高等代数 习题4.1 牛少明 3021233044.

1. 由于 $\forall n \in \mathbb{N}$，$(TAT^{-1})^n = \underbrace{(TAT^{-1})(TAT^{-1})\cdots(TAT^{-1})}_{n个}$

$= TA(T^{-1}T)A(T^{-1}T)\cdots(T^{-1}T)AT^{-1}$

$= TA^nT^{-1}$.

从而设 $f(x) = \sum_{k=0}^{n} a_k \cdot x^k$，有 $f(TAT^{-1}) = \sum_{k=0}^{n} a_k (TAT^{-1})^k = \sum_{k=0}^{n} a_k (TA^kT^{-1})$

$= T\left(\sum_{k=0}^{n} a_k A^k\right)T^{-1}$

$= Tf(A)T^{-1}$. 即证.

2. (1)

$A^2 = \begin{pmatrix} 4 & 4 & 0 \\ 0 & 4 & 0 \\ 3 & 4 & 1 \end{pmatrix}$ $A^3 = \begin{pmatrix} 8 & 12 & 0 \\ 0 & 8 & 0 \\ 7 & 12 & 1 \end{pmatrix}$

$f(A) = \begin{pmatrix} 8 & 12 & 0 \\ 0 & 8 & 0 \\ 7 & 12 & 1 \end{pmatrix} - 2\begin{pmatrix} 4 & 4 & 0 \\ 0 & 4 & 0 \\ 3 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 1 & 4 & 0 \end{pmatrix}$

(2) $A^3 = \begin{pmatrix} 22 & 21 & 21 \\ 21 & 22 & 21 \\ 21 & 21 & 22 \end{pmatrix}$

$f(A) = \begin{pmatrix} 22 & 21 & 21 \\ 21 & 22 & 21 \\ 21 & 21 & 22 \end{pmatrix} - 3\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 18 & 18 & 18 \\ 18 & 18 & 18 \\ 18 & 18 & 18 \end{pmatrix}$

3. $E_{ii}A = \begin{pmatrix} 0 \\ \vdots \\ E_{ii(i)}A \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ A_{(i)} \\ \vdots \\ 0 \end{pmatrix}$ ←第i行

$AE_{ii} = (0 \cdots AE_{ii}^{(i)} \cdots 0) = (0 \cdots A^{(i)} \cdots 0)$. 由 $E_{ii}A = AE_{ii}$. $\Rightarrow \begin{pmatrix} 0 \cdots & a_{1i} & \cdots 0 \\ & a_{2i} & \\ \vdots & \vdots & \vdots \\ 0 \cdots & a_{ni} & \cdots 0 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$

第i列.

$\Rightarrow a_{ij} = \begin{cases} 0, & i \neq j \\ a_{ii}, & i = j \end{cases}$ $a_{ji} = \begin{cases} 0 & i \neq j \\ a_{ii}, & i = j \end{cases}$

$\Rightarrow$ A为对角矩阵.

4. (1) 设 $AB = C = (c_{ij})_{n\times n}$. $BA = D = (d_{ij})_{n\times n}$.

则 $c_{ii} = \sum_{k=1}^{n} a_{ik}b_{ki}$，$d_{ii} = \sum_{k=1}^{n} b_{ik}a_{ki}$

$tr(AB) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ik}b_{ki} = tr(BA)$.

(2) 由(1)，知 $tr(TAT^{-1}) = tr(AT^{-1}T) = tr(A)$. 即证.

5. (1) $\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccccc} 6 & 6 & 6 & 1 & 1 & 1 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccccc} 1 & 1 & 1 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$

$\longrightarrow \left(\begin{array}{cccccc} 1 & 1 & 1 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 1 & 2 & 0 & -\frac{1}{6} & \frac{5}{6} & -\frac{1}{6} \\ 2 & 0 & 1 & -\frac{1}{6} & -\frac{1}{6} & \frac{5}{6} \end{array}\right)$

(1) 设 $X=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32}\end{pmatrix} \Rightarrow \begin{cases} a_{11}+2a_{21}+3a_{31}=0 \\ 2a_{11}+3a_{21}+a_{31}=1 \\ 3a_{11}+a_{21}+2a_{31}=0 \end{cases} \quad \begin{cases} a_{12}+2a_{22}+3a_{32}=0 \\ 2a_{12}+3a_{22}+a_{32}=0 \\ 3a_{12}+a_{22}+2a_{32}=1. \end{cases}$

$$\Rightarrow \begin{cases} a_{11}=\frac{1}{18} \\ a_{21}=\frac{7}{18} \\ a_{31}=-\frac{5}{18} \end{cases} \quad \begin{cases} a_{12}=\frac{7}{18} \\ a_{22}=-\frac{5}{18} \\ a_{32}=\frac{1}{18} \end{cases}$$

$$\Rightarrow X=\begin{pmatrix} \frac{1}{18} & \frac{7}{18} \\ \frac{7}{18} & -\frac{5}{18} \\ -\frac{5}{18} & \frac{1}{18}\end{pmatrix}$$

(2)

$$\begin{pmatrix} 1&1&0&1&0&0 \\ 0&1&2&0&1&0 \\ 0&1&1&0&0&1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1&1&0&1&0&0 \\ 0&1&2&0&1&0 \\ 0&0&-1&0&-1&1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1&1&0&1&0&0 \\ 0&1&0&0&-1&2 \\ 0&0&1&0&1&-1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1&0&0&1&1&-2 \\ 0&1&0&0&-1&2 \\ 0&0&1&0&1&-1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1&1&-2 \\ 0&-1&2 \\ 0&1&-1 \end{pmatrix}\begin{pmatrix} 1&1&0 \\ 0&1&2 \\ 0&1&1 \end{pmatrix} X = \begin{pmatrix} 1&1&-2 \\ 0&-1&2 \\ 0&1&-1 \end{pmatrix}\begin{pmatrix} 5 \\ -6 \\ -2 \end{pmatrix}$$

