\lambda,\mu,\nu<1$

则 AE,BF,CD 交于一点 $\Leftrightarrow$ A,G,E 共线 $\Leftrightarrow$ $\overrightarrow{AG},\overrightarrow{AE}$ 线性相关

$\overset{\overrightarrow{AE}\neq\vec{0}}{\Longleftrightarrow}\exists s\in\mathbb{R},\ \overrightarrow{AG}=s\overrightarrow{AE}$ (*)

$\because \overrightarrow{BE}=\mu\overrightarrow{EC}\Leftrightarrow\overrightarrow{BA}+\overrightarrow{AE}=\mu\overrightarrow{EA}+\mu\overrightarrow{AC}$

$\Leftrightarrow\overrightarrow{AE}=\frac{1}{1+\mu}\overrightarrow{AB}+\frac{\mu}{1+\mu}\overrightarrow{AC}$

而 $\begin{cases}D,G,C\text{ 共线}\\ G\neq D,C\end{cases}\Leftrightarrow t_1\in\mathbb{R},t_1\neq0$ 使 $\overrightarrow{DG}=t_1\overrightarrow{GC}$

$\Leftrightarrow\overrightarrow{DA}+\overrightarrow{AG}=t_1(\overrightarrow{GA}+\overrightarrow{AC})$

$\Leftrightarrow\overrightarrow{AG}=\frac{1}{1+t_1}\overrightarrow{AD}+\frac{t_1}{1+t_1}\overrightarrow{AC}$

同理 $\exists t_2\in\mathbb{R},t_2\neq0$，$\overrightarrow{AG}=\frac{t_2}{1+t_2}\overrightarrow{AB}+\frac{1}{1+t_2}\overrightarrow{AF}$

又 $\begin{cases}\overrightarrow{AD}=\lambda\overrightarrow{DB}\\ \overrightarrow{CF}=\nu\overrightarrow{FA}\end{cases}\Rightarrow\begin{cases}\overrightarrow{AD}=\frac{\lambda}{1+\lambda}\overrightarrow{AB}\\ \overrightarrow{AF}=\frac{1}{1+\nu}\overrightarrow{AC}\end{cases}$

$\therefore\overrightarrow{AG}=\frac{1}{1+t_1}\frac{\lambda}{1+\lambda}\overrightarrow{AB}+\frac{t_1}{1+t_1}\overrightarrow{AC}$

$\overrightarrow{AG}=\frac{t_2}{1+t_2}\overrightarrow{AB}+\frac{1}{1+t_2}\frac{1}{1+\nu}\overrightarrow{AC}$

又 $\overrightarrow{AB},\overrightarrow{AC}$ 线性无关，故 $\begin{cases}\frac{1}{1+t_1}\frac{\lambda}{1+\lambda}=\frac{t_2}{1+t_2}\\ \frac{t_1}{1+t_1}=\frac{1}{1+t_2}\frac{1}{1+\nu}\end{cases}$ 解得：$\begin{cases}t_1=\frac{1}{(1+\lambda)\nu}\\ t_2=\frac{\lambda\nu}{1+\nu}\end{cases}$

$\therefore\overrightarrow{AG}=\frac{\lambda\nu}{1+\nu+\lambda\nu}\overrightarrow{AB}+\frac{1}{1+\nu+\lambda\nu}\overrightarrow{AC}$

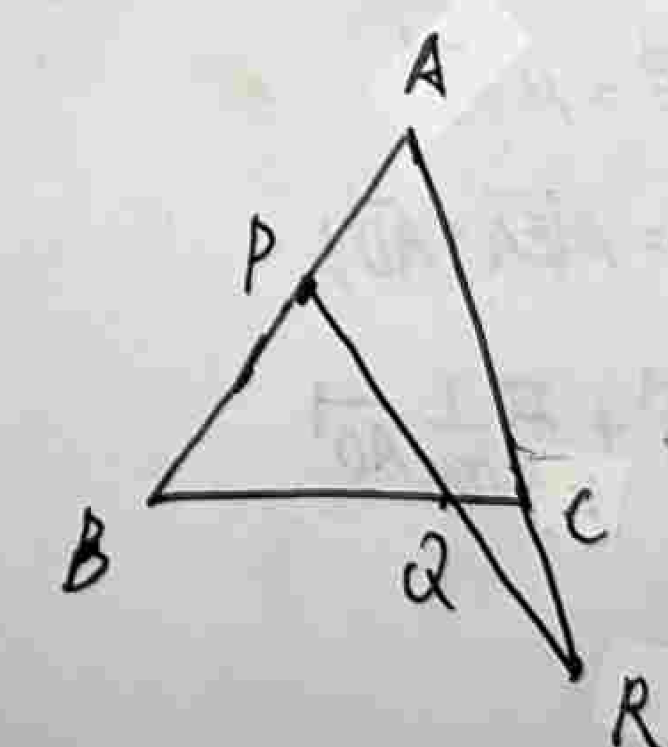
$\therefore(*)\Leftrightarrow\dfrac{\frac{\lambda\nu}{1+\nu+\lambda\nu}}{\frac{1}{1+\mu}}=\dfrac{\frac{1}{1+\nu+\lambda\nu}}{\frac{\mu}{1+\mu}}$

$\Leftrightarrow\lambda\nu=\frac{1}{\mu}$

$\Leftrightarrow\lambda\mu\nu=1$

即 AE,BF,CD 交于一点 $\Leftrightarrow\lambda\mu\nu=1$

1.13. 如图：



由题意：$\overrightarrow{PQ},\overrightarrow{PR},\overrightarrow{QR}\neq\vec{0}$，显然 A,P,R 不共线

$\therefore$ 由1.3已证：

P,Q,R 共线 $\Leftrightarrow\exists s\in\mathbb{R},\ \overrightarrow{AQ}=(1-s)\overrightarrow{AP}+s\overrightarrow{AR}$ (*)

而 $\begin{cases}\overrightarrow{AP}=\lambda\overrightarrow{PB}\\ \overrightarrow{CR}=\nu\overrightarrow{RA}\end{cases}\Leftrightarrow\begin{cases}\overrightarrow{AP}=\frac{\lambda}{1+\lambda}\overrightarrow{AB}\\ \overrightarrow{AR}=\frac{1}{1+\nu}\overrightarrow{AC}\end{cases}$

且 $\overrightarrow{BQ}=\mu\overrightarrow{QC}\Rightarrow\overrightarrow{BA}+\overrightarrow{AQ}=\mu\overrightarrow{QA}+\mu\overrightarrow{AC}$

$\Rightarrow\overrightarrow{AQ}=\frac{1}{1+\mu}\overrightarrow{AB}+\frac{\mu}{1+\mu}\overrightarrow{AC}$

∵ (*) ⟺ $\exists s\in\mathbb{R}$，$\vec{AQ} = (1-s)\frac{\lambda}{1+\lambda}\vec{AB} + s\cdot\frac{1}{1+\nu}\vec{AC}$

∵ $\vec{AB}$、$\vec{AC}$ 线性无关

∴ (*) ⟺ $\begin{cases}(1-s)\frac{\lambda}{1+\lambda} = \frac{1}{1+\mu}\\ s\cdot\frac{1}{1+\nu} = \frac{\mu}{1+\mu}\end{cases}$ 有解 ⟺ $\left(1-\frac{1+\lambda}{\lambda}\cdot\frac{1}{1+\mu}\right) = \frac{\mu}{1+\mu}\cdot(1+\nu)$

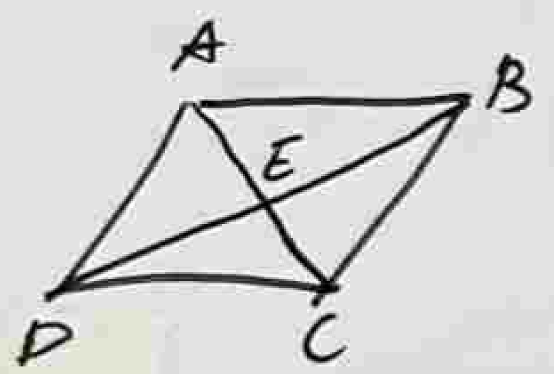
⟺ $\lambda(1+\mu)-(1+\lambda) = \mu\lambda(1+\nu)$

⟺ $\lambda\mu - 1 = \mu\lambda + \mu\lambda\nu$

⟺ $\lambda\mu\nu = -1$

即：P、Q、R 共线 ⟺ $\lambda\mu\nu = -1$

1.14. 证明：如图 □ABCD，设 $AC\cap BD = E$，则 $\vec{AE}$、$\vec{BE}$、$\vec{CE}$、$\vec{DE} \neq \vec{0}$



则 A、E、C 共线 ⇒ $\exists\lambda\in\mathbb{R}$，$\vec{AE} = \lambda\vec{EC}$

⇒ $\vec{AE} = \frac{\lambda}{1+\lambda}\vec{AC} = \frac{\lambda}{1+\lambda}\vec{AD} + \frac{\lambda}{1+\lambda}\vec{AB}$

B、D、E 共线 ⇒ $\exists\mu\in\mathbb{R}$，$\vec{BE} = \mu\vec{ED}$

⇒ $\vec{BA} + \vec{AE} = \mu(\vec{EA} + \vec{AD})$

⇒ $\vec{AE} = \frac{\mu}{1+\mu}\vec{AB} + \frac{1}{1+\mu}\vec{AD}$

∵ $\vec{AB}$、$\vec{AD}$ 线性无关

∴ $\begin{cases}\frac{\lambda}{1+\lambda} = \frac{\mu}{1+\mu}\\ \frac{\lambda}{1+\lambda} = \frac{1}{1+\mu}\end{cases}$ 解得 $\begin{cases}\lambda = 1\\ \mu = 1\end{cases}$

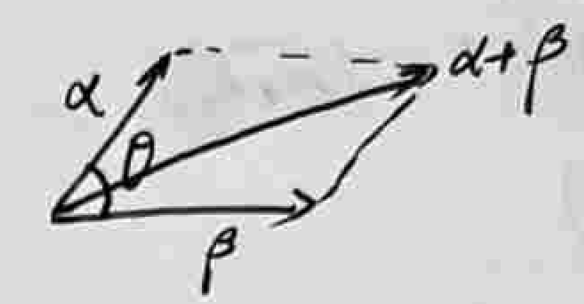
∴ $\vec{AE} = \frac{1}{2}\vec{AD} + \frac{1}{2}\vec{AB} = \frac{1}{2}(\vec{AD} + \vec{AB})$

由1.9已证：E为BD中点

同理E为AC中点

∴E平分AC、BD

1.15. 证明：如图：设α、β夹角为θ



由余弦定理：$|\alpha+\beta| = \sqrt{|\alpha|^2+|\beta|^2-2|\alpha||\beta|\cos(\pi-\theta)}$

$= \sqrt{|\alpha|^2+|\beta|^2+2|\alpha|\cdot|\beta|\cos\theta}$

$\leq \sqrt{|\alpha|^2+|\beta|^2+2|\alpha|\cdot|\beta|}$

$= \sqrt{(|\alpha|+|\beta|)^2}$

$= |\alpha|+|\beta|$

由推导知"="成立当且仅当 $\cos\theta = 1$

即 $\theta = 0$，即α、β同向

1.16. 证明：由1.15，$|\alpha+\beta|^2 = |\alpha|^2+|\beta|^2+2|\alpha||\beta|\cos\theta$ ①

同理 $|\alpha-\beta|^2 = |\alpha|^2+|\beta|^2-2|\alpha||\beta|\cos\theta$ ②

∴ ①+② 即有：$2|\alpha|^2+2|\beta|^2 = |\alpha+\beta|^2+|\alpha-\beta|^2$

对应几何定理为平行四边形四边对角线平方和定理

第10周作业(高代A)
第一次(2021 1020)

A

汪栩平
3019201080
转专业补修

2.1 证明: ~~[illegible]~~ $(\alpha+\beta)\cdot(\alpha+\beta)$

$= \alpha\cdot(\alpha+\beta)+\beta\cdot(\alpha+\beta)$

$=\alpha\cdot\alpha+\alpha\cdot\beta+\beta\cdot\alpha+\beta\cdot\beta$

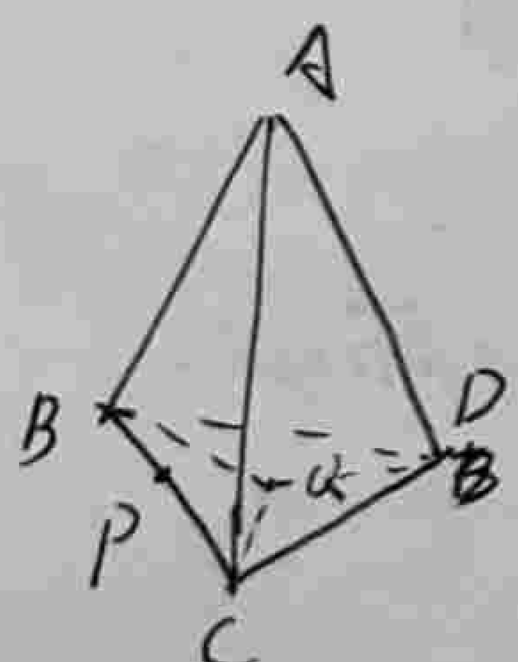
由对称性:

