hw1-solutions

Problem 1. 用归纳法证明Cauchy-Schwarz不等式.

Problem 2. (1)证明:存在$z \in \mathbb{C}$满足$|z-a|+|z+a|=2|c|$ $(a, c \in \mathbb{C})$当且仅当$|a| \leq |c|$.
(2)若上述条件满足, 试求$|z|$的最小值与最大值.

Problem 3. (1)证明：当$0<\theta<\frac{\pi}{2}$时，$\sin\left(2m+1\right)\theta=\left(\sin\theta\right)^{2m+1}P_m\left(\cot^2\theta\right)$ 其中$P_m\left(x\right)=\sum_{k=0}^{m}\left(-1\right)^k\begin{pmatrix}2m+1\\2k+1\end{pmatrix}x^{m-k}$.

证明：由DeMoive公式，得

$\cos\left(2m+1\right)\theta+i\sin\left(2m+1\right)\theta=\left(\cos\theta+i\sin\theta\right)^{2m+1}$

$=\sum_{j=0}^{2m+1}\begin{pmatrix}2m+1\\j\end{pmatrix}i^j\left(\sin\theta\right)^j\left(\cos\theta\right)^{2m+1-j}$

当j为奇数，即$j=2k+1, k=0,1,\cdots,m$时，

有$i\sin\left(2m+1\right)\theta=\sum_{k=0}^{m}\begin{pmatrix}2m+1\\2k+1\end{pmatrix}i^{2k+1}\left(\sin\theta\right)^{2k+1}\left(\cos\theta\right)^{2m-2k}$

$\therefore\sin\left(2m+1\right)\theta=\sum_{k=0}^{m}\left(-1\right)^k\begin{pmatrix}2m+1\\2k+1\end{pmatrix}\left(\sin\theta\right)^{2k+1}\left(\cos\theta\right)^{2m-2k}$

$=\left(\sin\theta\right)^{2m+1}\sum_{k=0}^{m}\left(-1\right)^k\begin{pmatrix}2m+1\\2k+1\end{pmatrix}\left(\sin^2\theta\right)^{k-m}\left(\cos^2\theta\right)^{m-k}$

$= (\sin\theta)^{2m+1} \sum_{k=0}^{m} (-1)^k \binom{2m+1}{2k+1} \left(\cot^2\theta\right)^{m-k}$

$= (\sin\theta)^{2m+1} P_m \left(\cot^2\theta\right)$

(2)证明：$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$.

提示：考虑方程$(z+1)^n - 1 = 0$不为零的$(n-1)$ 个根的乘积.

证明：设$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$,则$\omega - 1, \omega^2 - 1, \cdots, \omega^{n-1} - 1$为方程$(z+1)^n - 1 = 0$的$(n-1)$个非零根，

由韦达定理可得$(\omega-1)\left(\omega^2-1\right)\cdots\left(\omega^{n-1}-1\right) = (-1)^{n-1} n$,

$\therefore |\omega-1|\left|\omega^2-1\right|\cdots\left|\omega^{n-1}-1\right| = n$

$\because \left|\omega^k - 1\right| = 2\sin\frac{k\pi}{n}$

$\therefore \prod_{k=1}^{n-1} 2\sin\frac{k\pi}{n} = n$

即$2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = n$

$\therefore \sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$

Problem 4. 给定关于复变量z的方程$az + b\bar{z} + c = 0$, 其中$a, b, c \in \mathbb{C}$.

(1)找出这个方程只有一个解的条件，并计算这个解；

(2)找出这个方程有无穷解的条件；

(3)找出这个方程无解的条件.

解：由已知得

$$az + b\bar{z} + c = 0 \tag{0.1}$$

$$\bar{b}z + \bar{a}\bar{z} + \bar{c} = 0 \tag{0.2}$$

$(0.1) \times \bar{a} - (0.2) \times b$得

$$(|a|^2 - |b|^2)z + \bar{a}c - b\bar{c} = 0 \tag{0.3}$$

若$|a|^2 - |b|^2 \neq 0$,则有唯一解

$$z = \frac{b\bar{c} - \bar{a}c}{|a|^2 - |b|^2};$$

若$|a|^2 - |b|^2 = 0$, 则$\bar{a}c - b\bar{c} = 0$.
(i)当$|a|^2 = |b|^2 = 0$时, $c = 0$有无穷解; $c \neq 0$无解.
(ii)当$|a|^2 = |b|^2 \neq 0$时, 设$a = re^{\mathrm{i}\alpha}, b = re^{\mathrm{i}\beta}(r > 0), c = |c|e^{\mathrm{i}\theta}$, 则$\bar{a}c - b\bar{c} = 0$变成

$$|c|(e^{\mathrm{i}2\theta} - e^{\mathrm{i}(\alpha+\beta)}) = 0.$$

此时方程(0.1)变成

$$re^{\mathrm{i}\alpha}z + re^{\mathrm{i}\beta}\bar{z} + |c|e^{\mathrm{i}\theta} = 0.$$