$$\Rightarrow X=\begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}$$

(3)

$$\begin{pmatrix} 3&1&1&0 \\ 2&1&0&1 \end{pmatrix} \longrightarrow \begin{pmatrix} 3&1&1&0 \\ 0&\frac{1}{3}&-\frac{2}{3}&1 \end{pmatrix} \longrightarrow \begin{pmatrix} 3&1&1&0 \\ 0&1&-2&3 \end{pmatrix} \longrightarrow \begin{pmatrix} 3&0&3&-3 \\ 0&1&-2&3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1&0&1&-1 \\ 0&1&-2&3 \end{pmatrix}$$

$$\begin{pmatrix} 1&3&1&0 \\ 1&2&0&1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1&3&1&0 \\ 0&-1&-1&1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1&0&-2&3 \\ 0&1&1&-1 \end{pmatrix}$$

$$\Rightarrow X=\begin{pmatrix} 1&-1 \\ -2&3 \end{pmatrix}\begin{pmatrix} 3&3 \\ 2&2 \end{pmatrix}\begin{pmatrix} -2&3 \\ 1&-1 \end{pmatrix} = \begin{pmatrix} 1&1 \\ 0&0 \end{pmatrix}\begin{pmatrix} -2&3 \\ 1&-1 \end{pmatrix} = \begin{pmatrix} -1&2 \\ 0&0 \end{pmatrix}$$

6.

$$\begin{pmatrix} 1&2&0&0&1&0&0&0 \\ 2&3&0&0&0&1&0&0 \\ 1&-1&1&3&0&0&1&0 \\ 0&1&0&2&0&0&0&1 \end{pmatrix} \xrightarrow[③-①]{③-2\times①} \begin{pmatrix} 1&2&0&0&1&0&0&0 \\ 0&-1&0&0&-2&1&0&0 \\ 0&-3&1&3&-1&0&1&0 \\ 0&1&0&2&0&0&0&1 \end{pmatrix} \xrightarrow[④+②]{③-3\times②} \begin{pmatrix} 1&2&0&0&1&0&0&0 \\ 0&-1&0&0&-2&1&0&0 \\ 0&0&1&3&5&-3&1&0 \\ 0&0&0&2&-2&1&0&1 \end{pmatrix}$$

$$\xrightarrow[④\times\frac{1}{2}]{②\times(-1)} \begin{pmatrix} 1&2&0&0&1&0&0&0 \\ 0&1&0&0&2&-1&0&0 \\ 0&0&1&3&5&-3&1&0 \\ 0&0&0&1&-1&\frac{1}{2}&0&\frac{1}{2} \end{pmatrix} \xrightarrow[①-2\times②]{③-3\times④} \begin{pmatrix} 1&0&0&0&-3&2&0&0 \\ 0&1&0&0&2&-1&0&0 \\ 0&0&1&0&8&-\frac{9}{2}&1&-\frac{3}{2} \\ 0&0&0&1&-1&\frac{1}{2}&0&\frac{1}{2} \end{pmatrix}$$

$$\Rightarrow A^{-1}=\begin{pmatrix} -3&2&0&0 \\ 2&-1&0&0 \\ 8&-\frac{9}{2}&1&-\frac{3}{2} \\ -1&\frac{1}{2}&0&\frac{1}{2} \end{pmatrix}$$

$$A^{-1}=F_{4,3}(-3)\cdot F_{2,1}(-2)\cdot F_2(-1)F_4(\tfrac{1}{2})F_{2,3}(-3)F_{2,4}(1)F_{1,3}(-1)F_{1,2}(-2)$$

$$A=F_{1,2}(2)\cdot F_{1,3}(1)F_{2,4}(-1)F_{2,3}(3)F_4(2)F_2(-1)F_{2,1}(2)F_{4,3}(3)$$

(2)

$$\left(\begin{array}{cccc|cccc} 1&1&\cdots&1&1&0&\cdots&0\\ 0&1&\cdots&1&0&1&&0\\ \vdots&&\ddots&\vdots&\vdots&&\ddots&\vdots\\ 0&0&\cdots&1&0&0&\cdots&1\end{array}\right) \xrightarrow[\text{①}-\text{ⓝ}]{\text{ⓝ-1}-\text{ⓝ},\ \cdots} \left(\begin{array}{ccccc|cccc} 1&1&\cdots&1&0&1&0&\cdots&-1\\ 0&1&\cdots&1&0&0&1&&-1\\ \vdots&&&\vdots&\vdots&\vdots&&\ddots&\vdots\\ 0&0&\cdots&1&0&&&&-1\\ 0&0&\cdots&0&1&0&0&\cdots&1\end{array}\right)$$

$$\xrightarrow[\text{ⓝ-1}-\text{ⓝ-1}]{\text{ⓝ-2}-\text{ⓝ-1},\ \cdots}$$

$$\left(\begin{array}{cccc|cccc} 1&1&\cdots&1&1&0&\cdots&0\\ 0&1&\cdots&1&0&1&\cdots&0\\ \vdots&&\ddots&\vdots&\vdots&&&\vdots\\ 0&0&\cdots&1&0&0&\cdots&1\end{array}\right) \xrightarrow[\text{ⓝ-1}-\text{ⓝ}]{\text{①}-\text{②},\ \text{②}-\text{③},\ \cdots} \left(\begin{array}{cccc|cccc} 1&0&\cdots&0&1&-1&\cdots&0\\ 0&1&\cdots&0&0&1&\cdots&0\\ \vdots&&\ddots&\vdots&\vdots&&\ddots&-1\\ 0&0&\cdots&1&0&0&\cdots&1\end{array}\right)$$

$$A^{-1}=\begin{pmatrix} 1&-1&\cdots&\cdots&0\\ 0&1&-1&&0\\ \vdots&&\ddots&\ddots&\vdots\\ &&&1&-1\\ 0&\cdots&\cdots&&1\end{pmatrix} \qquad A^{-1}=\cancel{F_{2,1}(-1)}\,F_{n,n-1}(-1)\cdots F_{2,1}(-1).$$

$$\Rightarrow A=F_{2,1}(1)\,F_{3,2}(1)\cdots F_{n,n-1}(1).$$

7. (1) $(\lambda A)(\lambda^{-1}A^{-1})=(\lambda\cdot\lambda^{-1})A\cdot A^{-1}=I$