$(\alpha+\beta)\cdot(\alpha+\beta)=\alpha\cdot\alpha+2(\alpha\cdot\beta)+\beta\cdot\beta$

$\Rightarrow |\alpha+\beta|^2=|\alpha|^2+2(\alpha\cdot\beta)+|\beta|^2$

$\Rightarrow \alpha\cdot\beta=\frac{1}{2}(|\alpha+\beta|^2-|\alpha|^2-|\beta|^2)$

2.2. 解: 如示意图,



由重心定义: $\vec{QB}+\vec{QC}+\vec{QD}=\vec{0}$ ①

由P为中点: $2\vec{BP}=\vec{BC}$ ②

① $\Leftrightarrow$ $\vec{QA}+\vec{AB}+\vec{QA}+\vec{AC}+\vec{QA}+\vec{AD}=\vec{0}$

$\Leftrightarrow \vec{AQ}=\frac{1}{3}(\vec{AB}+\vec{AC}+\vec{AD})$

② $\Leftrightarrow 2\vec{BA}+2\vec{AP}=\vec{BA}+\vec{AC}$

$\Leftrightarrow \vec{AP}=\frac{1}{2}(\vec{AB}+\vec{AC})$

由定义: $\vec{AB}\cdot\vec{AC}=|AB|\cdot|AC|\cos\angle BAC=2\times2\times\cos\frac{\pi}{3}=2$

$\vec{AB}\cdot\vec{AD}=|AB|\cdot|AD|\cdot\cos\angle BAD=2\times1\times\cos\frac{\pi}{3}=1$

$\vec{AC}\cdot\vec{AD}=|AC|\cdot|AD|\cdot\cos\angle CAD=2\times1\times\cos\frac{\pi}{6}=\sqrt{3}$

且 $\vec{AB}\cdot\vec{AB}=|\vec{AB}|^2=4$, $\vec{AC}\cdot\vec{AC}=|\vec{AC}|^2=4$, $\vec{AD}\cdot\vec{AD}=|\vec{AD}|^2=1$

$\therefore \vec{AQ}\cdot\vec{AQ}=\frac{1}{9}(\vec{AB}\cdot\vec{AB}+\vec{AC}\cdot\vec{AC}+\vec{AD}\cdot\vec{AD}+2\vec{AB}\cdot\vec{AC}+2\vec{AB}\cdot\vec{AD}+2\vec{AC}\cdot\vec{AD})$

$=\frac{1}{9}[4+4+1+2\times(2+1+\sqrt{3})]$

$=\frac{5}{3}+\frac{2}{9\sqrt{3}}$

$\vec{AP}\cdot\vec{AP}=\frac{1}{4}(\vec{AB}\cdot\vec{AB}+2\vec{AB}\cdot\vec{AC}+\vec{AC}\cdot\vec{AC})=\frac{1}{4}\times(4+2\times2+4)=3$

$\vec{AP}\cdot\vec{AQ}=\frac{1}{6}(\vec{AB}\cdot\vec{AB}+\vec{AC}\cdot\vec{AC}+2\vec{AB}\cdot\vec{AC}+\vec{AB}\cdot\vec{AD}+\vec{AC}\cdot\vec{AD})$

$=\frac{1}{6}\times(4+4+2\times2+1+\sqrt{3})$

$=\frac{13+\sqrt{3}}{6}$

$\therefore |\vec{AQ}|=\sqrt{\vec{AQ}\cdot\vec{AQ}}=\frac{\sqrt{15+2\sqrt{3}}}{3}$

$|\vec{AP}|=\sqrt{\vec{AP}\cdot\vec{AP}}=\sqrt{3}$

$\vec{AP}\cdot\vec{AQ}=\frac{13+\sqrt{3}}{6}$

2.3. 证明: $\Rightarrow$): 显然可证

$\Leftarrow$): $\because$ α, β, γ不共面, 即其线性无关

$\therefore$ δ可由α、β、γ线性表示且表示方法唯一

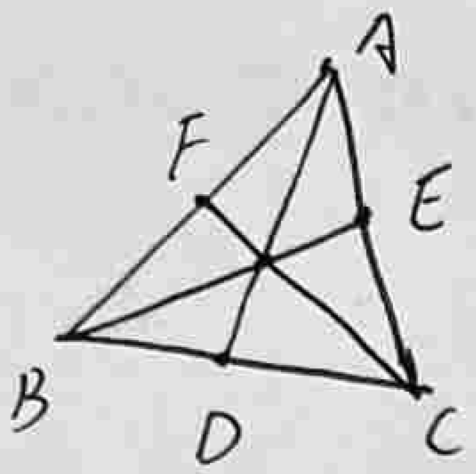
即 $\delta=a\alpha+b\beta+c\gamma$, 其中a, b, c由δ唯一确定

$\therefore \delta\cdot\delta=\delta\cdot(a\alpha+b\beta+c\gamma)=a(\delta\cdot\alpha)+b(\delta\cdot\beta)+c(\delta\cdot\gamma)$

$=a\times0+b\times0+c\times0=0$

$\therefore \delta=\vec{0}$

2.4. 证明：如示意图：



AB、BC、CA中点分别设为 F、D、E

则同2.3可推出：

$\vec{AD}=\frac{1}{2}(\vec{AB}+\vec{AC})$

$\vec{BE}=\frac{1}{2}(\vec{BA}+\vec{BC})$

$\vec{CF}=\frac{1}{2}(\vec{CA}+\vec{CB})$

$\therefore \vec{AD}\cdot\vec{AD}=\frac{1}{4}(\vec{AB}\cdot\vec{AB}+2\vec{AB}\cdot\vec{AC}+\vec{AC}\cdot\vec{AC})$

$\vec{BE}\cdot\vec{BE}=\frac{1}{4}(\vec{BA}\cdot\vec{BA}+2\vec{BA}\cdot\vec{BC}+\vec{BC}\cdot\vec{BC})$

$\vec{CF}\cdot\vec{CF}=\frac{1}{4}(\vec{CA}\cdot\vec{CA}+2\vec{CA}\cdot\vec{CB}+\vec{CB}\cdot\vec{CB})$

三式相加并注意到： $\vec{AB}\cdot\vec{AB}=\vec{BA}\cdot\vec{BA}$

$\vec{AC}\cdot\vec{AC}=\vec{CA}\cdot\vec{CA}$

$\vec{BC}\cdot\vec{BC}=\vec{CB}\cdot\vec{CB}$

$\vec{AB}\cdot\vec{AC}+\vec{BA}\cdot\vec{BC}=\vec{AB}\cdot(\vec{AC}-\vec{BC})=\vec{AB}\cdot\vec{AB}$

$\vec{AB}\cdot\vec{AC}+\vec{CA}\cdot\vec{CB}=(\vec{AB}-\vec{CB})\cdot\vec{AC}=\vec{AC}\cdot\vec{AC}$

$\vec{BA}\cdot\vec{BC}+\vec{CA}\cdot\vec{CB}=(\vec{BA}-\vec{CA})\cdot\vec{BC}=\vec{BC}\cdot\vec{BC}$

可得：$\vec{AD}\cdot\vec{AD}+\vec{BE}\cdot\vec{BE}+\vec{CF}\cdot\vec{CF}=\frac{3}{4}(\vec{AB}\cdot\vec{AB}+\vec{AC}\cdot\vec{AC}+\vec{BC}\cdot\vec{BC})$

即：$|\vec{AD}|^2+|\vec{BE}|^2+|\vec{CF}|^2=\frac{3}{4}(|\vec{AB}|^2+|\vec{AC}|^2+|\vec{BC}|^2)$

证毕

2.5. 证明：(1) $\vec{AB}\cdot\vec{CD}=\vec{AB}\cdot(\vec{CA}+\vec{AD})=-\vec{AB}\cdot\vec{AC}+\vec{AB}\cdot\vec{AD}$

$\vec{BC}\cdot\vec{AD}=(\vec{BA}+\vec{AC})\cdot\vec{AD}=-\vec{AB}\cdot\vec{AD}+\vec{AC}\cdot\vec{AD}$

$\vec{CA}\cdot\vec{BD}=-\vec{AC}(\vec{BA}+\vec{AD})=\vec{AB}\cdot\vec{AC}-\vec{AC}\cdot\vec{AD}$

叠加即得：$\vec{AB}\cdot\vec{CD}+\vec{BC}\cdot\vec{AD}+\vec{CA}\cdot\vec{BD}=0$，证毕

(2). $\because \vec{AB}^2+\vec{CD}^2-\vec{AC}^2-\vec{BD}^2$

$=\vec{AB}\cdot\vec{AB}+\vec{CD}\cdot\vec{CD}-\vec{AC}\cdot\vec{AC}-\vec{BD}\cdot\vec{BD}+0\times 2$

由(1)已证 $\vec{AB}\cdot\vec{AB}+\vec{CD}\cdot\vec{CD}-\vec{AC}\cdot\vec{AC}-\vec{BD}\cdot\vec{BD}+2\vec{AB}\cdot\vec{CD}+2\vec{BC}\cdot\vec{AD}+2\vec{CA}\cdot\vec{BD}$

$=(\vec{AB}+\vec{CD})\cdot(\vec{AB}+\vec{CD})-(\vec{AC}+\vec{BD})\cdot(\vec{AC}+\vec{BD})+2\vec{AD}\cdot\vec{BC}$

而 $(\vec{AC}+\vec{BD})-(\vec{AB}+\vec{CD})=(\vec{AC}-\vec{AB})+(\vec{BD}-\vec{CD})=2\vec{BC}$

$(\vec{AC}+\vec{BD})+(\vec{AB}+\vec{CD})=(\vec{AC}+\vec{CD})+(\vec{AB}+\vec{BD})=2\vec{AD}$

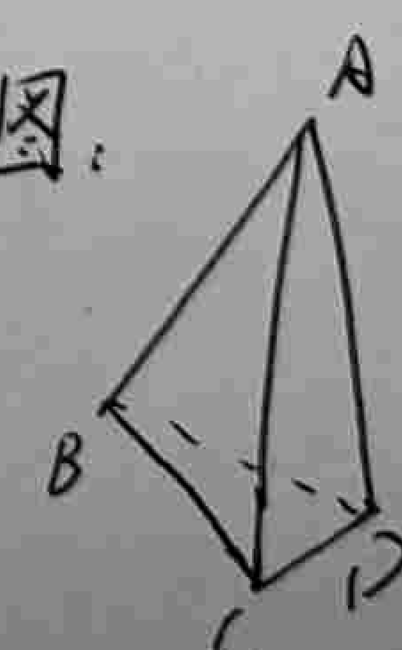
∴两式相乘得：$(\vec{AC}+\vec{BD})\cdot(\vec{AC}+\vec{BD})-(\vec{AB}+\vec{CD})\cdot(\vec{AB}+\vec{CD})=4\vec{AD}\cdot\vec{BC}$

$\therefore \vec{AB}^2+\vec{CD}^2-\vec{AC}^2-\vec{BD}^2=-4\vec{AD}\cdot\vec{BC}+2\vec{AD}\cdot\vec{BC}$

$=-2\vec{AD}\cdot\vec{BC}$

$\therefore \vec{AB}^2+\vec{CD}^2=\vec{AC}^2+\vec{BD}^2 \Leftrightarrow \vec{AB}^2+\vec{CD}^2-\vec{AC}^2-\vec{BD}^2=0 \Leftrightarrow \vec{AD}\cdot\vec{BC}=0$

(3) 如示意图：



不妨设 AB⊥CD，AD⊥BC

同(2)有：

$\vec{AC}^2+\vec{BD}^2=\vec{AD}^2+\vec{BC}^2 \Leftrightarrow \vec{AB}\cdot\vec{CD}=0$

$\vec{AD}^2+\vec{BC}^2=\vec{AB}^2+\vec{CD}^2 \Leftrightarrow \vec{AC}\cdot\vec{BD}=0$