若$c = 0$, 则方程(0.1)为

$$re^{\mathrm{i}\frac{\alpha-\beta}{2}}z + re^{\mathrm{i}\frac{\beta-\alpha}{2}}\bar{z} = 0$$

表示直线方程, 有无穷解.
若$c \neq 0$, 则$e^{\mathrm{i}2\theta} = e^{\mathrm{i}(\alpha+\beta)}$, 方程(0.1)为

$$re^{\mathrm{i}\frac{\alpha-\theta}{2}}z + re^{\mathrm{i}\frac{\beta-\theta}{2}}\bar{z} = 0$$

也表示直线方程, 有无穷解.
综合可得:
(1)$|a|^2 \neq |b|^2$时, 有唯一解$z = \frac{b\bar{c} - \bar{a}c}{|a|^2 - |b|^2}$;

(2)$|a|^2 = |b|^2 = 0, c = 0$或者$|a|^2 = |b|^2 \neq 0, \bar{a}c - b\bar{c} = 0$时, 有无穷解;
(3)$|a|^2 = |b|^2 = 0, c \neq 0$或者$|a|^2 = |b|^2 \neq 0, \bar{a}c - b\bar{c} \neq 0$时, 无解.

Problem 5. 写出复形式下的椭圆、双曲线和抛物线方程.

由$z = x + iy$,得$\begin{cases} x = \frac{z+\bar{z}}{2} \\ y = \frac{z-\bar{z}}{2i} \end{cases}$

椭圆：由$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$，得$\frac{(z+\bar{z})^2}{4a^2}-\frac{(z-\bar{z})^2}{4b^2}=1$

或$\left(b^2-a^2\right)z^2+\left(b^2-a^2\right)\bar{z}^2+2\left(b^2+a^2\right)z\bar{z}-4a^2b^2=0$

双曲线：由$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$，得$\frac{(z+\bar{z})^2}{4a^2}+\frac{(z-\bar{z})^2}{4b^2}=1$

或$\left(b^2+a^2\right)z^2+\left(b^2+a^2\right)\bar{z}^2+2\left(b^2-a^2\right)z\bar{z}-4a^2b^2=0$

抛物线：由$y^2=2px$，得$(z-\bar{z})^2+4p\left(z+\bar{z}\right)=0$或$z^2+\bar{z}^2-2z\bar{z}+4pz+4p\bar{z}=0$

由$x^2=2py$，得$(z+\bar{z})^2-4pi\left(z-\bar{z}\right)=0$或$z^2+\bar{z}^2+2z\bar{z}-4piz+4pi\bar{z}=0$

Problem 6. 证明: 在有限复平面$\mathbb{C}$上过点a and $\frac{1}{\bar{a}}$ $(a\in\mathbb{C},|a|\neq0,1)$的圆与单位圆周垂直相交.

证明:由于分式线性变换$w=f(z)=\frac{a-z}{1-\bar{a}z}$将单位圆$|z|=1$映成单位圆$|w|=1$,

将$a,\frac{1}{\bar{a}}$分别映成$0,\infty$, 故将过$a,\frac{1}{\bar{a}}$的圆映成过$w=0$的直线. 而在w平面中过$w=0$的直线与单位圆$|w|=1$正交, 由分式线性变换$z=f^{-1}(w)$的保角性知z平面上过$a,\frac{1}{\bar{a}}$的圆与单位圆$|z|=1$正交.

Problem 7. 证明: 点$z,z'\in\mathbb{C}$在球极投影下的像是直径的两个端点当且仅当$z\bar{z}'=-1$.

证明：当z和z'是位于黎曼球面上关于直径对称的两个点时，

有$d\left(z,z'\right)=\frac{\left|z-z'\right|}{\sqrt{\left(1+|z|^2\right)\left(1+|z'|^2\right)}}=1$

$\Longleftrightarrow|z-z'|^2=\left(1+|z|^2\right)\left(1+|z'|^2\right)$

$\Longleftrightarrow|z-z'|\left|\bar{z}-\bar{z'}\right|=\left(1+z\bar{z}\right)\left(1+z'\bar{z'}\right)$

$$\Longleftrightarrow z\bar{z}z'\bar{z}' + z'\bar{z} + z\bar{z}' + 1 = 0$$

$$\Longleftrightarrow \left(z\bar{z}' + 1\right)\left(z'\bar{z} + 1\right) = 0$$

$$\Longleftrightarrow z\bar{z}' = -1$$

Problem 8. 求复平面$\mathbb{C}$上以a为圆心R为半径的圆在球极投影逆映射下的像圆的半径.