$\Rightarrow (\lambda A)^{-1}=\lambda^{-1}A^{-1}$

(2) ${}^tA\cdot{}^t(A^{-1})={}^t(A^{-1}A)={}^tI=I$

$\Rightarrow ({}^tA)^{-1}={}^t(A^{-1})$

8. 设 $A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}$ $f(A)=A^2-(a_{11}+a_{22})A+(a_{11}a_{22}-a_{12}a_{21})\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$$=\begin{pmatrix}a_{11}^2+a_{12}a_{21}&a_{11}a_{12}+a_{12}a_{22}\\a_{11}a_{21}+a_{21}a_{22}&a_{21}a_{12}+a_{22}^2\end{pmatrix}-\begin{pmatrix}a_{11}(a_{11}+a_{22})&a_{12}(a_{11}+a_{22})\\a_{21}(a_{11}+a_{22})&a_{22}(a_{11}+a_{22})\end{pmatrix}$$

$$+(a_{11}a_{22}-a_{12}a_{21})\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$=\begin{pmatrix}0&0\\0&0\end{pmatrix}=0.$$

9. 设 $A^m=0$，$B^n=0$. 且易知 $\forall k\geq 0$，$k\in\mathbb{Z}$，有 $A^{m+k}=B^{n+k}=0$.

而由$AB=BA$，故有：$(A+B)^{m+n}=\sum_{k=0}^{m+n}C_{m+n}^{k}A^kB^{m+n-k}$

由于：$k\geq m$ 与 $m+n-k\geq n$ 至少一个成立.

从而 $A^kB^{m+n-k}=0$. $\Rightarrow (A+B)^{m+n}=0$.

$\Rightarrow A+B$ 也是幂零矩阵.

10. ~~设 $A{}^tA=I$~~ 设 $A^{-1}\cdot A=A\cdot A^{-1}=I_n$.

取转置 $\Rightarrow {}^t(A^{-1}\cdot A)={}^tI_n=I_n \Rightarrow {}^tA\,{}^t(A^{-1})=I_n$.

$\Rightarrow ({}^tA)^{-1}={}^t(A^{-1})$ ~~$\Rightarrow A^{-1}={}^t(A^{-1})$. 即 A^{-1} 为~~

若 ${}^tA=A$. $\Rightarrow A^{-1}={}^t(A^{-1}) \Rightarrow A^{-1}$ 为对称矩阵.

若 ${}^tA=-A \Rightarrow (-A)^{-1}={}^t(A^{-1}) \Rightarrow -A^{-1}={}^t(A^{-1}) \Rightarrow A^{-1}$ 为反对称矩阵.

11. 分两步说明:

1° $A^n=0$.

2° $I_n, A, \cdots, A^{n-1}$ 线性无关.

对于 1°, 考虑行向量标准基 $e_1,\cdots,e_n$ 与列向量标准基 $\alpha_1,\cdots,\alpha_n$.

则 $A=\begin{pmatrix} e_2\\ e_3\\ \vdots\\ e_n\\ 0\end{pmatrix}=(0\ \alpha_1\cdots\alpha_{n-1})$. 而 $e_i\alpha_j=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}$.

从而 $A^2=\begin{pmatrix} e_3\\ \vdots\\ e_n\\ 0\\ 0\end{pmatrix}$. 归纳证: $A^k=\begin{pmatrix} e_{k+1}\\ \vdots\\ e_n\\ 0\\ \vdots\\ 0\end{pmatrix}$. $k=1$ 成立. 设 k 成立.

$$A^{k+1}=\begin{pmatrix} e_{k+1}\\ \vdots\\ e_n\\ 0\\ \vdots\\ 0\end{pmatrix}(0\ \alpha_1\cdots\alpha_{n-1})=\begin{pmatrix}0&\cdots&0&1&0&\cdots&0\\ 0&\cdots&0&0&1&\cdots&0\\ & & & \vdots & & & \\ 0&\cdots&0&0&\cdots&\cdots&0\end{pmatrix}\begin{matrix}(\text{第}k+2\text{位为}1)\\ (\text{第}k+3\text{位为}1)\\ \\ \\ \end{matrix}=\begin{pmatrix} e_{k+2}\\ \vdots\\ e_n\\ 0\\ \vdots\\ 0\end{pmatrix}$$

从而成立. 即 $A^n=0$.

对于 2°. 考虑 $I_n, A, \cdots, A^{n-1}$. 它们有 1 的位置互不相同.

设 $a_0I_n+a_1A+\cdots+a_{n-1}A^{n-1}=0$. 分别考虑 $(1,1),(1,2),\cdots,(1,n)$ 处即知,

$a_0=a_1=\cdots=a_{n-1}=0 \Rightarrow I_n, A, \cdots, A^{n-1}$ 线性无关.

从而 A 的极小多项式为 $f(x)=x^n$.

12. 设 $A^n=0$, 则 $(I_n-A)(I_n+A+\cdots+A^{n-1})=I_n-A^n=I_n$

故 I_n-A 可逆.

13. 由 $(I_n+A)\left(-\frac{1}{2}(A-2I)\right)=-\frac{1}{2}(A^2-A-2I)=I$.

故 I_n+A 可逆.

14. $AX=B$ 有解 $\Longleftrightarrow \begin{cases}AX^{(1)}=B^{(1)}\\ AX^{(2)}=B^{(2)}\\ \quad\vdots\\ AX^{(k)}=B^{(k)}\end{cases}$ 有解 $\Longleftrightarrow \begin{cases}r(A)=r(A|B^{(1)})\\ r(A)=r(A|B^{(2)})\\ \quad\vdots\\ r(A)=r(A|B^{(k)})\end{cases}$.