$AB\perp CD$, $AD\perp BC$

∴ $\vec{AB}\cdot\vec{CD}=0$，$\vec{AD}\cdot\vec{BC}=0=\vec{BC}\cdot\vec{AD}$

∴由(1)已证，有：$\vec{CA}\cdot\vec{BD}=0$，即$\vec{CA}\cdot\vec{BD}$

由(1)已证及上推论：$\vec{AC}^2+\vec{BD}^2=\vec{AD}^2+\vec{BC}^2$

$\vec{AD}^2+\vec{BC}^2=\vec{AB}^2+\vec{CD}^2$

$\vec{AB}^2+\vec{CD}^2=\vec{AC}^2+\vec{BD}^2$

即 $\vec{AB}^2+\vec{CD}^2=\vec{AC}^2+\vec{BD}^2=\vec{AD}^2+\vec{BC}^2$

2.6.证明：(1) 设O为其中心

$\vec{A_1A_2}=\vec{A_1O}+\vec{OA_2}=-\vec{OA_1}+\vec{OA_2}$

$\vec{A_2A_3}=\vec{A_2O}+\vec{OA_3}=-\vec{OA_2}+\vec{OA_3}$

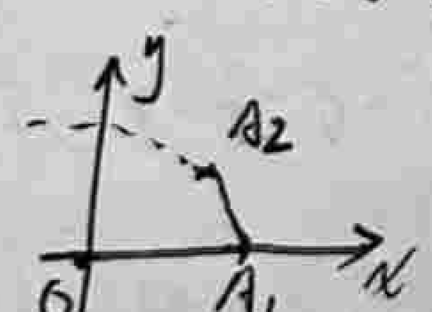
$\vdots$

$\vec{A_{n-1}A_n}=\vec{A_{n-1}O}+\vec{OA_n}=-\vec{OA_{n-1}}+\vec{OA_n}$

$\vec{A_nA_1}=\vec{A_nO}+\vec{OA_1}=-\vec{OA_n}+\vec{OA_1}$

叠加：$\vec{A_1A_2}+\vec{A_2A_3}+\cdots+\vec{A_nA_1}=-(\vec{OA_1}+\cdots+\vec{OA_n})+(\vec{OA_1}+\cdots+\vec{OA_n})=\vec{0}$

(2) 如图(局部)：以$\vec{OA_1}$（O为正多边形中心）为x轴单位方向向量建立右手笛卡尔坐标系



设$\vec{OA_1}$绕O逆时针旋转θ_k角后与$\vec{OA_k}$重合

则由正多边形性质：$\theta_k=0+(k-1)\frac{2\pi}{n}=\frac{k-1}{n}2\pi$

∴ $\vec{OA_k}=(\cos\theta_k)\vec{e_x}+(\sin\theta_k)\vec{e_y}$

$\vec{e_x}$、$\vec{e_y}$为x、y轴的单位方向向量

由1.10已证：$\vec{OA_1}+\cdots+\vec{OA_n}=\vec{0}$

∴ $\sum_{k=1}^{n}\cos\theta_k=\sum_{k=1}^{n}\sin\theta_k=0$

$\Rightarrow\begin{cases}\sum_{k=1}^{n}\cos\frac{2k-2}{n}\pi=0\\ \sum_{k=1}^{n}\sin\frac{2k-2}{n}\pi=0\end{cases}\Rightarrow\begin{cases}1+\cos\frac{2\pi}{n}+\cdots+\cos\frac{(2n-2)\pi}{n}=0\\ \sin\frac{2\pi}{n}+\cdots+\sin\frac{(2n-2)\pi}{n}=0\end{cases}$

2.7.证明：∵ $\alpha,\beta\perp\gamma$

∴ $\alpha,\beta,\gamma\neq\vec{0}$

① 若$\alpha\parallel\beta$，则$\alpha\times\beta=\vec{0}$

∴ $(\alpha\times\beta)\times\gamma=\vec{0}\times\gamma=\vec{0}$

② 若$\alpha\nparallel\beta$，则α、β、$(\alpha\times\beta)$线性无关（下证之：

设 $a\alpha+b\beta+c(\alpha\times\beta)=\vec{0}$，由于$\alpha\nparallel\beta$，$\alpha\neq\vec{0}$，$\beta\neq\vec{0}$，故$\alpha\times\beta\neq\vec{0}$

两边用$\alpha\times\beta$去乘点，注意到 $\alpha\cdot(\alpha\times\beta)=\beta\cdot(\alpha\times\beta)=0$

$c(\alpha\times\beta)\cdot(\alpha\times\beta)=0$

∴ $\alpha\times\beta\neq\vec{0}$ ∴ $(\alpha\times\beta)\cdot(\alpha\times\beta)>0$

∴ $c=0$

∴ $a\alpha+b\beta=\vec{0}$，又$\alpha\nparallel\beta$，故α、β线性无关，故$a,b=0$

∴ $a,b,c=0$，故$\alpha,\beta,(\alpha\times\beta)$线性无关）.

∴ γ可由$\alpha,\beta,\alpha\times\beta$线性表出且表示方式唯一

设 $\gamma=x_1\alpha+x_2\beta+x_3(\alpha\times\beta)$

则由于$\gamma\perp\alpha$、$\gamma\perp\beta$，故$\gamma\cdot\alpha=0$，$\gamma\cdot\beta=0$

2.4. 证明：构造参阅

$\therefore \begin{cases} x_1\,\alpha\cdot\alpha + x_2\,\alpha\cdot\beta = 0 \\ x_1\,\alpha\cdot\beta + x_2\,\beta\cdot\beta = 0 \end{cases} \Rightarrow \begin{cases} [(\alpha\cdot\alpha)(\beta\cdot\beta) - (\alpha\cdot\beta)^2]\,x_1 = 0 \\ [(\alpha\cdot\alpha)(\beta\cdot\beta) - (\alpha\cdot\beta)^2]\,x_2 = 0 \end{cases}$

考虑二次函数 $f(x) = (\alpha + x\beta)(\alpha + x\beta)$

$= (\alpha\cdot\alpha) + 2x(\alpha\cdot\beta) + x^2(\beta\cdot\beta)$

$f(x) \geq 0$，故 $\Delta = 4(\alpha\cdot\beta)^2 - 4(\alpha\cdot\alpha)(\beta\cdot\beta) \leq 0$

$\Rightarrow (\alpha\cdot\alpha)(\beta\cdot\beta) - (\alpha\cdot\beta)^2 \geq 0$

且"="成立 $\Leftrightarrow f(x) = 0 \Leftrightarrow \alpha + x\beta = \vec{0}$

但 α 与 β 线性无关，故"="不成立，即 $(\alpha\cdot\alpha)(\beta\cdot\beta) - (\alpha\cdot\beta)^2 > 0$

$\therefore x_1 = x_2 = 0$，故 $\gamma = c(\alpha\times\beta)$

$\therefore (\alpha\times\beta)\times\gamma = c[(\alpha\times\beta)\times(\alpha\times\beta)] = c\cdot\vec{0} = \vec{0}$

2.8. 证明：由题意 $\beta, \gamma \neq \vec{0}$，而 $\alpha = k\gamma$

① 若 $\alpha = \vec{0}$，则 $(\alpha\times\beta)\times\gamma = \vec{0}\times\gamma = \vec{0}$

$(\alpha\cdot\gamma)\cdot\beta = 0\cdot\beta = \vec{0}$

$\therefore (\alpha\times\beta)\times\gamma = (\alpha\cdot\gamma)\beta$

② 若 $\alpha \neq \vec{0}$，则 $\alpha = k\gamma$，$k \in \mathbb{R}$ 且 $k \neq 0$

则可设 $\delta = (\gamma\times\beta)\times\gamma$

则 $\delta \perp (\gamma\times\beta)$，$\delta \perp \gamma$，又 $(\gamma\times\beta) \perp \gamma$

$\therefore$ $(\gamma\times\beta)$，γ，δ 构成一右手正交系

$\therefore (\gamma\times\beta)$，$\gamma$，$\delta$ 线性无关，故 β 可由其线性表示且方式唯一

设 $\beta = a(\gamma\times\beta) + b\gamma + c\delta$，$a, b, c \in \mathbb{R}$

由于 $(\gamma\times\beta) \perp \beta$，$\underbrace{}_{(\gamma\times\beta)\perp\delta}$ 故两边点乘 $(\gamma\times\beta)$ 得：

$0 = a(\gamma\times\beta)^2$

而 $\gamma\times\beta \neq \vec{0}$，故 $a = 0$

$\therefore \beta = b\gamma + c\delta$，而 $\beta \perp \gamma$，且 $\delta = [(\gamma\times\beta)\times\gamma] \perp \gamma$

$\therefore$ 两边点乘 γ 得 $b\gamma^2 = 0$，同理 $b = 0$

$\therefore \beta = c\delta$，而分析知 β 与 δ 同向，故 $c > 0$

$\therefore c = \dfrac{|\beta|}{|\delta|}$，而 $|\delta| = |\gamma\times\beta|\cdot|\gamma| = |\gamma|^2\cdot|\beta| \Rightarrow c = \dfrac{1}{|\gamma|^2}$

$\therefore \beta = \dfrac{1}{|\gamma|^2}\delta \Rightarrow k|\gamma|^2\beta = k\delta = (k\gamma\times\beta)\times\gamma = (\alpha\times\beta)\times\gamma$

$\Rightarrow (k\gamma)\gamma\,\beta = (\alpha\times\beta)\times\gamma \Rightarrow (\alpha\cdot\gamma)\beta = (\alpha\times\beta)\times\gamma$

$(\hat{\alpha}\cdot\beta)\hat{\alpha}$

2.9. 证明：设 β 在 α 上的投影为 β'，令 $\beta - \beta' = \bar{\beta}$，则 $\bar{\beta}\cdot\alpha = 0$

令 $\beta' = \langle\alpha\cdot\beta\rangle\hat{\alpha}$

设 γ 在 α、$\bar{\beta}$ 上的投影分别为 γ'、γ''，令 $\bar{\gamma} = \gamma - \gamma' - \gamma''$；

则 $\bar{\beta} = \beta - \beta'$

（注：若 α，$\bar{\beta}$ 中某個为 $\vec{0}$，则规定其它向量在它上面的投影也为 $\vec{0}$）

(更常用这样的表示)

则 $\gamma\cdot\alpha = \gamma'\cdot\alpha$，$\gamma\cdot\bar{\beta} = \gamma''\cdot\bar{\beta}$，$\gamma''\cdot\alpha = \gamma'\cdot\bar{\beta} = 0$

$\therefore \bar{\gamma}\cdot\alpha = \gamma\cdot\alpha - \gamma'\cdot\alpha - \gamma''\cdot\alpha = 0$

$\bar{\gamma}\cdot\beta = \gamma\cdot\beta - \gamma'\cdot\beta - \gamma''\cdot\beta = 0$

$\therefore (\alpha\times\beta)\times\gamma = [\alpha\times(\beta' + \bar{\beta})]\times\gamma = (\alpha\times\beta')\times\gamma + (\alpha\times\bar{\beta})\times\gamma$

$(\alpha\times\beta)\times\gamma=\vec{0}\times\gamma=\vec{0}$

$(\alpha\times\bar{\beta})\times\gamma=(\alpha\times\bar{\beta})\times(\gamma'+\gamma''+\bar{\gamma})$

$=(\alpha\times\bar{\beta})\times\gamma'+(\alpha\times\bar{\beta})\times\gamma''+(\alpha\times\bar{\beta})\times\bar{\gamma}$

$=(\alpha\times\bar{\beta})\times\gamma'-(\bar{\beta}\times\alpha)\times\gamma''+(\alpha\times\bar{\beta})\times\bar{\gamma}$

由2.7已证：$\alpha\cdot\bar{\gamma}=0$，$\bar{\beta}\cdot\bar{\gamma}=0$

故$(\alpha\times\bar{\beta})\times\bar{\gamma}=\vec{0}$

由2.8已证：$\gamma'//\alpha$，$\gamma'\cdot\bar{\beta}=0$，故$(\alpha\times\bar{\beta})\times\gamma'=(\alpha\cdot\gamma')\bar{\beta}=(\alpha\cdot\gamma)\bar{\beta}$

且 $\gamma''//\bar{\beta}$，$\gamma''\cdot\alpha=0$，故 $(\bar{\beta}\times\alpha)\times\gamma''=(\bar{\beta}\cdot\gamma'')\alpha=(\bar{\beta}\cdot\gamma)\alpha$

∴ $(\alpha\times\bar{\beta})\times\gamma=(\alpha\cdot\gamma)\bar{\beta}-(\bar{\beta}\cdot\gamma)\alpha$ (*)，代入 $\bar{\beta}=\beta-\beta'$：

$(\alpha\cdot\gamma)\bar{\beta}=(\alpha\cdot\gamma)(\beta-\beta')=(\alpha\cdot\gamma)\beta-(\alpha\cdot\gamma)\beta'$ ①

$(\bar{\beta}\cdot\gamma)\alpha=(\beta-\beta',\gamma)\alpha=(\beta\cdot\gamma)\alpha-(\beta'\cdot\gamma)\alpha$ ②

而 $\alpha//\beta'$，故：$\alpha=\vec{0}$时 $(\beta'\cdot\gamma)\alpha-(\alpha\cdot\gamma)\beta'=\vec{0}$

$\alpha\neq\vec{0}$时 $\beta'=k\alpha$，$k\in\mathbb{R}$，

故 $(\beta'\cdot\gamma)\alpha-(\alpha\cdot\gamma)\beta'$

$=k(\alpha\cdot\gamma)\alpha-(\alpha\cdot\gamma)k\alpha$

$=\vec{0}$

∴ ①-②代入(*)得：$(\alpha\times\beta)\times\gamma=(\alpha\cdot\gamma)\beta-(\beta\cdot\gamma)\alpha$

证毕

且 $\vec{AB}\cdot\vec{AB}=|\vec{AB}|^2=4$，$\vec{AC}\cdot\vec{AC}=|\vec{AC}|^2=4$，$\vec{AD}\cdot\vec{AD}=|\vec{AD}|^2=1$