解：（法一）直接计算

$$|z-a| = R \Leftrightarrow z\bar{z} - \bar{a}z - a\bar{z} + |a|^2 = R^2,$$

将$z = \frac{x_1+ix_2}{1-x_3}$代入，可得在球极投影逆映射下的像为

$$\begin{cases} -\left(a+\bar{a}\right)x_1 + i\left(a-\bar{a}\right)x_2 + \left(1+R^2-|a|^2\right)x_3 = R^2-1-|a|^2 \\ x_1^2 + x_2^2 + \left(x_3 - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 \end{cases}$$

$\therefore$此圆半径为

$$\sqrt{\tfrac{1}{4} - \tfrac{\left(R^2-1-|a|^2\right)^2}{(a+\bar{a})^2-(a-\bar{a})^2+\left(1+R^2-|a|^2\right)^2}}$$

$$= \sqrt{\tfrac{1}{4} - \tfrac{\left(R^2-1-|a|^2\right)^2}{4|a|^2+\left(1+R^2-|a|^2\right)^2}}$$

$$= \tfrac{R}{\sqrt{4|a|^2+\left(1+R^2-|a|^2\right)^2}}$$

（法二）利用距离公式

设$\varphi:\mathbb{C}_\infty\to S^2$是球极投影逆映射，则$d\left(\varphi\left(z\right),\varphi\left(z'\right)\right)=\frac{\left|z-z'\right|}{\sqrt{\left(|z|^2+1\right)\left(|z'|^2+1\right)}}$

(i)若$a\in\mathbb{R}^+$,下证象圆直径$D=d\left(\varphi\left(a-R\right),\varphi\left(a+R\right)\right)$

设$D=\sup\limits_{|z-a|=R}d\left(\varphi\left(z\right),\varphi\left(a+R\right)\right)$,$z=a+R\left(\cos\theta+i\sin\theta\right)$,$\theta\in[0,2\pi]$

则$d\left(\varphi\left(z\right),\varphi\left(a+R\right)\right)=\frac{|z-(a+R)|}{\sqrt{\left(1+|z|^2\right)\left(1+|a+R|^2\right)}}$

$=\frac{R\sqrt{(\cos\theta-1)^2+\sin^2\theta}}{\sqrt{(1+a^2+R^2+2aR\cos\theta)\left(1+|a+R|^2\right)}}$

$=\frac{R\sqrt{2-2\cos\theta}}{\sqrt{(1+a^2+R^2+2aR\cos\theta)\left(1+|a+R|^2\right)}}$

可以看到$\cos\theta=-1$,即$z=a-R$时，$d\left(\varphi\left(z\right),\varphi\left(a+R\right)\right)$取到最大值$D=\frac{2R}{\sqrt{1+|a-R|^2}\sqrt{1+|a+R|^2}}$.

(ii)若$a\in\mathbb{C}\left(a\neq0\right)$,由于球面距离在旋转作用下保持不变，这是因为$d\left(\varphi\left(\lambda z\right),\varphi\left(\lambda z'\right)\right)=d\left(\varphi\left(z\right),\varphi\left(z'\right)\right)$,$|\lambda|=1$,故可将$a$旋转至正实轴上$\left(\cdot\frac{\bar{a}}{|a|}\right)$,从而得到象圆半径为$\tilde{R}=\frac{R}{\sqrt{\left[1+(|a|-R)^2\right]\left[1+(|a|+R)^2\right]}}$.

Problem 9.设$T(z)=\frac{az-b}{\bar{b}z+\bar{a}}$是$\mathbb{C}_\infty$上的一个变换, 其中$a,b\in\mathbb{C}$满足$|a|^2+|b|^2=1$.
证明T保持$\mathbb{C}_\infty$上的球面距离不变.

作业2-解答

1. 讨论下列函数的复可微性:
(i)$f(z) = |z|$; (ii)$f(z) = \bar{z}$; (iii)$f(z) = \mathrm{Re} z$.

2. 证明: 函数$f(z) = \frac{xy}{|z|^2}$在$\mathbb{C} \setminus 0$上连续. 试考虑f是否可以连续延拓至整个复平面$\mathbb{C}$上?

3. 设$u(x, y) = e^x \cos y$, 找一个函数$v(x, y)$使得u和v满足柯西-黎曼方程.

4. 证明:若函数$f(z)$在$\mathbb{C}$上全纯, 则$\overline{f(\bar{z})}$也在$\mathbb{C}$上全纯; 反之亦然.

证明：令$f(z) = u(x, y) + \mathtt{i}v(x, y)$,则$\overline{f(\bar{z})} = u(x, -y) - \mathtt{i}v(x, -y)$

由$f(z)$在$\mathbb{C}$上全纯知：$u(x, y)$,$v(x, y)$在$\mathbb{R}$上实可微，且$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

由复合函数的性质可知：：$u(x, -y)$,$-v(x, -y)$在$\mathbb{R}$上实可微，

且$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial(-v)}{\partial(-y)}, \frac{\partial u}{\partial(-y)} = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = -\frac{\partial(-v)}{\partial x}$.