设 A 的列向量为 $\alpha_1,\cdots,\alpha_n$. 设 $\alpha_1,\cdots\alpha_r$ ~~线~~ 为一组极大线性无关组, 则

由于 $B^{(1)},\cdots,B^{(k)}$ 可被 $\alpha_1,\cdots\alpha_r$ 线性表出. $\Rightarrow r(A|B)=r(A)$.

另一方面，若 $r(A|B)=r(A)$. 则~~有~~同样有 $B^{(1)},\cdots,B^{(k)}$ 可由 $\alpha_1,\cdots,\alpha_r$ 线性表出.

从而 $r(A|B^{(i)})=r(A)$. $\forall 1\le i\le k$.

即证.

15. 设 $(I_n+BA)(I_n+X)=I_n$

$\Rightarrow BA+X+BAX=0$

~~$\Rightarrow$~~ 令 $X=BYA$

$\Rightarrow BA+BYA+BABYA=0$

$\Rightarrow B(I_n+Y+ABY)A=0$

$\Rightarrow B(I_n+(I_n+AB)Y)A=0$.

只用取 $Y=-(I_n+AB)^{-1}$ 即可.

即有 $(I_n+BA)^{-1}=I_n-B(I_n+AB)^{-1}A$ ，I_n+BA 可逆.

16. 由于 $A=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}(y_1\cdots y_n)$ $\Rightarrow r(A)\le\min\left\{r\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix},\ r(y_1,\cdots,y_n)\right\}$.

若 $\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$, $(y_1\cdots y_n)$ 有一个为零矩阵，则 $r(A)=0$.

若 $\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$, $(y_1,\cdots,y_n)$ 均不为0，则 $r(A)=1$.

不妨 $x_1\neq 0$. 则作初等行变换:

$$\begin{pmatrix}x_1y_1 & \cdots & x_1y_n\\ \vdots & & \vdots\\ x_ny_1 & \cdots & x_ny_n\end{pmatrix}\xleftrightarrow[\substack{\vdots\\ ⓝ-\frac{x_n}{x_1}①}]{②-\frac{x_2}{x_1}①}\begin{pmatrix}x_1y_1 & \cdots & x_1y_n\\ 0 & \cdots & 0\\ \vdots & & \vdots\\ 0 & \cdots & 0\end{pmatrix}$$，从而 $r(A)=1$.

17. $f(A)=(A^3+2A^2+3A+1)(A^2-A+2)$

$=A^5+A^4+3A^3+2A^2+5A+2$

18. 易知 I_n+E_{ij} 为上三角或下三角矩阵，从而

$\det(I_n+E_{ij})=1\Rightarrow I_n+E_{ij}$ 是可逆矩阵.

19. 取 $B=I_n+E_{ij}$. $\Rightarrow A(I_n+E_{ij})=(I_n+E_{ij})A$

$\Rightarrow AE_{ij}=E_{ij}A$.

从而对任意矩阵 B，有 $AB=BA$.

从而 $A=\lambda I_n$.

~~20.~~

20. 取 $X=E_{ij}$. 则 $A=(a_{ij})_{n\times n}=\begin{pmatrix}\alpha_1 & \cdots & \alpha_n\end{pmatrix}$

$$AE_{ij}=(0,\cdots\alpha_i,\cdots 0)$$

$\uparrow$ 第j列.

$tr(AE_{ij})=a_{ji}=0$.

从而 $\forall\ 1\le i,j\le n$. $a_{ij}=0$. 即 $A=0$.

21. $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}\end{pmatrix}$ $\quad {}^tA=\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{12} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn}\end{pmatrix}$

由 $A\cdot{}^tA={}^tAA$. 考虑 $A\,{}^tA$ 第 (i,j) 元 c_{ij} ~~$(i\le j)$~~ $i=j$

$$c_{ij}=(0,\cdots,0,a_{ii},\cdots,a_{in})\begin{pmatrix}0\\ \vdots\\ 0\\ a_{jj}\\ \vdots\\ a_{jn}\end{pmatrix}=\cancel{\begin{cases}a_{ii}a_{ji}+\cdots+a_{in}a_{jn}, & i>j\\ a_{ij}a_{jj}+\cdots+a_{in}a_{jn}, & i<j\end{cases}}\ \cancel{a_{ii}a_{in}+\cdots+a_{in}a_{in}}\ a_{ii}^2+\cdots+a_{in}^2$$

考虑 ${}^tA\cdot A$ 第 (i,j) 元 d_{ij} ~~$(i\ne j)$~~

$$d_{ij}=(a_{1i}\ a_{2i}\ \cdots\ a_{ii}\ 0\cdots 0)\begin{pmatrix}a_{1j}\\ \vdots\\ a_{jj}\\ 0\\ \vdots\\ 0\end{pmatrix}=\begin{cases}a_{1i}a_{1j}+\cdots+a_{ii}a_{ij}, & i<j\\ a_{1i}^2+\cdots+a_{ii}^2, & i=j\\ a_{1i}a_{1j}+\cdots+a_{ji}a_{jj}, & i>j.\end{cases}$$

由 $c_{ii}=d_{ii}\Rightarrow \sum_{k=i}^{n}a_{ik}^2=\sum_{k=1}^{i}a_{ki}^2$

依次取 $i=1,2,\cdots,n$

$\Rightarrow a_{12}^2+\cdots+a_{1n}^2=0\Rightarrow a_{12}=\cdots=a_{1n}=0$.