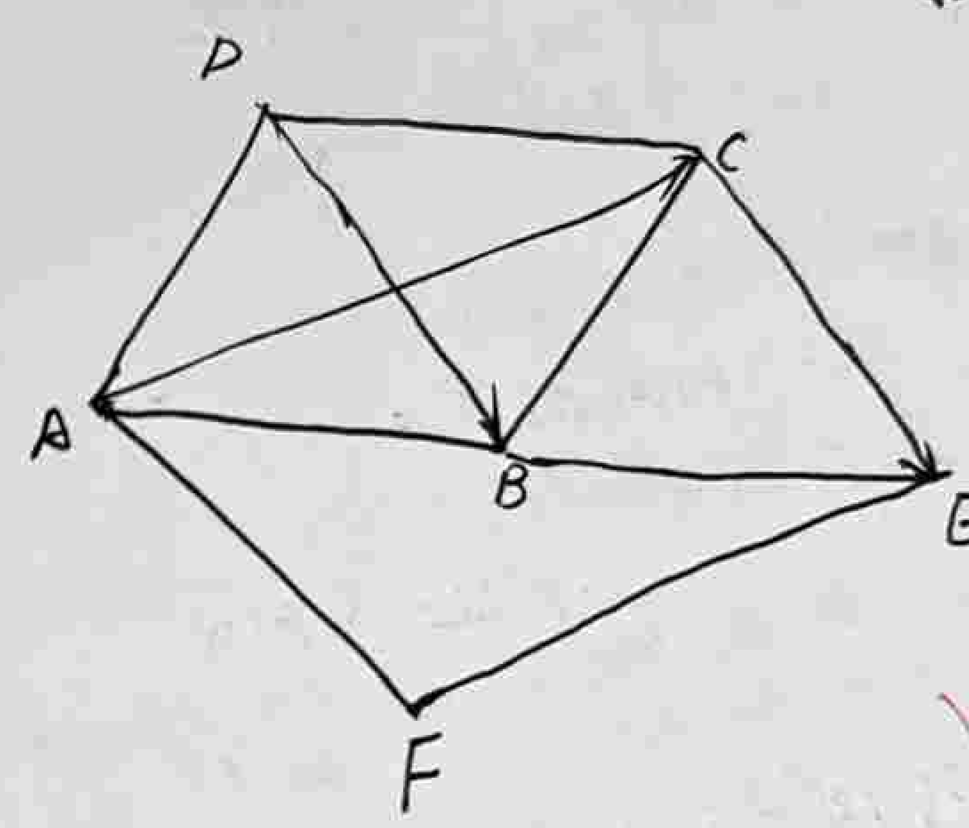
2.10. 证明：$(\alpha-\beta)\times(\alpha+\beta)=\alpha\times\alpha-\beta\times\alpha+\alpha\times\beta-\beta\times\beta$

$=\vec{0}+\alpha\times\beta+\alpha\times\beta-\vec{0}$

$=2(\alpha\times\beta)$

几何意义：如示意图 ▱ABCD，$\vec{AB}=\alpha$，$\vec{AD}=\beta$

则 $\vec{AC}=\alpha+\beta$，$\vec{DB}=\alpha-\beta$



再作 $\vec{CE}=\vec{DB}$

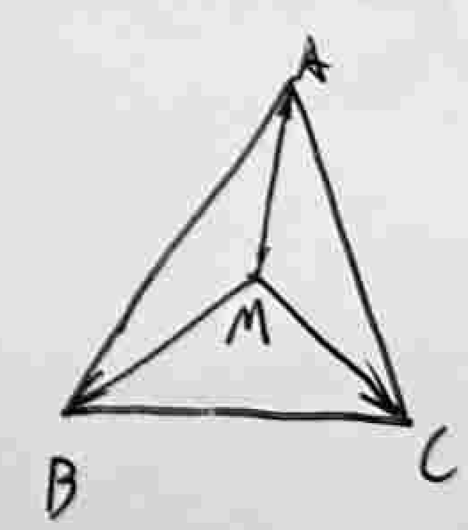
以 AC、DE 为邻边作平行四边形

则：① $(\alpha-\beta)\times(\alpha+\beta)$ 与 $(\alpha\times\beta)$ 方向相同

② $S_{▱ACEF}=2S_{▱ABCD}$

(S 为面积)

2.11. 证明：如示意图，$\vec{MA}+\vec{MB}+\vec{MC}=\vec{0}$



∴ $(\vec{MA}+\vec{MB}+\vec{MC})\times\vec{MA}=\vec{0}\times\vec{MA}=\vec{0}$

$\Rightarrow \vec{0}+\vec{MB}\times\vec{MA}+\vec{MC}\times\vec{MA}=\vec{0}$

$\Rightarrow \vec{MB}\times\vec{MA}=-\vec{MC}\times\vec{MA}$

$\Rightarrow |\vec{MB}\times\vec{MA}|=|\vec{MC}\times\vec{MA}|$

$\Rightarrow 2S_{\triangle MBA}=2S_{\triangle MAC}$

$\Rightarrow S_{\triangle MBA}=S_{\triangle MAC}$

同理 $S_{\triangle MBA}=S_{\triangle MBC}$，$S_{\triangle MBC}=S_{\triangle MAC}$

∴ $S_{\triangle MBA}=S_{\triangle MAC}=S_{\triangle MCB}$，证毕

2.12. 证明：设 e_1, e_2, e_3 为构成右手系的两两垂直单位向量

而 $\alpha = a_{11}e_1 + a_{12}e_2 + a_{13}e_3$

$\beta = a_{21}e_1 + a_{22}e_2 + a_{23}e_3$

$\gamma = a_{31}e_1 + a_{32}e_2 + a_{33}e_3$

$\therefore \alpha\cdot\beta = (a_{11}, a_{12}, a_{13})\begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix}$，其余同理

$\therefore$ 不妨记 $b_1 = (a_{11}\ a_{12}\ a_{13})$

$b_2 = (a_{21}\ a_{22}\ a_{23})$

$b_3 = (a_{31}\ a_{32}\ a_{33})$

设 $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$，$B = \begin{pmatrix} \alpha\cdot\alpha & \alpha\cdot\beta & \alpha\cdot\gamma \\ \beta\cdot\alpha & \beta\cdot\beta & \beta\cdot\gamma \\ \gamma\cdot\alpha & \gamma\cdot\beta & \gamma\cdot\gamma \end{pmatrix}$

则 $A A^T = \begin{pmatrix} \alpha\cdot\alpha & \alpha\cdot\beta & \alpha\cdot\gamma \\ \beta\cdot\alpha & \beta\cdot\beta & \beta\cdot\gamma \\ \gamma\cdot\alpha & \gamma\cdot\beta & \gamma\cdot\gamma \end{pmatrix} = B$

$\therefore |A|\cdot|A^T| = |B|$

又 $|A^T| = |A|$

$\therefore |B| = |A|^2$

$\therefore \alpha, \beta, \gamma$ 共面 $\Leftrightarrow |A| = 0 \Leftrightarrow |A|^2 = 0 \Leftrightarrow |B| = 0$

2.13. 证明：$(\alpha_1\times\alpha_2)\cdot(\alpha_3\times\alpha_4) = [(\alpha_3\times\alpha_4)\times\alpha_1]\cdot\alpha_2$

由2.9已证：$(\alpha_3\times\alpha_4)\times\alpha_1 = (\alpha_3\cdot\alpha_1)\alpha_4 - (\alpha_4\cdot\alpha_1)\alpha_3$

$\therefore (\alpha_1\times\alpha_2)\cdot(\alpha_3\times\alpha_4) = (\alpha_3\cdot\alpha_1)(\alpha_4\cdot\alpha_2) - (\alpha_4\cdot\alpha_1)(\alpha_3\cdot\alpha_2)$

亦即 $(\alpha_1\times\alpha_2)\cdot(\alpha_3\times\alpha_4) = \begin{vmatrix} \alpha_1\cdot\alpha_3 & \alpha_2\cdot\alpha_3 \\ \alpha_1\cdot\alpha_4 & \alpha_2\cdot\alpha_4 \end{vmatrix}$

2.14. 证明：题意即证：$(\frac{1}{2}|\vec{OA}||\vec{OB}|)^2 + (\frac{1}{2}|\vec{OB}||\vec{OC}|)^2 + (\frac{1}{2}|\vec{OC}||\vec{OA}|)^2 = (\frac{1}{2}|\vec{AB}\times\vec{AC}|)^2$

即：$(\vec{OA}\times\vec{OB})\cdot(\vec{OA}\times\vec{OB}) + (\vec{OB}\times\vec{OC})\cdot(\vec{OB}\times\vec{OC}) + (\vec{OC}\times\vec{OA})\cdot(\vec{OC}\times\vec{OA}) = (\vec{AB}\times\vec{AC})\cdot(\vec{AB}\times\vec{AC})$ (*)

记 $\vec{OA} = \alpha$，$\vec{OB} = \beta$，$\vec{OC} = \gamma$，则 $\vec{AB} = \beta - \alpha$，$\vec{AC} = \gamma - \alpha$，且 $\alpha\cdot\beta = \beta\cdot\gamma = 0$，$\alpha\cdot\gamma = 0$

$\therefore$ (*) $\Leftrightarrow (\alpha\times\beta)\cdot(\alpha\times\beta) + (\beta\times\gamma)\cdot(\beta\times\gamma) + (\gamma\times\alpha)\cdot(\gamma\times\alpha) = [(\beta-\alpha)\times(\gamma-\alpha)]\cdot[(\beta-\alpha)\times(\gamma-\alpha)]$

$\xLeftrightarrow[\text{恒等式}]{\text{Lagrange}} (\alpha\cdot\alpha)(\beta\cdot\beta) - (\alpha\cdot\beta)^2 + (\beta\cdot\beta)(\gamma\cdot\gamma) - (\beta\cdot\gamma)^2 + (\gamma\cdot\gamma)(\alpha\cdot\alpha) - (\gamma\cdot\alpha)^2$

$= [(\beta-\alpha)\cdot(\beta-\alpha)][(\gamma-\alpha)\cdot(\gamma-\alpha)] - [(\beta-\alpha)\cdot(\gamma-\alpha)]^2$

$\Leftrightarrow (\alpha\cdot\alpha)(\beta\cdot\beta) + (\beta\cdot\beta)(\gamma\cdot\gamma) + (\gamma\cdot\gamma)(\alpha\cdot\alpha) = [(\beta\cdot\beta) - 2(\alpha\cdot\beta) + (\alpha\cdot\alpha)][(\gamma\cdot\gamma) - 2(\alpha\cdot\gamma) + (\alpha\cdot\alpha)]$

$- (\beta\cdot\gamma - \alpha\cdot\gamma - \beta\cdot\alpha + \alpha\cdot\alpha)^2$

$\Leftrightarrow (\alpha\cdot\alpha)(\beta\cdot\beta) + (\beta\cdot\beta)(\gamma\cdot\gamma) + (\gamma\cdot\gamma)(\alpha\cdot\alpha) = [(\beta\cdot\beta) + (\alpha\cdot\alpha)][(\gamma\cdot\gamma) + (\alpha\cdot\alpha)] - (\alpha\cdot\alpha)^2$ (**)

而(**)右侧 $= (\beta\cdot\beta)(\gamma\cdot\gamma) + (\alpha\cdot\alpha)(\gamma\cdot\gamma) + (\beta\cdot\beta)(\alpha\cdot\alpha) + (\alpha\cdot\alpha)^2 - (\alpha\cdot\alpha)^2$

$= (\beta\cdot\beta)(\gamma\cdot\gamma) + (\alpha\cdot\alpha)(\gamma\cdot\gamma) + (\beta\cdot\beta)(\alpha\cdot\alpha) =$ (**)左侧

即(**)为恒等式

$\therefore$ (*) 成立，证毕

第11周作业(高代A)　　A

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第一次(20211025)

3.1. (1) 证明 $\tau=\mathcal{A}\cdot\mathcal{B}$ 是 $V\to V$ 的线性映射：

设 $\forall\alpha,\beta\in V,\ \lambda\in\mathbb{R}$

则 $\tau(\alpha)=\mathcal{A}[\mathcal{B}(\alpha)]\in V$

$\tau(\alpha+\beta)=\mathcal{A}[\mathcal{B}(\alpha+\beta)]=\mathcal{A}(\mathcal{B}\alpha+\mathcal{B}\beta)=\mathcal{A}\cdot\mathcal{B}\alpha+\mathcal{A}\cdot\mathcal{B}\beta$

$\tau(k\alpha)=\mathcal{A}[\mathcal{B}(k\alpha)]=\mathcal{A}[k\mathcal{B}(\alpha)]=k\,\mathcal{A}\cdot\mathcal{B}(\alpha)$