$\therefore$ $\overline{f(\bar{z})}$也在$\mathbb{C}$上全纯.

同理可证，若函数$\overline{f(\bar{z})}$在$\mathbb{C}$上全纯, 则$f(z)$也在$\mathbb{C}$上全纯.

5.下面这族映射在复分析中起着重要的作用.
(1) 设z, w 是两个复数满足$\bar{z}w \neq 1$. 证明

$$\left| \frac{w - z}{1 - \bar{w}z} \right| < 1 \text{ if } |z| < 1 \text{ and } |w| < 1,$$

且

$$\left| \frac{w - z}{1 - \bar{w}z} \right| = 1 \text{ if } |z| = 1 \text{ and } |w| = 1.$$

证明：要证$|\frac{w-z}{1-\bar{w}z}| < 1$,只需证$|w - z| < |1 - \bar{w}z|$.

即证$(w - z)(\bar{w} - \bar{z}) < (1 - \bar{w}z)(1 - w\bar{z})$

即证$|w|^2+|z|^2<1+|wz|^2$

由$|z|<1$,$|w|<1$可知$(1-|w|^2)(1-|z|^2)>0$,即$1-|w|^2-|z|^2+|w|^2|z|^2>0$,

$\therefore |w|^2+|z|^2<1+|wz|^2$

当$|z|=1$且$|w|=1$时, $|\frac{w-z}{1-\bar{w}z}|=|\frac{w-z}{z(\bar{z}-\bar{w})}|=1$

(2)证明:对于确定的$w\in\mathbb{D}=\{z\in\mathbb{C},|z|<1\}$, 映射

$$F:z\mapsto\frac{w-z}{1-\bar{w}z}$$

满足下面的条件:
(i)F 将开圆盘映到开圆盘, 且在$\mathbb{D}$上复可微.
(ii)$F(0)=w$ and $F(w)=0$.
(iii)$|F(z)|=1$ if $|z|=1$.
(iv)$F:\mathbb{D}\to\mathbb{D}$ 是一个双射.

证明：(i)由(1)知，当$|z|<1$且$|w|<1$时,$|F(z)|<1$.

设$F(z)=\frac{w-z}{1-\bar{w}z}=c\in\mathbb{D}$,则$z=\frac{w-c}{1-\bar{w}c}\in\mathbb{D}$

$\therefore F:\mathbb{D}\to\mathbb{D}$.

$\because \omega\in\mathbb{D}$, $\therefore \frac{1}{\bar{\omega}}\notin\mathbb{D}$.

$\therefore F(z)=\frac{w-z}{1-\bar{w}z}=\frac{w-z}{\bar{w}(\frac{1}{\bar{w}}-z)}$ 在$\mathbb{D}$上处处复可微.

(ii)由定义立即可得$F(0)=w$,$F(w)=0$

(iii)当$|z|=1$时,$z\bar{z}=1$, $F(z)=\frac{w-z}{1-\bar{w}z}=\frac{w-z}{z\bar{z}-\bar{w}z}=\frac{w-z}{z(\bar{z}-\bar{w})}$

$\therefore |F(z)|=\frac{1}{|z|}|\frac{w-z}{\bar{z}-\bar{w}}|=1$

(iv)$\because F\circ F(z)=\frac{w-\frac{w-z}{1-\bar{w}z}}{1-\bar{w}\frac{w-z}{1-\bar{w}z}}=z$,

$\therefore F : \mathbb{D} \to \mathbb{D}$是双射.

6.考虑极坐标(r, θ) 使得$x = r\cos\theta$ 且$y = r\sin\theta$, 从而

$$z = x + \mathtt{i}y = r(\cos\theta + \mathtt{i}\sin\theta).$$

证明:在极坐标(r, θ)下, 柯西黎曼方程表示如下

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

利用这些方程证明如下定义的对数函数

$$\log z = \log r + \mathtt{i}\theta \text{ with } -\pi < \theta < \pi$$

在区域$\Omega = \{r > 0, -\pi < \theta < \pi\}$上复可微.

证明：由复合函数求偏导法则可知：

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}$.

$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}$.

$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial v}{\partial x} + \sin\theta\frac{\partial v}{\partial y}$.

$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial v}{\partial x} + r\cos\theta\frac{\partial v}{\partial y}$.

应用$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$立即可得$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$.