$a_{22}^2+a_{23}^2+\cdots+a_{2n}^2=a_{12}^2+a_{22}^2\Rightarrow a_{23}^2+\cdots+a_{2n}^2=0\Rightarrow a_{23}=\cdots=a_{2n}=0.\ \cdots$

依此类推 $\Rightarrow a_{ij}=\begin{cases}a_{ii}, & i=j\\ 0, & i\ne j.\end{cases}$ 即 A为对角矩阵. 即证

高等代数　习题4.2.　　牛少明　3021233044

1. (a). $\begin{vmatrix} 2 & -3 & 4 & 1 \\ 4 & -2 & 3 & 2 \\ a & b & c & d \\ 3 & -1 & 4 & 3 \end{vmatrix} = a\begin{vmatrix} -3 & 4 & 1 \\ -2 & 3 & 2 \\ -1 & 4 & 3 \end{vmatrix} - b\begin{vmatrix} 2 & 4 & 1 \\ 4 & 3 & 2 \\ 3 & 4 & 3 \end{vmatrix} + c\begin{vmatrix} 2 & -3 & 1 \\ 4 & -2 & 2 \\ 3 & -1 & 3 \end{vmatrix} - d\begin{vmatrix} 2 & -3 & 4 \\ 4 & -2 & 3 \\ 3 & -1 & 4 \end{vmatrix}$

$= 8a + 15b + 12c - 19d.$

(b) $\begin{vmatrix} 5 & a & 2 & -1 \\ 4 & b & 4 & -3 \\ 2 & c & 3 & -2 \\ 4 & d & 5 & -4 \end{vmatrix} = -a\begin{vmatrix} 4 & 4 & -3 \\ 2 & 3 & -2 \\ 4 & 5 & -4 \end{vmatrix} + b\begin{vmatrix} 5 & 2 & -1 \\ 2 & 3 & -2 \\ 4 & 5 & -4 \end{vmatrix} - c\begin{vmatrix} 5 & 2 & -1 \\ 4 & 4 & -3 \\ 4 & 5 & -4 \end{vmatrix} + d\begin{vmatrix} 5 & 2 & -1 \\ 4 & 4 & -3 \\ 2 & 3 & -2 \end{vmatrix}$

$= a\begin{vmatrix} 4 & 4 & 3 \\ 2 & 3 & 2 \\ 4 & 5 & 4 \end{vmatrix} - b\begin{vmatrix} 5 & 2 & 1 \\ 2 & 3 & 2 \\ 4 & 5 & 4 \end{vmatrix} + c\begin{vmatrix} 5 & 2 & 1 \\ 4 & 4 & 3 \\ 4 & 5 & 4 \end{vmatrix} - d\begin{vmatrix} 5 & 2 & 1 \\ 4 & 4 & 3 \\ 2 & 3 & 2 \end{vmatrix}$

$= 2a - 8b + c + 5d.$

2. (a) $\begin{vmatrix} x & y & 0 & \cdots & 0 & 0 \\ 0 & x & y & \cdots & 0 & 0 \\ \cdots & & & & \cdots & \\ 0 & 0 & 0 & \cdots & x & y \\ y & 0 & 0 & \cdots & 0 & x \end{vmatrix} = x\begin{vmatrix} x & y & \cdots & 0 & 0 \\ \cdots & & & & \\ 0 & 0 & \cdots & x & y \\ 0 & 0 & \cdots & 0 & x \end{vmatrix} \cancel{- y\begin{vmatrix} 0 & y & \cdots & 0 & 0 \\ \vdots & & & \vdots & \\ 0 & 0 & \cdots & x & y \\ y & 0 & \cdots & 0 & x \end{vmatrix}} + (-1)^{n+1} y\begin{vmatrix} y & 0 & \cdots & 0 & 0 \\ x & y & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & y & 0 \\ 0 & 0 & \cdots & x & y \end{vmatrix}$

$= x^n + (-1)^{n+1} y^n$

(b) $A_n \triangleq \begin{vmatrix} a_0 & 1 & 1 & \cdots & 1 \\ 1 & a_1 & 0 & \cdots & 0 \\ 1 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & a_n \end{vmatrix} = \begin{vmatrix} a_0 & 1 & \cdots & 1 \\ 1 & a_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & a_{n-1} \end{vmatrix} \cdot a_n + (-1)^{n+1}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{n-1} \; 0 \end{vmatrix}$

$= a_n A_{n-1} + (-1)^{n+2} a_1 a_2 \cdots a_{n-1} \cdot (-1)^{n-1}$

$= a_n A_{n-1} - a_1 a_2 \cdots a_{n-1}$

$\Rightarrow \dfrac{A_n}{a_1 \cdots a_n} = \dfrac{A_{n-1}}{a_1 \cdots a_{n-1}} - \dfrac{1}{a_n}. \quad \Rightarrow \dfrac{A_n}{a_1 \cdots a_n} = \dfrac{A_1}{a_1} - \dfrac{1}{a_2} - \cdots - \dfrac{1}{a_n}.$

$\cancel{\Rightarrow \dfrac{A_n}{a_1 \cdots a_n} = \dfrac{A_1}{a_1} + n - 1 = \dfrac{1}{a_1}\begin{vmatrix} a_0 & 1 \\ 1 & a_1 \end{vmatrix} + n - 1 = \dfrac{a_0 a_1 - 1}{a_1} + n - 1.}$

$\cancel{\Rightarrow A_n = a_1 \cdots a_n\left(\dfrac{a_0 a_1 - 1}{a_1} + n - 1\right) = a_0 a_1 \cdots a_n - a_2 \cdots a_n + (n-1) a_1 \cdots a_n.}$

$= \dfrac{a_0 a_1 - 1}{a_1} - \dfrac{1}{a_2} - \cdots - \dfrac{1}{a_n} = a_0 - \dfrac{1}{a_1} - \cdots - \dfrac{1}{a_n}$

$\Rightarrow A_n = a_1 \cdots a_n\left(a_0 - \dfrac{1}{a_1} - \cdots - \dfrac{1}{a_n}\right).$

(C). $A=\begin{pmatrix}\cos(\alpha_1-\beta_1) & \cdots & \cos(\alpha_1-\beta_n)\\ \vdots & \ddots & \vdots\\ \cos(\alpha_n-\beta_1) & \cdots & \cos(\alpha_n-\beta_n)\end{pmatrix}=\begin{pmatrix}\cos\alpha_1\cos\beta_1+\sin\alpha_1\sin\beta_1 & \cdots & \cos\alpha_1\cos\beta_n+\sin\alpha_1\sin\beta_n\\ \vdots & & \vdots\\ \cos\alpha_n\cos\beta_1+\sin\alpha_n\sin\beta_1 & \cdots & \cos\alpha_n\cos\beta_n+\sin\alpha_n\sin\beta_n\end{pmatrix}$.