$\therefore$ τ 为 $V\to V$ 的一个线性映射

(2) 由题意：$(\mathcal{A}(e_1),\mathcal{A}(e_2),\mathcal{A}(e_3))=(e_1,e_2,e_3)A$

$(\mathcal{B}(e_1),\mathcal{B}(e_2),\mathcal{B}(e_3))=(e_1,e_2,e_3)B$

设 $A=[a_{ij}]_{3\times3}$，$B=[b_{ij}]_{3\times3}$

则 $\tau(e_1)=\mathcal{A}\mathcal{B}(e_1)=\mathcal{A}(b_{11}e_1+b_{21}e_2+b_{31}e_3)$

$=b_{11}\mathcal{A}(e_1)+b_{21}\mathcal{A}(e_2)+b_{31}\mathcal{A}(e_3)$

$=b_{11}\sum_{i=1}^{3}a_{i1}e_i+b_{21}\sum_{i=1}^{3}a_{i2}e_i+b_{31}\sum_{i=1}^{3}a_{i3}e_i$

$=(\sum_{k=1}^{3}a_{1k}b_{k1})e_1+(\sum_{k=1}^{3}a_{2k}b_{k1})e_2+(\sum_{k=1}^{3}a_{3k}b_{k1})e_3$

$=(e_1,e_2,e_3)\begin{pmatrix}\sum_{k=1}^{3}a_{1k}b_{k1}\\ \sum_{k=1}^{3}a_{2k}b_{k1}\\ \sum_{k=1}^{3}a_{3k}b_{k1}\end{pmatrix}=(e_1,e_2,e_3)\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}b_{11}\\b_{21}\\b_{31}\end{pmatrix}$

同理得 $\tau(e_2)=(e_1,e_2,e_3)\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}b_{12}\\b_{22}\\b_{32}\end{pmatrix}$

$\tau(e_3)=(e_1,e_2,e_3)\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}b_{13}\\b_{23}\\b_{33}\end{pmatrix}$

$\therefore$ 有 $(\tau(e_1),\tau(e_2),\tau(e_3))=(e_1,e_2,e_3)\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{pmatrix}$

$=(e_1,e_2,e_3)AB$

即 $\tau=\mathcal{A}\cdot\mathcal{B}$ 在 e_1,e_2,e_3 下坐标矩阵为 AB

3.2. 解：将各等式展开有：$e_i'=a_{1i}e_1+a_{2i}e_2+a_{3i}e_3$

由 $(O';e_1',e_2',e_3')$ 为直角坐标架有：$e_i'\cdot e_j'=\delta_{ij}=\begin{cases}1, & i=j\\0, & i\neq j\end{cases}$

由 $(O';e_1',e_2',e_3')$ 为直角坐标架有：

$e_i'\cdot e_j'=a_{1i}a_{1j}+a_{2i}a_{2j}+a_{3i}a_{3j}$

$\therefore a_{1i}a_{1j}+a_{2i}a_{2j}+a_{3i}a_{3j}=\delta_{ij}\Rightarrow\sqrt{a_{1i}a_{1j}+a_{2i}a_{2j}+a_{3i}a_{3j}}=\delta_{ij}$

$\therefore A$ 的列向量两两正交且长度为1

故 A 为正交阵

3.3. 解：由题意 $\overrightarrow{A_1A}=\frac{\lambda_1}{\lambda_2}\overrightarrow{AA_2}\Leftrightarrow\overrightarrow{A_1O}+\overrightarrow{OA}=\frac{\lambda_1}{\lambda_2}\overrightarrow{AO}+\frac{\lambda_1}{\lambda_2}\overrightarrow{OA_2}$

$\Leftrightarrow\overrightarrow{OA}=\frac{\lambda_2}{\lambda_1+\lambda_2}\overrightarrow{OA_1}+\frac{\lambda_1}{\lambda_1+\lambda_2}\overrightarrow{OA_2}$

而 $\overrightarrow{OA_1}=x_1e_1+x_2e_2+x_3e_3$

$\overrightarrow{OA_2}=y_1e_1+y_2e_2+y_3e_3$

$\therefore \vec{OA} = \frac{\lambda_2}{\lambda_1+\lambda_2}(x_1e_1+x_2e_2+x_3e_3) + \frac{\lambda_1}{\lambda_1+\lambda_2}(y_1e_1+y_2e_2+y_3e_3)$

$= \frac{\lambda_2x_1+\lambda_1y_1}{\lambda_1+\lambda_2}e_1 + \frac{\lambda_2x_2+\lambda_1y_2}{\lambda_1+\lambda_2}e_2 + \frac{\lambda_2x_3+\lambda_1y_3}{\lambda_1+\lambda_2}e_3$

$\therefore$ A 在 (O,e_1,e_2,e_3) 下坐标为 $\left(\frac{\lambda_2x_1+\lambda_1y_1}{\lambda_1+\lambda_2}, \frac{\lambda_2x_2+\lambda_1y_2}{\lambda_1+\lambda_2}, \frac{\lambda_2x_3+\lambda_1y_3}{\lambda_1+\lambda_2}\right)$

3.4. 由重心定义：$\vec{MA_1}+\vec{MA_2}+\cdots+\vec{MA_n}=\vec{0}$

$\Rightarrow \sum_{k=1}^{n}(\vec{MO}+\vec{OA_k})=\vec{0}$

$\Rightarrow n\vec{OM} = \sum_{k=1}^{n}\vec{OA_k}$

$= \sum_{k=1}^{n}(x_ke_1+y_ke_2+z_ke_3)$

$= (\sum_{k=1}^{n}x_k)e_1 + (\sum_{k=1}^{n}y_k)e_2 + (\sum_{k=1}^{n}z_k)e_3$

$\Rightarrow \vec{OM} = (\frac{1}{n}\sum_{k=1}^{n}x_k)e_1 + (\frac{1}{n}\sum_{k=1}^{n}y_k)e_2 + (\frac{1}{n}\sum_{k=1}^{n}z_k)e_3$

即 M 坐标为 $(\frac{1}{n}\sum_{k=1}^{n}x_k, \frac{1}{n}\sum_{k=1}^{n}y_k, \frac{1}{n}\sum_{k=1}^{n}z_k)$

~~3.5. 证明：A、B、C 共线~~ $\Leftrightarrow \vec{AB} \parallel \vec{AC}$

~~①若 $\vec{AC}=\vec{0}$，则 $x_i=z_i$，$i=1,2,3$~~

~~显然~~ $\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0$

~~②若 $\vec{AC}\neq\vec{0}$，则 A、B、C 共线 $\Leftrightarrow \vec{AB}=\lambda\vec{AC}$，$\lambda\in R$~~

~~同前题解法求得：$y_i = x_i+\lambda(z_i-x_i) = (1-\lambda)x_i+\lambda z_i$~~

$\therefore \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_3 \\ (1-\lambda)x_1+\lambda z_1 & (1-\lambda)x_2+\lambda z_2 & (1-\lambda)x_3+\lambda z_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$

$= (1-\lambda)\begin{vmatrix} x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \\ z_1 & z_2 & z_3 \end{vmatrix} + \lambda \begin{vmatrix} x_1 & x_2 & x_3 \\ z_1 & z_2 & z_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$

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3.5. 证明：取右手系标架 正交单位 (O,e_1',e_2',e_3')，则 $\vec{OO'}=0$，设 $(e_1,e_2,e_3)=(e_1',e_2',e_3')A$

$A=(a_{ij})_{3\times3}$

又设 A、B、C 在 e_1',e_2',e_3' 下坐标分别为 (x_1',x_2',x_3')、(y_1',y_2',y_3')、(z_1',z_2',z_3')

则：$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} a_{11}x_1+a_{12}x_2+a_{13}x_3 \\ a_{21}x_1+a_{22}x_2+a_{23}x_3 \\ a_{31}x_1+a_{32}x_2+a_{33}x_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

同理求得：$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$\begin{pmatrix} z_1' \\ z_2' \\ z_3' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$

$\Rightarrow \therefore \begin{pmatrix} x_1' & y_1' & z_1' \\ x_2' & y_2' & z_2' \\ x_3' & y_3' & z_3' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$

为方便起见将上式记为 $M_1 = AM$

两边取行列式：$|M_1| = |AM| = |A||M|$

而 $\begin{cases} e_1 = a_{11}e_1'+a_{21}e_2'+a_{31}e_3' \\ e_2 = a_{12}e_1'+a_{22}e_2'+a_{32}e_3' \\ e_3 = a_{13}e_1'+a_{23}e_2'+a_{33}e_3' \end{cases}$

~~若~~又 e_1',e_2',e_3' 为右手标架 且相互垂直，长度为1

$\therefore D(e_1,e_2$

由于 e_1,e_2

$D(e_1,e_2,e_3)=\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}=|A|$ $\cancel{\Rightarrow |A|\neq}$

由于e_1,e_2,e_3为标架，其应线性无关

∴ $D(e_1,e_2,e_3)\neq 0 \Rightarrow |A|\neq 0$

∵A、B、C 共线

∴$\vec{OA}$、$\vec{OB}$、$\vec{OC}$共面

∴$D(\vec{OA},\vec{OB},\vec{OC})=0$

而同前可算得：$D(\vec{OA},\vec{OB},\vec{OC})=\begin{vmatrix} x_1' & x_2' & x_3'\\ y_1' & y_2' & y_3'\\ z_1' & z_2' & z_3'\end{vmatrix}$

∴$|{}^tM_1|=0 \Rightarrow |M_1|=0$

而$|A|\neq 0$，故$|M|=\frac{|M_1|}{|A|}=0$

∴ $|{}^tM|=0$，即：$\begin{vmatrix} x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ z_1 & z_2 & z_3\end{vmatrix}=0$

3.6. 证明：设(O,e_1'',e_2'',e_3'')为单位正交右手系标架

且$(e_1,e_2,e_3)=(e_1'',e_2'',e_3'')B$，$B=(b_{ij})_{3\times3}$

$(e_1',e_2',e_3')=(e_1'',e_2'',e_3'')C$，$C=(c_{ij})_{3\times3}$

则 $\begin{cases} e_1=b_{11}e_1''+b_{21}e_2''+b_{31}e_3''\\ e_2=b_{12}e_1''+b_{22}e_2''+b_{32}e_3''\\ e_3=b_{13}e_1''+b_{23}e_2''+b_{33}e_3''\end{cases}$ $\begin{cases} e_1'=c_{11}e_1''+c_{21}e_2''+c_{31}e_3''\\ e_2'=c_{12}e_1''+c_{22}e_2''+c_{32}e_3''\\ e_3'=c_{13}e_1''+c_{23}e_2''+c_{33}e_3''\end{cases}$

又：$(e_1',e_2',e_3')=(e_1,e_2,e_3)A$

即：$\begin{cases} e_1'=a_{11}e_1+a_{21}e_2+a_{31}e_3\\ e_2'=a_{12}e_1+a_{22}e_2+a_{32}e_3\\ e_3'=a_{13}e_1+a_{23}e_2+a_{33}e_3\end{cases}$

∴ $e_1'=a_{11}(\sum_{i=1}^{3}b_{i1}e_i'')+a_{21}(\sum_{i=1}^{3}b_{i2}e_i'')+a_{31}(\sum_{i=1}^{3}b_{i3}e_i'')$

$=(\sum_{k=1}^{3}a_{k1}b_{1k})e_1''+(\sum_{k=1}^{3}a_{k1}b_{2k})e_2''+(\sum_{k=1}^{3}a_{k1}b_{3k})e_3''$

$=(e_1'',e_2'',e_3'')\begin{pmatrix}\sum_{k=1}^{3}a_{k1}b_{1k}\\ \sum_{k=1}^{3}a_{k1}b_{2k}\\ \sum_{k=1}^{3}a_{k1}b_{3k}\end{pmatrix}=(e_1'',e_2'',e_3'')\begin{pmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix}\begin{pmatrix}a_{11}\\ a_{21}\\ a_{31}\end{pmatrix}$

同理求得 $e_2'=(e_1'',e_2'',e_3'')\begin{pmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix}\begin{pmatrix}a_{12}\\ a_{22}\\ a_{32}\end{pmatrix}$

$e_3'=(e_1'',e_2'',e_3'')\begin{pmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix}\begin{pmatrix}a_{13}\\ a_{23}\\ a_{33}\end{pmatrix}$

∴$(e_1',e_2',e_3')=(e_1'',e_2'',e_3'')\begin{pmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$

$=(e_1'',e_2'',e_3'')BA$

∴ $C=BA$，两边取行列式：$|C|=|B||A|$

而同3.5可证得：$\begin{cases}|B|=D(e_1,e_2,e_3)\\ |C|=D(e_1',e_2',e_3')\end{cases}$

∴$|A|=\frac{|C|}{|B|}\neq 0$

$\vec{OA_2}=y_1e_1+y_2e_2+y_3e_3$

3

3.7. 证明：取 e_1'', e_2'', e_3'' 为 V的一组单位正交基且构成右手系

又设 $(e_1, e_2, e_3) = (e_1'', e_2'', e_3'')B$

$(e_1', e_2', e_3') = (e_1'', e_2'', e_3'')C$

则同3.8得：$C = BA$，$|C| = |B||A|$

$D(e_1, e_2, e_3) = |B| \neq 0$

$D(e_1', e_2', e_3') = |C|$

∴ e_1', e_2', e_3' 为V的一组基 $\Leftrightarrow D(e_1', e_2', e_3') \neq 0$

$\Leftrightarrow |C| \neq 0$

$\Leftrightarrow |B||A| \neq 0$

$\Leftrightarrow |A| \neq 0$

3.8. 证明：设所选标架为 $\{O; e_1, e_2, e_3\}$

则 $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (y_1 - x_1)e_1 + (y_2 - x_2)e_2 + (y_3 - x_3)e_3$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (z_1 - x_1)e_1 + (z_2 - x_2)e_2 + (z_3 - x_3)e_3$