下面说明:设$f(z) = u(r, \theta) + \mathrm{i}v(r, \theta), z = x + \mathrm{i}y = re^{\mathrm{i}\theta}$, 若$u(r, \theta), v(r, \theta)$在点$(r, \theta)$可微, 且满足以上极坐标的柯西-黎曼方程, 则$f(z)$在点$z$可微. 由于$x = x(r, \theta), y = y(r, \theta)$在$(r, \theta)$具有连续一阶偏导, 且雅可比

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = r > 0,$$

由隐函数定理知, 存在$r = r(x, y), \theta = \theta(x, y)$在$(x, y)$有一阶连续偏导,且

$$\frac{\partial r}{\partial x} = \cos\theta, \frac{\partial r}{\partial y} = \sin\theta, \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}, \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r},$$

故u, v在(x, y)可微且满足

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

因此$f(z)$在点z可微.

设$\log z = u + \mathtt{i}v$,则$\frac{\partial u}{\partial r} = \frac{1}{r}$,$\frac{\partial v}{\partial \theta} = 1$,$\frac{\partial u}{\partial \theta} = 0$,$\frac{\partial v}{\partial r} = 0$.

满足$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$,$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$ 且$\frac{\partial u}{\partial r}$,$\frac{\partial v}{\partial \theta}$,$\frac{\partial u}{\partial \theta}$,$\frac{\partial v}{\partial r}$在区域$r > 0$,$-\pi < \theta < \pi$上连续,

$\therefore \log z$在区域$r > 0$,$-\pi < \theta < \pi$上可微.

7.设$f = u + \mathrm{i}v$在$\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$上复可微. 证明: 若$f$满足下面任一条件则在$\mathbb{D}$上恒为常值.
(i)Ref在$\mathbb{D}$上恒为常值;
(ii)Imf在$\mathbb{D}$上恒为常值;
(iii)$|f|$在$\mathbb{D}$上恒为常值;

证明：设$f = u + \mathtt{i}v$,由f复可微可知,u,v实可微且满足$u_x = v_y$,$u_y = -v_x$.

(i)若$u = C$,则$u_x = u_y = 0$,$\therefore v_y = u_x = 0$,$v_x = -u_y = 0$.

$\therefore u$,v在$\mathbb{D}$内为常数.故f为常数.

(ii)若$v = C$,则$v_x = v_y = 0$,$\therefore u_x = v_y = 0$,$u_y = -v_x = 0$.

$\therefore u$,v在$\mathbb{D}$内为常数.故f为常数.

(iii)若$|f| = C = 0$,显然有$f = 0$.

若$|f| = C \neq 0$,则$f \neq 0$,$\therefore u^2 + v^2 = C^2 \neq 0$,

分别对x，y微分,再应用C.-R.方程,可得$\begin{cases} vv_x + uv_y = 0 \\ -uv_x + vv_y = 0 \end{cases}$

此二元一次齐次方程组系数矩阵行列式不为0,

故$v_x = v_y = 0$,$\therefore u_x = v_y = 0$,$u_y = -v_x = 0$.

$\therefore u$,v在$\mathbb{D}$内为常数.故f为常数.综上,f为常数.

8. (1)设Q 是多项式有n个不同根$\alpha_1, \cdots, \alpha_n$, 且$P$ 是度$< n$的多项式, 证明:

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)}.$$

证明：设

$$Q(z) = \alpha \prod_{i=1}^{n}(z-\alpha_i),$$

则

$$Q'(z) = \alpha \sum_{j=1}^{n} \prod_{i \neq j}^{n}(z-\alpha_i),$$

$\therefore$

$$Q'(\alpha_k) = \alpha \prod_{i \neq k}^{n}(\alpha_k-\alpha_i).$$

$\therefore$

$$P(z) - \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)} Q(z) = \sum_{k=1}^{n} \frac{P(\alpha_k)\prod_{i=1}^{n}(z-\alpha_i)}{\prod_{i\neq k}^{n}(\alpha_k-\alpha_i)(z-\alpha_k)} = \sum_{k=1}^{n} \frac{P(\alpha_k)\prod_{i\neq k}^{n}(z-\alpha_i)}{\prod_{i\neq k}^{n}(\alpha_k-\alpha_i)}$$

当$z = \alpha_1, \cdots, \alpha_n$时,上式等于0,即$\alpha_1, \cdots, \alpha_n$是上式的n个根.

又$\because$ `deg`$P < n$，故上式的次数$< n$,且有n个根.

故上式恒为0,即

$$P(z) = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)} Q(z),$$

$\therefore$

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)}.$$

(2)利用(1) 中的公式证明: 对于给定的复数c_k, 存在唯一的度小于n的多项式P满足$P(\alpha_k) = c_k$.

证明：由(1)知

$$P(z)=\sum_{k=1}^{n}\frac{c_k\prod_{i\neq k}^{n}(z-\alpha_i)}{\prod_{i\neq k}^{n}(\alpha_k-\alpha_i)},$$

且满足$P(\alpha_k)=c_k,k=,\cdots,n$,故存在性得证.