$=\begin{pmatrix}\cos\alpha_1 & \sin\alpha_1\\ \vdots & \vdots\\ \cos\alpha_n & \sin\alpha_n\end{pmatrix}\begin{pmatrix}\cos\beta_1 & \cdots & \cos\beta_n\\ \sin\beta_1 & \cdots & \sin\beta_n\end{pmatrix}$.

$n>3$时，$\operatorname{rank} A\le 2$. ~~而~~ $\Rightarrow |A|=0$

$n=1$时，$|A|=\cos(\alpha_1-\beta_1)$.

$n=2$时，$|A|=\begin{vmatrix}\cos\alpha_1 & \sin\alpha_1\\ \cos\alpha_2 & \sin\alpha_2\end{vmatrix}\cdot\begin{vmatrix}\cos\beta_1 & \cos\beta_2\\ \sin\beta_1 & \sin\beta_2\end{vmatrix}=\sin(\alpha_2-\alpha_1)\cdot\sin(\beta_2-\beta_1)$

3. $\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)-\left(\sum_{i=1}^n a_ib_i\right)^2=\begin{vmatrix}\sum_{i=1}^n a_i^2 & \sum_{i=1}^n a_ib_i\\ \sum_{i=1}^n a_ib_i & \sum_{i=1}^n b_i^2\end{vmatrix}=|AB|$.

其中 $A=\begin{pmatrix}a_1 & \cdots & a_n\\ b_1 & \cdots & b_n\end{pmatrix}$. $B=\begin{pmatrix}a_1 & b_1\\ \vdots & \vdots\\ a_n & b_n\end{pmatrix}$.

由Cauchy-Binet公式，

$|AB|=\sum_{1\le i_1<i_2\le n}\begin{vmatrix}a_{i_1} & a_{i_2}\\ b_{i_1} & b_{i_2}\end{vmatrix}\cdot\begin{vmatrix}a_{i_1} & b_{i_1}\\ a_{i_2} & b_{i_2}\end{vmatrix}$

$=\sum_{1\le i<j\le n}(a_ib_j-a_jb_i)\cdot(a_ib_j-a_jb_i)=\sum_{1\le i<j\le n}(a_ib_j-a_jb_i)^2$. 即证

4. (1) 若$|A|\ne 0$，则 $A\cdot A^*=\det A\cdot I_n\Rightarrow |A|\cdot|A^*|=|A|^n\Rightarrow |A^*|=|A|^{n-1}$.

若$|A|=0$. 设$A=(a_{ij})_{n\times n}$. $A^*=\begin{pmatrix}A_{11} & \cdots & A_{n1}\\ \vdots & & \vdots\\ A_{1n} & \cdots & A_{nn}\end{pmatrix}$.

~~设$(a_{n1}$~~，由$|A|=0\Rightarrow A$的行向量线性相关，则存在一个可由其余线性表出.

不妨设 $(a_{n1},\cdots,a_{nn})=\lambda_1(a_{11},\cdots,a_{1n})+\cdots+\lambda_{n-1}(a_{n-1,1},\cdots,a_{n-1,n})$

则 $M_{nk}=\begin{vmatrix}a_{11} & \cdots & a_{1\,k-1} & a_{1\,k+1} & \cdots & a_{1n}\\ \vdots & & \vdots & \vdots & & \vdots\\ a_{n-1,1} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n}\end{vmatrix}$

$M_{1k}=\begin{vmatrix}a_{21} & \cdots & a_{2\,k-1} & a_{2\,k+1} & \cdots & a_{2n}\\ \vdots & & & & & \vdots\\ a_{n1} & \cdots & a_{n\,k-1} & a_{n\,k+1} & \cdots & a_{nn}\end{vmatrix}=\sum_{i=1}^{n-1}\lambda_i\begin{vmatrix}a_{21} & \cdots & a_{2\,k-1} & a_{2\,k+1} & \cdots & a_{2n}\\ \vdots & & \vdots & \vdots & & \vdots\\ a_{i1} & \cdots & a_{i\,k-1} & a_{i\,k+1} & \cdots & a_{in}\end{vmatrix}$

$= \cancel{\lambda}(-1)^{n-2}\lambda_1 \begin{vmatrix} a_{11} & \cdots & a_{1k-1} & a_{1k+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n-11} & \cdots & a_{n-1k-1} & a_{n-1k+1} & \cdots & a_{n-1n} \end{vmatrix} = (-1)^{n-2}\lambda_1 M_{nk}.$

即 $(-1)^{1+k} M_{1k} = (-1)^{1+k+n}\lambda_1 M_{nk}$

$\Rightarrow A_{1k} = -\lambda_1 A_{nk}.\ \forall\, 1 \le k \le n.$

$\Rightarrow A^*$ 中第1列与第n列线性相关. 从而 $|A^*| = 0 = |A|^{n-1}$. 即证.

(2) 由于 $A^* \cdot (A^*)^* = \det A^* \cdot I_n$.

$\Rightarrow A \cdot A^* \cdot (A^*)^* = A|A^*| \cdot I_n$

$\Rightarrow |A| \cdot I_n \cdot (A^*)^* = |A^*| A = |A|^{n-1} A$

若 $|A| \ne 0 \Rightarrow (A^*)^* = |A|^{n-2} A$.

若 $|A| = 0$. 由于 $A \cdot A^* = 0 \Rightarrow \operatorname{rank} A + \operatorname{rank} A^* \le n$.

若 $n=2$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $A^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ $(A^*)^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

若 $n \ge 3$, 则:

① 若 $\operatorname{rank} A = n-1$. 则 A 存在 $n-1$ 阶非零子式. $\Rightarrow A^* \ne 0$.