∴ A,B,C共线 $\Leftrightarrow \overrightarrow{AB} \parallel \overrightarrow{AC} \Leftrightarrow \overrightarrow{AB} = \vec{0}$ 或 $\overrightarrow{AB} = k\overrightarrow{AC}$，$k \in \mathbb{R}$

$$\Leftrightarrow \begin{cases} y_1 - x_1 = 0 \\ y_2 - x_2 = 0 \\ y_3 - x_3 = 0 \end{cases} \text{或} \begin{cases} y_1 - x_1 = k(z_1 - x_1) \\ y_2 - x_2 = k(z_2 - x_2) \\ y_3 - x_3 = k(z_3 - x_3) \end{cases}$$

而 $\begin{vmatrix} y_2 - x_2 & y_3 - x_3 \\ z_2 - x_2 & z_3 - x_3 \end{vmatrix} = 0 \Leftrightarrow (y_2 - x_2)(z_3 - x_3) - (y_3 - x_3)(z_2 - x_2) = 0$

$$\Leftrightarrow \begin{cases} y_2 - x_2 = 0 \\ y_3 - x_3 = 0 \end{cases} \text{或} \begin{cases} y_2 - x_2 = \lambda_1(z_2 - x_2) \\ y_3 - x_3 = \lambda_1(z_3 - x_3) \end{cases}, \lambda_1 \in \mathbb{R}$$

同理 $\begin{vmatrix} y_3 - x_3 & y_1 - x_1 \\ z_3 - x_3 & z_1 - x_1 \end{vmatrix} = 0 \Leftrightarrow \begin{cases} y_3 - x_3 = 0 \\ y_1 - x_1 = 0 \end{cases}$ 或 $\begin{cases} y_3 - x_3 = \lambda_2(z_3 - x_3) \\ y_1 - x_1 = \lambda_2(z_1 - x_1) \end{cases}, \lambda_2 \in \mathbb{R}$

$\begin{vmatrix} y_1 - x_1 & y_2 - x_2 \\ z_1 - x_1 & z_2 - x_2 \end{vmatrix} = 0 \Leftrightarrow \begin{cases} y_1 - x_1 = 0 \\ y_2 - x_2 = 0 \end{cases}$ 或 $\begin{cases} y_1 - x_1 = \lambda_3(z_1 - x_1) \\ y_2 - x_2 = \lambda_3(z_2 - x_2) \end{cases}, \lambda_3 \in \mathbb{R}$

$$\therefore \begin{vmatrix} y_2 - x_2 & y_3 - x_3 \\ z_2 - x_2 & z_3 - x_3 \end{vmatrix} = \begin{vmatrix} y_3 - x_3 & y_1 - x_1 \\ z_3 - x_3 & z_1 - x_1 \end{vmatrix} = \begin{vmatrix} y_1 - x_1 & y_2 - x_2 \\ z_1 - x_1 & z_2 - x_2 \end{vmatrix} = 0 \Leftrightarrow \begin{cases} y_1 - x_1 = 0 \\ y_2 - x_2 = 0 \\ y_3 - x_3 = 0 \end{cases}$$

$$\text{或} \begin{cases} y_1 - x_1 = \lambda(z_1 - x_1) \\ y_2 - x_2 = \lambda(z_2 - x_2) \\ y_3 - x_3 = \lambda(z_3 - x_3) \end{cases}, \lambda \in \mathbb{R}$$

由上分析即有：A,B,C共线 $\Leftrightarrow \begin{vmatrix} y_2 - x_2 & y_3 - x_3 \\ z_2 - x_2 & z_3 - x_3 \end{vmatrix} = \begin{vmatrix} y_3 - x_3 & y_1 - x_1 \\ z_3 - x_3 & z_1 - x_1 \end{vmatrix} = \begin{vmatrix} y_1 - x_1 & y_2 - x_2 \\ z_1 - x_1 & z_2 - x_2 \end{vmatrix} = 0$

3.9. 证明：选 $e_1 = \overrightarrow{AB}$，$e_2 = \overrightarrow{AC}$，$e_3 = e_1 \times e_2$，$O = A$

则A,B,C坐标为 $(0,0,0)$，$(1,0,0)$，$(0,1,0)$

由 $\begin{cases} \overrightarrow{AP} = \lambda\overrightarrow{PB} \\ \overrightarrow{BQ} = \mu\overrightarrow{QC} \\ \overrightarrow{CR} = \nu\overrightarrow{RA} \end{cases} \Rightarrow \begin{cases} \overrightarrow{AP} = \lambda\overrightarrow{PA} + \lambda\overrightarrow{AB} \\ \overrightarrow{BA} + \overrightarrow{AQ} = \mu\overrightarrow{QA} + \mu\overrightarrow{AC} \\ \overrightarrow{CA} + \overrightarrow{AR} = \nu\overrightarrow{RA} \end{cases} \Rightarrow \begin{cases} \overrightarrow{AP} = \frac{\lambda}{1+\lambda}\overrightarrow{AB} = \frac{\lambda}{1+\lambda}e_1 \\ \overrightarrow{AQ} = \frac{1}{1+\mu}\overrightarrow{AB} + \frac{\mu}{1+\mu}\overrightarrow{AC} = \frac{1}{1+\mu}e_1 + \frac{\mu}{1+\mu}e_2 \\ \overrightarrow{AR} = \frac{1}{1+\nu}\overrightarrow{AC} = \frac{1}{1+\nu}e_2 \end{cases}$

∴ P,Q,R坐标为：$P(\frac{\lambda}{1+\lambda}, 0, 0)$，$Q(\frac{1}{1+\mu}, \frac{\mu}{1+\mu}, 0)$，$R(0, \frac{1}{1+\nu}, 0)$

∴ 由3.8，P,Q,R共线 $\Leftrightarrow \begin{vmatrix} \frac{\mu}{1+\mu} & 0 \\ \frac{1}{1+\nu} & 0 \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{1+\mu} - \frac{\lambda}{1+\lambda} \\ 0 & -\frac{\lambda}{1+\lambda} \end{vmatrix} = \begin{vmatrix} \frac{1}{1+\mu} - \frac{\lambda}{1+\lambda} & \frac{\mu}{1+\mu} \\ -\frac{\lambda}{1+\lambda} & \frac{1}{1+\nu} \end{vmatrix} = 0$

$\Leftrightarrow \begin{vmatrix} \frac{1}{1+\mu} - \frac{\lambda}{1+\lambda} & \frac{\mu}{1+\mu} \\ -\frac{\lambda}{1+\lambda} & \frac{1}{1+\nu} \end{vmatrix} = 0 \Leftrightarrow \frac{1}{1+\nu}\left(\frac{1}{1+\mu} - \frac{\lambda}{1+\lambda}\right) + \frac{\mu\lambda}{(1+\mu)(1+\lambda)} = 0$

$\Leftrightarrow 1 + \lambda - \lambda(1+\mu) + \mu\lambda(1+\nu) = 0 \Leftrightarrow 1 - \lambda\mu + \mu\lambda + \mu\lambda\nu = 0 \Leftrightarrow \lambda\mu\nu = -1$

第12周作业（高代A）　　　　汪翊丰
第一次（20221103）　　　　3019201080
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A

4.1、证明：由题意：$\rho(L_1,L_2)=\min\{\rho(A_1,L_2)\mid A_1\in L_1\}$

设L_1方程：$\begin{cases}x=x_1+v_1t\\ y=y_1+v_2t\\ z=z_1+v_3t\end{cases}$ 而$L_1/\!/L_2$，故L_2方程可设为 $\begin{cases}x=x_2+v_1t\\ y=y_2+v_2t\\ z=z_2+v_3t\end{cases}$

其中均有$t\in\mathbb{R}$

设$A_1(x_0,y_0,z_0)\in L_1$，对应参数为t_0

又 $\rho(A_1,L_2)=\dfrac{|(x_0-x_2,\ y_0-y_2,\ z_0-z_2)\times(v_1,v_2,v_3)|}{|(v_1,v_2,v_3)|}$

$=\dfrac{|(x_1-x_2+v_1t_0,\ y_1-y_2+v_2t_0,\ z_1-z_2+v_3t_0)\times(v_1,v_2,v_3)|}{|(v_1,v_2,v_3)|}$

$=\dfrac{\sqrt{\begin{vmatrix}y_1-y_2+v_2t_0 & z_1-z_2+v_3t_0\\ v_2 & v_3\end{vmatrix}^2+\begin{vmatrix}z_1-z_2+v_3t_0 & x_1-x_2+v_1t_0\\ v_3 & v_1\end{vmatrix}^2+\begin{vmatrix}x_1-x_2+v_1t_0 & y_1-y_2+v_2t_0\\ v_1 & v_2\end{vmatrix}^2}}{\sqrt{v_1^2+v_2^2+v_3^2}}$

$=\dfrac{\sqrt{[v_3(y_1-y_2)-v_2(z_1-z_2)]^2+[v_1(z_1-z_2)-v_3(x_1-x_2)]^2+[v_2(x_1-x_2)-v_1(y_1-y_2)]^2}}{\sqrt{v_1^2+v_2^2+v_3^2}}$

即 $\rho(A_1,L_2)$为一常数

$\therefore \rho(L_1,L_2)=\rho(A_1,L_2),\ \forall A_1\in L_1$

4.2. 证明：设L_1,L_2方程分别为：$\begin{cases}x=x_1+u_1t\\ y=y_1+u_2t\\ z=z_1+u_3t\end{cases}$，$\begin{cases}x=x_2+v_1t\\ y=y_2+v_2t\\ z=z_2+v_3t\end{cases}$，$A_1(x_1,y_1,z_1)\in L_1$，$A_2(x_2,y_2,z_2)\in L_2$

由L_1,L_2异面得：$D(\overrightarrow{A_1A_2},\vec{s_1},\vec{s_2})\neq0$，其中：$\begin{cases}\vec{s_1}=(u_1,u_2,u_3)\\ \vec{s_2}=(v_1,v_2,v_3)\end{cases}$

即：$\begin{vmatrix}x_1-x_2 & y_1-y_2 & z_1-z_2\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}\neq0$，故$(u_1,u_2,u_3)\neq\lambda(v_1,v_2,v_3)$，$\forall\lambda\in\mathbb{R}$

设$A(x_0,y_0,z_0)\in L_1$，过A作$AB\perp L_2$ 且 $AB\cap L_2=B(x_0',y_0',z_0')$

由于L_1,L_2异面，故$A\neq B$，设A，B分别对应参数t_0,t_0'

考察方程组：$\begin{cases}\overrightarrow{AB}\cdot\vec{s_1}=0\\ \overrightarrow{AB}\cdot\vec{s_2}=0\end{cases}$ 即：$\begin{cases}u_1(x_0-x_0')+u_2(y_0-y_0')+u_3(z_0-z_0')=0\\ v_1(x_0-x_0')+v_2(y_0-y_0')+v_3(z_0-z_0')=0\end{cases}$

代入$\begin{cases}x_0-x_0'=(x_1-x_2)+(u_1t_0-v_1t_0')\\ y_0-y_0'=(y_1-y_2)+(u_2t_0-v_2t_0')\\ z_0-z_0'=(z_1-z_2)+(u_3t_0-v_3t_0')\end{cases}$ 得：

$$\begin{cases}(u_1^2+u_2^2+u_3^2)t_0-(u_1v_1+u_2v_2+u_3v_3)t_0'=-u_1(x_1-x_2)-u_2(y_1-y_2)-u_3(z_1-z_2)\\ (u_1v_1+u_2v_2+u_3v_3)t_0-(v_1^2+v_2^2+v_3^2)t_0'=-v_1(x_1-x_2)-v_2(y_1-y_2)-v_3(z_1-z_2)\end{cases}\quad(*)$$

该方程组为线性方程组，其系数行列式：

$$\begin{vmatrix}u_1^2+u_2^2+u_3^2 & -(u_1v_1+u_2v_2+u_3v_3)\\ (u_1v_1+u_2v_2+u_3v_3) & -(v_1^2+v_2^2+v_3^2)\end{vmatrix}=-(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)+(u_1v_1+u_2v_2+u_3v_3)^2\overset{\text{Cauchy不等式}}{<}0$$

而"="成立 $\Leftrightarrow$ $u_1=u_2=u_3=0$ 或 $(v_1,v_2,v_3)=\lambda(u_1,u_2,u_3)$

但由前分析二者均不可能，故"="不成立

$$\therefore \begin{vmatrix} u_1^2+u_2^2+u_3^2 & -(u_1v_1+u_2v_2+u_3v_3) \\ (u_1v_1+u_2v_2+u_3v_3) & -(v_1^2+v_2^2+v_3^2) \end{vmatrix} < 0$$