下证唯一性:若有$R(z)$满足$R(\alpha_k)=c_k,k=,\cdots,n$,则由(1)可知

$$R(z)=\sum_{k=1}^{n}\frac{R(\alpha_k)Q(z)}{Q'(\alpha_k)(z-\alpha_k)}=\sum_{k=1}^{n}\frac{c_kQ(z)}{Q'(\alpha_k)(z-\alpha_k)}=\sum_{k=1}^{n}\frac{c_k\prod_{i\neq k}^{n}(z-\alpha_i)}{\prod_{i\neq k}^{n}(\alpha_k-\alpha_i)}=P(z)$$

综上,$P(z)$存在且唯一.

9.设有理函数$R(z)$满足：当$|z|=1$时$|R(z)|=1$,讨论$R(z)$的零点和极点怎样分布？给出$R(z)$的一般形式.

解:(i)设有理函数$R(z)$满足:当$|z|=1$时,$|R(z)|^2=1$,则有理函数$M(z)=R(z)\overline{R(\frac{1}{\bar{z}})}$满足:当$|z|=1$时,$M(z)=1$.

因为非常值的有理函数取每一值有限次,故$M(z)=\text{const}$,即$M(z)=1$.

从而有$R(\frac{1}{\bar{z}})=\frac{1}{\overline{R(z)}}.\forall z\in\mathbb{C}$

这表明,z是$R(z)$的k阶零点$\Leftrightarrow$ $\frac{1}{\bar{z}}$是$R(z)$的k阶极点.(若z在单位圆盘内,则$\frac{1}{\bar{z}}$在单位圆盘外).

(ii)设$(a_n)_{0\leq n\leq N}$是$R(z)$在单位圆盘内的相异的零点和极点,其阶为m_n(若是零点,则$m_n>0$,若是极点,则$m_n<0$).不妨设$a_n=0$(其中$m_0=0$是可能的),

则

$$S(z)=z^{m_0}\prod_{n=1}^{N}(\frac{z-a_n}{1-\overline{a_n}z})^{m_n}$$

是一个有理函数，且与$R(z)$有相同的零点和极点,并满足当$|z|=1$时,$|S(z)|=1$.

因此,$\frac{R(z)}{S(z)}$是一个无零点或极点的有理函数，故为常数.

$\therefore$

$$R(z) = \lambda S(z) = \lambda z^{m_0} \prod_{n=1}^{N} (\frac{z - a_n}{1 - \overline{a_n} z})^{m_n}.$$

其中$\lambda \in \mathbb{C}$,$|\lambda| = 1$.

10.将有理函数$R(z) = \frac{1}{z(z+1)^2(z+2)^3}$展开成部分分式之和.

作业3-解答

1.讨论全纯函数列$\{f_n(z) = nz^n, n \geq 1\}$的收敛性与一致收敛性.

2.若$\lim\limits_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = R$, 试证明$\sum\limits_{n=0}^{\infty} a_n z^n$的收敛半径为$R$.

3. 决定幂级数$\sum\limits_{n=1}^{\infty} a_n z^n$的收敛半径, 其中:

(1) $a_n = (\log n)^2$

(2) $a_n = n!$

(3) $a_n = \frac{n^2}{4^n+3n}$

(4) $a_n = \frac{(n!)^3}{(3n)!}$

Hint: $n! \sim cn^{n+\frac{1}{2}}e^{-n}\ c > 0$.

(5)求下面超几何级数的收敛半径

$$F(\alpha, \beta, \gamma; z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)} z^n.$$

Here $\alpha, \beta \in \mathbb{C}$ and $\gamma \neq 0, -1, -2, \cdots$.

(6)求下面r阶$Bessel\ function$的收敛半径:

$$J_r(z) = \left(\frac{z}{2}\right)^r \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{z}{2}\right)^{2n},$$

其中r 是整数.

解：(1)当$n \to \infty$时,有 $\sqrt[n]{1} \leq \sqrt[n]{(logn)^2} \leq \sqrt[n]{n^2}$,

又$\because \lim_{n\to\infty} \sqrt[n]{n^2} = 1$,$\therefore \lim_{n\to\infty} \sqrt[n]{(logn)^2} = 1$.