从而 $\operatorname{rank} A^* \ge 1 \Rightarrow \operatorname{rank} A^* = 1 < n-1$.

② 若 $\operatorname{rank} A < n-1$, 则 A 的所有 $n-1$ 阶子式均为 $0 \Rightarrow A^* = 0$.

从而 $\operatorname{rank} A^* < n-1$. ~~ran~~ $\Rightarrow (A^*)^* = 0 = |A|^{n-2} A$. 即证.

(3) 由 $A \cdot A^* = |A| \cdot I_n$. $\Rightarrow A \cdot A^* \cdot (A^*)^{-1} = |A|(A^*)^{-1} \Rightarrow A = |A|(A^*)^{-1}$.

从而有: $A \cdot \frac{1}{|A|} \cdot A^* = I_n \Rightarrow A^{-1} = \frac{1}{|A|} \cdot A^*$.

5. 设 $A = (a_{ij})_{m \times n}$. 由 Cauchy-Binet 公式. ${}^tA = \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix}$.

$\cancel{|{}^tA \cdot A| = \sum_{1 \le i_1 < \cdots < i_n \le m} \begin{vmatrix} a_{1i_1} & \cdots & a_{ni_1} \\ \vdots & & \vdots \\ a_{1i_n} & \cdots & a_{ni_n} \end{vmatrix}}$

$\sum_{1 \le i_1 < \cdots < i_n \le m} \begin{vmatrix} a_{i_11} & \cdots & a_{i_n1} \\ \vdots & & \vdots \\ a_{i_1n} & \cdots & a_{i_nn} \end{vmatrix} \cdot \begin{vmatrix} a_{i_11} & \cdots & a_{i_1n} \\ \vdots & & \vdots \\ a_{i_n1} & \cdots & a_{i_nn} \end{vmatrix}$.

$= \sum_{1 \le i_1 < \cdots < i_n \le m} \begin{vmatrix} a_{i_11} & \cdots & a_{i_1n} \\ \vdots & & \vdots \\ a_{i_n1} & \cdots & a_{i_nn} \end{vmatrix}^2$ 即证(明显遍历n阶子式).

$A \cdot {}^tA = |A| \cdot I_n$.

从而考虑 $A \cdot {}^tA$ 对角线上的元素，为：$\sum_{j=1}^{n} a_{ij}^2$，$1 \le i \le n$.

又 $A \cdot {}^tA = |A| \cdot I_n \Rightarrow \sum_{j=1}^{n} a_{ij}^2 = |A|$. 即证.

7. 考虑 A_{ij}，

若 $i > j+1$ 则有 $a_{ij} = 0$. 考虑第 j 列中最下方的 1（即为 a_{jj}）

则在 A 划掉第 i 行第 j 列时，有原第 j 行与第 $j+1$ 行相同.

$\Rightarrow A_{ij} = 0$

若 $j > i$，则有 $a_{ij} = 1$ 考虑第 j 列最下方的 1（即为 a_{jj}）.

则在 A 划掉第 i 行第 j 列时，原矩阵第 j 行与第 $j+1$ 行相同.

$\Rightarrow A_{ij} = 0$.

下面考虑 $A_{i+1,i}$ 与 $A_{i,i}$.

$$A_{i+1,i} = (-1) \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = -1.$$

$$A_{i,i} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = 1$$

$$\Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} = n + (-1)\cdot(n-1) = 1.$$

8. 由于 $AX = 0$ 的基础解系有 $n-m$ 个元素.

从而 $\operatorname{rank} A = m$. 亦即 A 行满秩.

由于 $A \begin{pmatrix} b_{11} & \cdots & b_{n-m\,1} \\ \vdots & & \vdots \\ b_{1n} & \cdots & b_{n-m\,n} \end{pmatrix} = 0$. 取转置.

$$\Rightarrow \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n-m\,1} & \cdots & b_{n-m\,n} \end{pmatrix} {}^tA = 0.$$

从而 方程组 $\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n-m\,1} & \cdots & b_{n-m\,n} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0$ 有解：${}^tA^{(1)}, \cdots, {}^tA^{(m)}$

而由 A 行满秩 $\Rightarrow {}^tA^{(1)}, \cdots, {}^tA^{(m)}$ 线性无关.

从而一个基础解系为：$\alpha_i = [a_{i1}, \cdots, a_{in}]$，$1 \le i \le m$

9. $\Delta(x_1,\cdots,x_n)\Delta(y_1,\cdots,y_n)=\det\begin{pmatrix}1 & x_1 & \cdots & x_1^{n-1}\\ 1 & x_2 & \cdots & x_2^{n-1}\\ \vdots & \vdots & & \vdots\\ 1 & x_n & \cdots & x_n^{n-1}\end{pmatrix}\det\begin{pmatrix}1 & 1 & \cdots & 1\\ y_1 & y_2 & \cdots & y_n\\ \vdots & \vdots & & \vdots\\ y_1^{n-1} & y_2^{n-1} & \cdots & y_n^{n-1}\end{pmatrix}$

$$=\det\left((1+x_iy_j+\cdots+x_i^{n-1}y_j^{n-1})_{n\times n}\right)=\det\left(\left(\frac{1-x_i^ny_j^n}{1-x_iy_j}\right)_{n\times n}\right).$$

左右均可看作关于 x_i, y_i 的多项式.($1\le i\le n$).

$\Delta(x_1,\cdots,x_n)\Delta(y_1,\cdots,y_n)=\prod_{1\le i<j\le n}(x_j-x_i)\prod_{1\le i<j\le n}(y_j-y_i)$.

有~~根~~因式 x_i-x_j, y_i-y_j, $\forall i\ne j$.

而对于右边.若 $x_i=x_j$,则矩阵 $A=\left(\frac{1}{1-x_iy_j}\right)_{n\times n}$ 的第 i 行与第 j 列相同.