故方程组(*)存在唯一一组解 $\begin{cases} t_0=\lambda_0 \\ t_0'=\lambda_0' \end{cases}$

这组解使 $\overrightarrow{AB}\perp\vec{s_1}$，$\overrightarrow{AB}\perp\vec{s_2}$ 即 $AB\perp L_1$，$AB\perp L_2$

故综上，存在唯一一条直线与 L_1, L_2 均垂直相交，其即为公垂线 L

4.3 证明：同4.1得：$\rho(L_1,L_2)=\min\{\rho(A_1,L_2)\mid A_1\in L_1\}$

设 L_1 方程：$\begin{cases} x=x_1+u_1t \\ y=y_1+u_2t \\ z=z_1+u_3t \end{cases}$，$L_2$ 方程：$\begin{cases} x=x_2+v_1t \\ y=y_2+v_2t \\ z=z_2+v_3t \end{cases}$

设 $A_1(x,y,z)\in L_1$，对应参数 t

$$\rho(A_1,L_2)=\frac{|(x-x_2,y-y_2,z-z_2)\times(v_1,v_2,v_3)|}{\sqrt{v_1^2+v_2^2+v_3^2}}$$

$$=\sqrt{\frac{[(y-y_2)v_3-(z-z_2)v_2]^2+[(z-z_2)v_1-(x-x_2)v_3]^2+[(x-x_2)v_2-(y-y_2)v_1]^2}{v_1^2+v_2^2+v_3^2}}$$

代入 $\begin{cases} x-x_2=(x_1-x_2)+u_1t \\ y-y_2=(y_1-y_2)+u_2t \\ z-z_2=(z_1-z_2)+u_3t \end{cases}$

记 $\vec{s}=(x_1-x_2, y_1-y_2, z_1-z_2)$，$\vec{s_1}=(u_1,u_2,u_3)$，$\vec{s_2}=(v_1,v_2,v_3)$

得：$\rho(A_1,L_2)=\dfrac{|(\vec{s}+t\vec{s_1})\times\vec{s_2}|}{|\vec{s_2}|}$

$=\dfrac{|\vec{s}\times\vec{s_2}+t\vec{s_1}\times\vec{s_2}|}{|\vec{s_2}|}$

4.3 证明：设 $\forall B_1\in L_1$，$B_2\in L_2$

则 $\overrightarrow{B_1B_2}=\overrightarrow{B_1A_1}+\overrightarrow{A_1A_2}+\overrightarrow{A_2B_2}$

设 L_1, L_2 方向向量分别为 $\vec{s_1}, \vec{s_2}$

则 $\overrightarrow{B_1A_1}\parallel\vec{s_1}$

$\overrightarrow{B_2A_2}\parallel\vec{s_2}$

$\overrightarrow{A_1A_2}\cdot\vec{s_1}=0$

$\overrightarrow{A_1A_2}\cdot\vec{s_2}=0$

$\therefore \overrightarrow{B_1A_1}\cdot\overrightarrow{A_1A_2}=0$

$\overrightarrow{B_2A_2}\cdot\overrightarrow{A_1A_2}=0$

则 $|\overrightarrow{B_1B_2}|^2=\overrightarrow{B_1B_2}\cdot\overrightarrow{B_1B_2}=(\overrightarrow{B_1A_1}+\overrightarrow{A_1A_2}+\overrightarrow{A_2B_2})\cdot(\overrightarrow{B_1A_1}+\overrightarrow{A_1A_2}+\overrightarrow{A_2B_2})$

$=|\overrightarrow{B_1A_1}|^2+|\overrightarrow{A_1A_2}|^2+|\overrightarrow{A_2B_2}|^2+2\overrightarrow{B_1A_1}\cdot\overrightarrow{A_2B_2}$

$=|\overrightarrow{A_1A_2}|^2+(\overrightarrow{B_1A_1}+\overrightarrow{A_2B_2})\cdot(\overrightarrow{B_1A_1}+\overrightarrow{A_2B_2})$

$=|\overrightarrow{A_1A_2}|^2+|\overrightarrow{B_1A_1}+\overrightarrow{A_2B_2}|^2\geqslant|\overrightarrow{A_1A_2}|^2$

~~平面, 故 $\overrightarrow{B_1A_1}+\overrightarrow{A_1B_2}$ ...~~

即: $\rho(B_1,B_2)\geq\rho(A_1,A_2)$, $\forall B_1\in L_1, B_2\in L_2$

$\therefore \rho(L_1,L_2)=\rho(A_1,A_2)$

4.4. 证明: 设 L_1,L_2 的公垂线为 L, $L\cap L_1=A_1$, $L\cap L_2=A_2$

由4.4已证, $\rho(L_1,L_2)=\rho(A_1,A_2)=|\overrightarrow{A_1A_2}|$

$\because \overrightarrow{A_1A_2}\perp\vec{v_1}$, $\overrightarrow{A_1A_2}\perp\vec{v_2}$

$\therefore \overrightarrow{A_1A_2}\parallel(\vec{v_1}\times\vec{v_2})$

$$\therefore |\overrightarrow{A_1A_2}|=\frac{|\overrightarrow{A_1A_2}|\cdot|(\vec{v_1}\times\vec{v_2})|}{|\vec{v_1}\times\vec{v_2}|}=\frac{|\overrightarrow{A_1A_2}\cdot(\vec{v_1}\times\vec{v_2})|}{(\vec{v_1}\times\vec{v_2})}$$

而 $\overrightarrow{M_1M_2}=\overrightarrow{M_1A_1}+\overrightarrow{A_1A_2}+\overrightarrow{A_2M_2}$

又 $\overrightarrow{M_1A_1}\parallel\vec{v_1}$, $\overrightarrow{A_2M_2}\parallel\vec{v_2}$

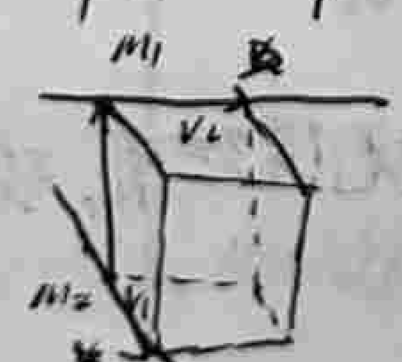
$$\therefore \overrightarrow{M_1M_2}\cdot(\vec{v_1}\times\vec{v_2})=\overrightarrow{M_1A_1}\cdot(\vec{v_1}\times\vec{v_2})+\overrightarrow{A_1A_2}\cdot(\vec{v_1}\times\vec{v_2})-\overrightarrow{M_2A_2}\cdot(\vec{v_2}\times\vec{v_1})$$

$$=\overrightarrow{A_1A_2}\cdot(\vec{v_1}\times\vec{v_2})$$

$$\therefore |\overrightarrow{A_1A_2}|=\frac{|\overrightarrow{M_1M_2}\cdot(\vec{v_1}\times\vec{v_2})|}{|\vec{v_1}\times\vec{v_2}|}$$

$$即\ \rho(L_1,L_2)=\frac{|\overrightarrow{M_1M_2}\cdot(\vec{v_1}\times\vec{v_2})|}{|\vec{v_1}\times\vec{v_2}|}$$

几何意义如示意图:



$\rho(L_1,L_2)=h_{平行六面体}=\frac{V_{六面体}}{S_{底}}$

$V_{六面体}=|\overrightarrow{M_1M_2}\cdot(\vec{v_1}\times\vec{v_2})|$, $S_{底}=|\vec{v_1}\times\vec{v_2}|$

4.5. 证明: 由题意 L_1 过 $M_1(x_0,y_0,z_0)$, L_2 过 $M_2(x_0',y_0',z_0')$, L_1,L_2 方向向量分别为 $\vec{v_1}=(v_1,v_2,v_3)$, $\vec{v_2}=(v_1',v_2',v_3')$, 故由4.4已证:

$$\rho(L_1,L_2)=\frac{|\overrightarrow{M_1M_2}\cdot(\vec{v_1}\times\vec{v_2})|}{|\vec{v_1}\times\vec{v_2}|}$$

$$而\ \overrightarrow{M_1M_2}\cdot(\vec{v_1}\times\vec{v_2})=\begin{vmatrix} x_0'-x_0 & y_0'-y_0 & z_0'-z_0\\ v_1 & v_2 & v_3\\ v_1' & v_2' & v_3'\end{vmatrix}$$

$$\vec{v_1}\times\vec{v_2}=\left(\begin{vmatrix} v_2 & v_3\\ v_2' & v_3'\end{vmatrix},\begin{vmatrix} v_3 & v_1\\ v_3' & v_1'\end{vmatrix},\begin{vmatrix} v_1 & v_2\\ v_1' & v_2'\end{vmatrix}\right)$$

$$\therefore \rho(L_1,L_2)=\left|\frac{\begin{vmatrix} x_0'-x_0 & y_0'-y_0 & z_0'-z_0\\ v_1 & v_2 & v_3\\ v_1' & v_2' & v_3'\end{vmatrix}}{\sqrt{\left(\begin{vmatrix} v_2 & v_3\\ v_2' & v_3'\end{vmatrix}\right)^2+\left(\begin{vmatrix} v_3 & v_1\\ v_3' & v_1'\end{vmatrix}\right)^2+\left(\begin{vmatrix} v_1 & v_2\\ v_1' & v_2'\end{vmatrix}\right)^2}}\right|$$

4.6. 证明: 定义函数 $l: V\to\mathbb{R}$

$$\overrightarrow{OA}\mapsto a e_1^*(\overrightarrow{OA})+b e_2^*(\overrightarrow{OA})+c e_3^*(\overrightarrow{OA})$$

显然 $A\in H\Leftrightarrow l(\overrightarrow{OA})+d=a e_1^*(\overrightarrow{OA})+b e_2^*(\overrightarrow{OA})+c e_3^*(\overrightarrow{OA})+d=0$

故只需证 l 是线性函数即可:

设 $\forall\overrightarrow{OA},\overrightarrow{OA'}\in V$, $\forall\lambda\in\mathbb{R}$

由 e_1^*、e_2^*、e_3^* 为 $V\to\mathbb{R}$ 的线性函数有:

$$l(\vec{OA}+\vec{OA'}) = ae_1^*(\vec{OA}+\vec{OA'}) + be_2^*(\vec{OA}+\vec{OA'}) + ce_3^*(\vec{OA}+\vec{OA'})$$

$$= [ae_1^*(\vec{OA}) + be_2^*(\vec{OA}) + ce_3^*(\vec{OA})] + [ae_1^*(\vec{OA'}) + be_2^*(\vec{OA'}) + ce_3^*(\vec{OA'})]$$

$$= l(\vec{OA}) + l(\vec{OA'})$$

$$l(\lambda\vec{OA}) = ae_1^*(\lambda\vec{OA}) + be_2^*(\lambda\vec{OA}) + ce_3^*(\lambda\vec{OA})$$

$$= \lambda[ae_1^*(\vec{OA}) + be_2^*(\vec{OA}) + ce_3^*(\vec{OA})]$$

$$= \lambda l(\vec{OA})$$

∴ l 为线性函数, 证毕

4.7. 证明: 定义 $l_1: V\to\mathbb{R}$

$$\vec{OA}\mapsto a_1e_1^*(\vec{OA}) + b_1e_2^*(\vec{OA}) + c_1e_3^*(\vec{OA})$$

$l_2: V\to\mathbb{R}$

$$\vec{OA}\mapsto a_2e_1^*(\vec{OA}) + b_2e_2^*(\vec{OA}) + c_2e_3^*(\vec{OA})$$

显然 $A\in L \Leftrightarrow \begin{cases} l_1(\vec{OA})+d_1=0 \\ l_2(\vec{OA})+d_2=0 \end{cases}$

同4.8 可证 l_1, l_2 为线性函数

4.8. 解: 由 $(4,-1,3)\times(1,5,-1) = \left(\begin{vmatrix}-1&3\\5&-1\end{vmatrix}, \begin{vmatrix}3&4\\-1&1\end{vmatrix}, \begin{vmatrix}4&-1\\1&5\end{vmatrix}\right)$

$$= (-14,7,21) \parallel (-2,1,3)$$

解及原线性方程组得一组特解: $\begin{cases} x = x_0 = \frac{1}{7} \\ y = y_0 = -\frac{3}{7} \\ z = z_0 = 0 \end{cases}$

可得 L 的参数方程: $\begin{cases} x = \frac{1}{7} - 2t \\ y = -\frac{3}{7} + t \\ z = 3t \end{cases}$, $t\in\mathbb{R}$ 为参数

4.9. 证明: 设平面: $H_1: a_1x+b_1y+c_1z+d_1=0$

$H_2: a_2x+b_2y+c_2z+d_2=0$

$H_3: a_3x+b_3y+c_3z+d_3=0$

由其两两不平行有 $(a_1,b_1,c_1)\times(a_2,b_2,c_2)\neq\vec{0}$, $(a_2,b_2,c_2)\times(a_3,b_3,c_3)\neq\vec{0}$, $(a_3,b_3,c_3)\times(a_1,b_1,c_1)\neq\vec{0}$