$\therefore$

$$R = \frac{1}{\lim_{n\to\infty} sup \sqrt[n]{|a_n|}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{(logn)^2}} = 1.$$

(2)$\because$

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{(n+1)!}{(n)!}=\lim_{n\to\infty}(n+1)=+\infty$$

由Problem 2.可知,

$$R=0.$$

(3)$\because$

$$\lim_{n\to\infty}\sqrt[n]{\frac{n^2}{4^n+3n}}=\lim_{n\to\infty}\frac{\sqrt[n]{n^2}}{\sqrt[n]{4^n+3n}}=\frac{1}{4}$$

$\therefore$

$$R=\frac{1}{\lim_{n\to\infty} sup\sqrt[n]{|a_n|}}=\frac{1}{\lim_{n\to\infty}\sqrt[n]{\frac{n^2}{4^n+3n}}}=4.$$

(4)由Stirling公式可知,

$$\lim_{n\to\infty}\sqrt[n]{\frac{(n!)^3}{(3n)!}}=\lim_{n\to\infty}\sqrt[n]{\frac{(cn^{n+\frac{1}{2}}e^{-n})^3}{c(3n)^{3n+\frac{1}{2}}e^{-3n}}}=\lim_{n\to\infty}\sqrt[n]{\frac{c^2n}{3^{3n+\frac{1}{2}}}}=\lim_{n\to\infty}\frac{\sqrt[n]{c^2n}}{\sqrt[n]{3^{3n+\frac{1}{2}}}}=\frac{1}{27}$$

$\therefore$

$$R=\frac{1}{\lim_{n\to\infty} sup\sqrt[n]{|a_n|}}=\frac{1}{\lim_{n\to\infty}\sqrt[n]{\frac{(n!)^3}{(3n)!}}}=27.$$

(5)$\because$

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)(\alpha+n)\beta(\beta+1)\cdots(\beta+n-1)(\beta+n)}{(n+1)!\gamma(\gamma+1)\cdots(\gamma+n-1)(\gamma+n)}}{\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)}}=\lim_{n\to\infty}\frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)}=1$$

由Problem 2.可知,

$$R = 1.$$

(6)

$$J_r(z) = \left(\frac{z}{2}\right)^r \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{z}{2}\right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!2^{2n+r}} z^{2n+r}$$

$\because$

$$\lim_{n\to\infty} \sqrt[n]{\frac{(-1)^n}{n!(n+r)!2^{2n+r}}} = 0$$

(根据$\lim_{n\to\infty} \sqrt[n]{n!} = +\infty$)

$\therefore$

$$R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{\frac{(-1)^n}{n!(n+r)!2^{2n+r}}}} = +\infty.$$

4.确定$\sum\limits_{n=0}^{\infty} \left(\frac{z}{1+z}\right)^n$ 的收敛范围.(答案:$\{Rez > \frac{-1}{2}\}$).

5.若$f(z) = \sum\limits_{n=0}^{\infty} a_n z^n,\ |z| < R$, 求$\sum\limits_{n=0}^{\infty} n^3 a_n z^n$.

6.若$f(z) = \sum\limits_{n=0}^{\infty} a_n z^n,\ a_n \in \mathbb{R},\ |z| < R$, 且$f(-z) = f(z)$, 证明$f(z)$在虚轴上取实值.

7. 设$f(z) = \sum\limits_{n=0}^{\infty} a_n z^n,\ |z| < R$, 证明$f(z)$ 在收敛圆盘的任一点处有幂级数展开.
Hint: 考虑$z = z_0 + (z - z_0)$,

$$z^n = (z_0 + (z - z_0))^n = \sum_{k=0}^{n} \binom{n}{k} z_0^k (z - z_0)^{n-k},$$

其中$\binom{n}{k}=\frac{n!}{k!(n-k)!}$. 注意幂级数的重排.

证明:

法一:(注意:实数项级数绝对收敛的重排定理可以推广到复数项级数)

设

$$f(z)=\sum_{n=0}^{+\infty}a_nz^n=\sum_{n=0}^{+\infty}\sum_{k=0}^{n}a_n\binom{n}{k}z_0^k(z-z_0)^{n-k},|z|<R$$

若$|z-z_0|<R-|z_0|$,则$|z-z_0|+|z_0|<R$,

$$\sum_{n=0}^{+\infty}\sum_{k=0}^{n}|a_n\binom{n}{k}z_0^k(z-z_0)^{n-k}|=\sum_{n=0}^{+\infty}\sum_{k=0}^{n}|a_n|\binom{n}{k}|z_0|^k|z-z_0|^{n-k}=\sum_{n=0}^{+\infty}|a_n|(|z_0|+|z-z_0|)^n$$

$\because \sum_{n=0}^{+\infty}a_nz^n$在$\{|z|<R\}$内绝对收敛,故上式收敛.

$\therefore$

$$\sum_{n=0}^{+\infty}\sum_{k=0}^{n}a_n\binom{n}{k}z_0^k(z-z_0)^{n-k}$$

在$\{|z-z_0|<R-|z_0|\}$上绝对收敛,可以重排.