从而 $\det A=0$. 同理 $y_i=y_j$ 时 $\det A=0$.

从而右边多项式亦有~~根~~因式 ~~$x_i=x_j$~~ x_i-x_j, ~~$y_i=y_j$~~ y_i-y_j.

即 $\prod_{1\le i<j\le n}(x_j-x_i)(y_j-y_i)\ \Big|\ \det\left(\frac{1}{1-x_iy_j}\right)_{n\times n}\cdot\prod_{i,j}(1-x_iy_j)$

由于左右两边关于 $x_1,\cdots,x_n$ 对称

10. $\det\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & -x_1 & -x_4 & x_3 \\ x_3 & x_4 & -x_1 & -x_2 \\ x_4 & -x_3 & x_2 & -x_1 \end{pmatrix}\begin{pmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & -y_1 & y_4 & y_3 \\ y_3 & y_4 & -y_1 & -y_2 \\ y_4 & -y_3 & y_2 & -y_1 \end{pmatrix}$

由 $\det\begin{pmatrix} a & b & c & d \\ b & -a & -d & c \\ c & d & -a & -b \\ d & -c & b & -a \end{pmatrix} = -(a^2+b^2+c^2+d^2)^2$

=

$$\det\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & -x_1 & -x_4 & x_3 \\ x_3 & x_4 & -x_1 & -x_2 \\ x_4 & -x_3 & x_2 & -x_1 \end{pmatrix} = -(x_1^2+x_2^2+x_3^2+x_4^2)^2$$

$$\det\begin{pmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & -y_1 & -y_4 & y_3 \\ y_3 & y_4 & -y_1 & -y_2 \\ y_4 & -y_3 & y_2 & -y_1 \end{pmatrix} = -(y_1^2+y_2^2+y_3^2+y_4^2)^2$$

$$\Rightarrow \det\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & -x_1 & -x_4 & x_3 \\ x_3 & x_4 & -x_1 & -x_2 \\ x_4 & -x_3 & x_2 & -x_1 \end{pmatrix}\begin{pmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & -y_1 & -y_4 & y_3 \\ y_3 & y_4 & -y_1 & -y_2 \\ y_4 & -y_3 & y_2 & -y_1 \end{pmatrix} = (x_1^2+x_2^2+x_3^2+x_4^2)^2(y_1^2+y_2^2+y_3^2+y_4^2)^2$$

~~另一方面~~ $\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & -x_1 & -x_4 & x_3 \\ x_3 & x_4 & -x_1 & -x_2 \\ x_4 & -x_3 & x_2 & -x_1 \end{pmatrix}\begin{pmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & -y_1 & -y_4 & y_3 \\ y_3 & y_4 & -y_1 & -y_2 \\ y_4 & -y_3 & y_2 & -y_1 \end{pmatrix}$

$= \cancel{x_1y_1+x_2y_2+x_3y_3+x_4y_4 \quad x_1y_2-x_2y_1+x_3y_4-x_4y_3 \quad x_1y_3-x_2y_4-x_3y_1+x_4y_2}$ x

$$= \begin{vmatrix} x_1y_1+x_2y_2+x_3y_3+x_4y_4 & x_1y_2-x_2y_1+x_3y_4-x_4y_3 & x_1y_3-x_2y_4-x_3y_1+x_2y_2 & x_1y_4+x_2y_3-x_3y_2-x_4y_1 \\ x_2y_1-x_1y_2-x_4y_3+x_3y_4 & x_2y_2+x_1y_1 \pm x_4y_4 \pm x_3y_3 & & \\ x_3y_1+x_4y_2-x_1y_3-x_2y_4 & & x_3y_3-x_4y_4+x_1y_1-x_2y_2 & \\ x_4y_1-x_3y_2+x_2y_3-x_1y_4 & & & x_4y_4-x_3y_3-x_2y_2+x_1y_1 \end{vmatrix}$$

另一方面，
$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & -x_1 & -x_4 & x_3 \\ x_3 & x_4 & -x_1 & -x_2 \\ x_4 & -x_3 & x_2 & -x_1 \end{pmatrix}\begin{pmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & -y_1 & y_4 & -y_3 \\ y_3 & -y_4 & -y_1 & y_2 \\ y_4 & y_3 & -y_2 & -y_1 \end{pmatrix}.$$

$$= \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ -\alpha_2 & \alpha_1 & -\alpha_4 & \alpha_3 \\ -\alpha_3 & \alpha_4 & \alpha_1 & -\alpha_2 \\ -\alpha_4 & -\alpha_3 & \alpha_2 & \alpha_1 \end{pmatrix}$$

其中 $\alpha_1 = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$

$\alpha_2 = x_1y_2 - x_2y_1 - x_3y_4 + x_4y_3$

$\alpha_3 = x_1y_3 + x_2y_4 - x_3y_1 - x_4y_2$

$\alpha_4 = x_1y_4 - x_2y_3 + x_3y_2 - x_4y_1$.

而 $\det\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ -\alpha_2 & \alpha_1 & -\alpha_4 & \alpha_3 \\ -\alpha_3 & \alpha_4 & \alpha_1 & -\alpha_2 \\ -\alpha_4 & -\alpha_3 & \alpha_2 & \alpha_1 \end{pmatrix} = \cancel{\text{do}}(-1)\cdot\det\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 & -\alpha_1 & \alpha_4 & -\alpha_3 \\ \alpha_3 & -\alpha_4 & -\alpha_1 & \alpha_2 \\ \alpha_4 & \alpha_3 & -\alpha_2 & -\alpha_1 \end{pmatrix}$

$$\overset{\text{转置}}{=} (-1)\det\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 & -\alpha_1 & -\alpha_4 & \alpha_3 \\ \alpha_3 & \alpha_4 & -\alpha_1 & -\alpha_2 \\ \alpha_4 & -\alpha_3 & \alpha_2 & -\alpha_1 \end{pmatrix}$$

$$= (-1)\cdot(-1)\cdot(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2)^2$$

从而有：$(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2)$. 即证.