先证 $\forall A(x_0,y_0,z_0)$ 到任一平面 $H: ax+by+cz+d=0$

距离公式为: $\rho(A,H) = \dfrac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$

由题意 H 法向量 $\vec{n}=(a,b,c)$

取 $\forall M(x_0',y_0',z_0')\in H$, 则 $\rho(A,H) = \dfrac{|\vec{MA}\cdot\vec{n}|}{|\vec{n}|} = \dfrac{|a(x_0-x_0')+b(y_0-y_0')+c(z_0-z_0')|}{\sqrt{a^2+b^2+c^2}}$

而 $M\in H$, 故 $ax_0'+by_0'+cz_0'+d=0 \Rightarrow -ax_0'-by_0'-cz_0'=d$

$$\therefore \rho(A,H) = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

设 $B(x,y,z)$ 满足 $\rho(B,H_1)=\rho(B,H_2)=\rho(B,H_3)$

则 $\rho^2(B,H_1)=\rho^2(B,H_2)=\rho^2(B,H_3)$

$$\Leftrightarrow \frac{(a_1x+b_1y+c_1z+d_1)^2}{a_1^2+b_1^2+c_1^2} = \frac{(a_2x+b_2y+c_2z+d_2)^2}{a_2^2+b_2^2+c_2^2} = \frac{(a_3x+b_3y+c_3z+d_3)^2}{a_3^2+b_3^2+c_3^2} \quad (*)$$

由于平面方程各项同除一非零数无影响, 故不妨设 $a_1^2+b_1^2+c_1^2=1$, $a_2^2+b_2^2+c_2^2=1$, $a_3^2+b_3^2+c_3^2=1$

∴ (*) $\iff (a_1x+b_1y+c_1z+d_1)^2=(a_2x+b_2y+c_2z+d_2)^2=(a_3x+b_3y+c_3z+d_3)^2$

$$\iff \begin{cases}(a_1x+b_1y+c_1z+d_1)^2-(a_2x+b_2y+c_2z+d_2)^2=0\\(a_2x+b_2y+c_2z+d_2)^2-(a_3x+b_3y+c_3z+d_3)^2=0\end{cases}$$

$$\iff \begin{cases}[(a_1-a_2)x+(b_1-b_2)y+(c_1-c_2)z+(d_1-d_2)][(a_1+a_2)x+(b_1+b_2)y+(c_1+c_2)z+(d_1+d_2)]=0\\ [(a_2-a_3)x+(b_2-b_3)y+(c_2-c_3)z+(d_2-d_3)][(a_2+a_3)x+(b_2+b_3)y+(c_2+c_3)z+(d_2+d_3)]=0\end{cases}$$

$$\iff \begin{cases}(a_1-a_2)x+(b_1-b_2)y+(c_1-c_2)z+(d_1-d_2)=0 \text{ 或 } (a_1+a_2)x+(b_1+b_2)y+(c_1+c_2)z+(d_1+d_2)=0\\ (a_2-a_3)x+(b_2-b_3)y+(c_2-c_3)z+(d_2-d_3)=0 \text{ 或 } (a_2+a_3)x+(b_2+b_3)y+(c_2+c_3)z+(d_2+d_3)=0\end{cases}$$

由于 $(a_1,b_1,c_1)\times(a_2,b_2,c_2)\neq\vec{0}$，故 a_1-a_2、b_1-b_2、c_1-c_2不全为0

同理 a_1+a_2、b_1+b_2、c_1+c_2不全为0

a_2-a_3、b_2-b_3、c_2-c_3不全为0

a_2+a_3、b_2+b_3、c_2+c_3不全为0

设 $H_1^*: (a_1-a_2)x+(b_1-b_2)y+(c_1-c_2)z+(d_1-d_2)=0$

$H_2^*: (a_1+a_2)x+(b_1+b_2)y+(c_1+c_2)z+(d_1+d_2)=0$

$H_3^*: (a_2-a_3)x+(b_2-b_3)y+(c_2-c_3)z+(d_1-d_2)=0$

$H_4^*: (a_2+a_3)x+(b_2+b_3)y+(c_2+c_3)z+(d_2+d_3)=0$

下证 $H_i^*\not\parallel H_j^*$ $(\forall i,j\in\{1,2,3,4\}, i\neq j)$.

只需证 $H_1^*\not\parallel H_2^*$，其余同理：反证：由于 $(a_1-a_2,b_1-b_2,c_1-c_2)$、$(a_1+a_2,b_1+b_2,c_1+c_2)\neq\vec{0}$

故假设 $H_1^*\parallel H_2^*$，则 $(a_1-a_2,b_1-b_2,c_1-c_2)=\lambda(a_1+a_2,b_1+b_2,c_1+c_2)$，$\lambda\in\mathbb{R},\lambda\neq0$

∴ $(1-\lambda)a_1=(1+\lambda)a_2$，$(1-\lambda)b_1=(1+\lambda)b_2$，$(1-\lambda)c_1=(1+\lambda)c_2$

∴ $(1-\lambda)(a_1,b_1,c_1)=(1+\lambda)(a_2,b_2,c_2)$，故 $(a_1,b_1,c_1)\parallel(a_2,b_2,c_2)$，矛盾

∴ $H_1^*\not\parallel H_2^*$

记 $L_1=H_1^*\cap H_3^*$、$L_2=H_1^*\cap H_4^*$、$L_3=H_2^*\cap H_3^*$、$L_4=H_2^*\cap H_4^*$

由上分析即知 $\rho(B,H_1)=\rho(B,H_2)=\rho(B,H_3)\iff B\in L_1\cup L_2\cup L_3\cup L_4$

∴ B的轨迹为直线

4.10. 证明：设 $A(a_1,a_2,a_3)$、$B(b_1,b_2,b_3)$、$C(c_1,c_2,c_3)$为$\triangle ABC$三个顶点

设 $D(x,y,z)$ 满足 $\rho(A,D)=\rho(B,D)=\rho(C,D)$ (*)

分析方程 $\rho(A,D)=\rho(B,D)$.

$\rho(A,D)=\rho(B,D)\iff\rho^2(A,D)=\rho^2(B,D)$

$\iff(x-a_1)^2+(y-a_2)^2+(z-a_3)^2=(x-b_1)^2+(y-b_2)^2+(z-b_3)^2$

$\iff(a_1-b_1)x+(a_2-b_2)y+(a_3-b_3)z=\dfrac{(a_1^2+a_2^2+a_3^2)-(b_1^2+b_2^2+b_3^2)}{2}$

∵ $A\neq B$ ∴ a_1-b_1，a_2-b_2，a_3-b_3 不全为0

故 $(a_1-b_1)x+(a_2-b_2)y+(a_3-b_3)z=\dfrac{(a_1^2+a_2^2+a_3^2)-(b_1^2+b_2^2+b_3^2)}{2}$ 为一平面，记为H_1

∴ $\rho(A,D)=\rho(B,D)\iff D\in H_1$

同理：$\rho(B,D)=\rho(C,D)\iff D\in$平面$H_2$：$(b_1-c_1)x+(b_2-c_2)y+(b_3-c_3)z=\dfrac{\sum_{i=1}^3 b_i^2-\sum_{i=1}^3 c_i^2}{2}$

$\rho(A,D)=\rho(C,D)\iff D\in$平面$H_3$：$(c_1-a_1)x+(c_2-a_2)y+(c_3-a_3)z=\dfrac{\sum_{i=1}^3 c_i^2-\sum_{i=1}^3 a_i^2}{2}$

下证 H_1,H_2,H_3交于一直线L.

∵ A,B,C不共线

∴ $\vec{BA}\not\parallel\vec{CB}$，即 $(a_1-b_1,a_2-b_2,a_3-b_3)\not\parallel(b_1-c_1,b_2-c_2,b_3-c_3)$，即 $H_1\not\parallel H_2$

$\therefore H_1 \cap H_2 = L_1$

$\because \forall (x,y,z) \in L_1$，有：$\begin{cases} (a_1-b_1)x+(a_2-b_2)y+(a_3-b_3)z = \dfrac{(a_1^2+a_2^2+a_3^2)-(b_1^2+b_2^2+b_3^2)}{2} \\ (b_1-c_1)x+(b_2-c_2)y+(b_3-c_3)z = \dfrac{(b_1^2+b_2^2+b_3^2)-(c_1^2+c_2^2+c_3^2)}{2} \end{cases}$

相加取负有：$(c_1-a_1)x+(c_2-a_2)y+(c_3-a_3)z = \dfrac{(c_1^2+c_2^2+c_3^2)-(a_1^2+a_2^2+a_3^2)}{2}$

$\therefore (x,y,z) \in H_3$，故 $L_1 \subset H_3$

$\therefore L = H_1 \cap H_2 = H_1 \cap H_2 \cap H_3$

$\therefore (*)$ 成立 $\Leftrightarrow D \in L$，即 D 的轨迹为一直线

4.11. 证明：设 $a_1x+b_1y+c_1z+d_1=0$ 和 $a_2x+b_2y+c_2z+d_2=0$ 分别定义平面 H_1、H_2

若 $H_1=H_2$，则 $(a_1,b_1,c_1) \parallel (a_2,b_2,c_2)$（否则 $H_1 \cap H_2 = L$ 为直线）

$\therefore a_2=b_2=c_2=0$ 或 $(a_1,b_1,c_1)=\lambda(a_2,b_2,c_2)$，$\lambda \in \mathbb{R}$

但 (a_2,b_2,c_2) 为 π_2 法向量，故前者不可能

$\therefore H_1$：$\lambda a_2x+\lambda b_2y+\lambda c_2z+d_1=0$

由于 $H_1=H_2$，故 $\forall (x_0,y_0,z_0) \in H_2$，有 $(x_0,y_0,z_0) \in H_1$

即：$\begin{cases} \lambda a_2x_0+\lambda b_2y_0+\lambda c_2z_0+d_1=0 \\ a_2x_0+b_2y_0+c_2z_0+d_2=0 \end{cases} \Rightarrow d_1-\lambda d_2=0 \Rightarrow d_1=\lambda d_2$

$\therefore \exists \lambda \in \mathbb{R}$，$a_1x+b_1y+c_1z+d_1=\lambda(a_2x+b_2y+c_2z+d_2)$

4.12 证明：只证前者即可，后者由对称性即得

设两方程组分别定义直线 L_1，L_2，由 $L_1=L_2$ 有：

$[(a_1,b_1,c_1)\times(a_2,b_2,c_2)] \parallel [(a_3,b_3,c_3)\times(a_4,b_4,c_4)]$

由方向向量非零性，设 L_1、$L_2 \perp \pi$ $(L_1=L_2)$

则向量 (a_1,b_1,c_1)、(a_2,b_2,c_2)、(a_3,b_3,c_3)、$(a_4,b_4,c_4) \in \pi$，即四向量共面

又 (a_1,b_1,c_1)，(a_2,b_2,c_2) 线性无关

(a_3,b_3,c_3)、(a_4,b_4,c_4) 线性无关

$\therefore \exists \lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$，使：

$(a_3,b_3,c_3)=\lambda_1(a_1,b_1,c_1)+\lambda_2(a_2,b_2,c_2)$

$(a_4,b_4,c_4)=\mu_1(a_1,b_1,c_1)+\mu_2(a_2,b_2,c_2)$

$\because L_1=L_2$ $\therefore \forall (x_0,y_0,z_0) \in L_1$，$(x_0,y_0,z_0) \in L_2$

$\therefore \begin{cases} a_1x_0+b_1y_0+c_1z_0+d_1=0 \\ a_2x_0+b_2y_0+c_2z_0+d_2=0 \\ (\lambda_1a_1+\lambda_2a_2)x_0+(\lambda_1b_1+\lambda_2b_2)y_0+(\lambda_1c_1+\lambda_2c_2)z_0+d_3=0 \\ (\mu_1a_1+\mu_2a_2)x_0+(\mu_1b_1+\mu_2b_2)y_0+(\mu_1c_1+\mu_2c_2)z_0+d_4=0 \end{cases}$

$\Rightarrow \begin{cases} d_3-\lambda_1d_1-\lambda_2d_2=0 \\ d_4-\mu_1d_1-\mu_2d_2=0 \end{cases} \Rightarrow \begin{cases} d_3=\lambda_1d_1+\lambda_2d_2 \\ d_4=\mu_1d_1+\mu_2d_2 \end{cases}$

$\therefore \delta=(a_3,b_3,c_3,d_3)=\lambda_1(a_1,b_1,c_1,d_1)+\lambda_2(a_2,b_2,c_2,d_2)=\lambda_1\alpha+\lambda_2\beta$

$\gamma=(a_4,b_4,c_4,d_4)=\mu_1(a_1,b_1,c_1,d_1)+\mu_2(a_2,b_2,c_2,d_2)=\mu_1\alpha+\mu_2\beta$

证毕

4.13. 解：该方程组为齐次方程组，故对其系数矩阵作初等行变换：

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & -1 & 3 \\ 1 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \end{bmatrix}$$