从而有

$$f(z)=\sum_{n=0}^{+\infty}\sum_{k=0}^{n}a_n\binom{n}{k}z_0^k(z-z_0)^{n-k}=\sum_{k=0}^{+\infty}\sum_{n=k}^{+\infty}a_n\binom{n}{k}z_0^k(z-z_0)^{n-k}=\sum_{m=0}^{+\infty}(\sum_{n=m}^{+\infty}a_n\binom{n}{m}z_0^{n-m})(z-z_0)^m.$$

法二:利用全纯函数泰勒展开的证明过程.

由

$$f(z)=\sum_{n=0}^{+\infty}a_nz^n,|z|<R$$

可知$f(z)$在$D_R(0)$上全纯.

固定$z_0\in D_R(0)$,取$r<R-|z_0|$,则$\overline{D_r(z_0)}\subset D_R(0)$.

由柯西积分公式,对任意$z\in D_r(z_0)$.

$$f(z)=\frac{1}{2\pi\mathtt{i}}\int_{|\xi-z_0|=r}\frac{f(\xi)}{\xi-z}d\xi=\frac{1}{2\pi\mathtt{i}}\int_{|\xi-z_0|=r}f(\xi)\sum_{n=0}^{+\infty}\frac{(z-z_0)^n}{(\xi-z_0)^{n+1}}d\xi$$

$$=\sum_{n=0}^{+\infty}(\frac{1}{2\pi\mathtt{i}}\int_{|\xi-z_0|=r}\frac{f(\xi)}{(\xi-z_0)^{n+1}}d\xi)(z-z_0)^n.$$

8. (1)设$z=x+\mathtt{i}y$, 证明: $|y|\leq|\sin z|\leq e^{|y|}$.
(2)求$2^{\mathtt{i}},\mathtt{i}^{\mathtt{i}},(-1)^{2\mathtt{i}}$的值.
(3)设全纯函数$f(z)$是$z^{\frac{1}{3}}$在区域$\mathbb{C}\setminus\{z\leq 0\}$内的一个单值全纯分支, 且$f(\mathtt{i})=-\mathtt{i}$, 求$f(-\mathtt{i})$的值.

解：(1)证明$\because sinz=sin(x+\mathtt{i}y)=sinxcos\mathtt{i}y+cosxsin\mathtt{i}y=sinxcoshy+\mathtt{i}cosxsinhy$

$\therefore|sinz|=\sqrt{sin^2xcosh^2y+cos^2xsinh^2y}=\sqrt{sin^2xcosh^2y+(1-sin^2x)sinh^2y}=\sqrt{sin^2x+sinh^2y}$

$\therefore|sinz|=\sqrt{sin^2x+sinh^2y}\geq\sqrt{sinh^2y}=|sinhy|\geq|y|$

$\therefore |sinz| = \sqrt{sin^2x + sinh^2y} \leq \sqrt{sin^2x + cos^2x + sinh^2y} = |\frac{e^y + e^{-y}}{2}| \leq e^{|y|}$

综上,$|y| \leq |\sin z| \leq e^{|y|}$.

$(2) 2^{\mathtt{i}} = e^{\mathtt{i}log2} = e^{\mathtt{i}(log2 + \mathtt{i}2k\pi)} = e^{\mathtt{i}log2} \cdot e^{-2k\pi} (k \in \mathbb{Z})$

$\mathtt{i}^{\mathtt{i}} = e^{\mathtt{i}log\mathtt{i}} = e^{\mathtt{i}[(log|\mathtt{i}| + \mathtt{i}(\frac{\pi}{2} + 2k\pi)]} = e^{-\frac{\pi}{2} - 2k\pi} (k \in \mathbb{Z})$

$(-1)^{2\mathtt{i}} = e^{2\mathtt{i}log(-1)} = e^{2\mathtt{i}[(log|-1| + \mathtt{i}(\pi + 2k\pi)]} = e^{-2\pi - 4k\pi} (k \in \mathbb{Z})$

$(3) f(z) = z^{\frac{1}{3}} = e^{\frac{1}{3}logz} = e^{\frac{1}{3}[(log|z| + \mathtt{i}(argz + 2k\pi)]} = e^{\frac{log|z|}{3}} e^{\mathtt{i}(\frac{argz}{3} + \frac{2k\pi}{3})} (k \in \mathbb{Z})$

由$f(\mathtt{i}) = e^{\frac{log|\mathtt{i}|}{3}} e^{\mathtt{i}(\frac{arg\mathtt{i}}{3} + \frac{2k\pi}{3})} = e^{\mathtt{i}(\frac{\pi}{6} + \frac{2k\pi}{3})} = -\mathtt{i} = e^{\mathtt{i}\frac{3\pi}{2}}$可知$k = 2$

$\therefore f(z) = e^{\frac{log|z|}{3}} e^{\mathtt{i}(\frac{argz}{3} + \frac{4\pi}{3})}$

$\therefore f(-\mathtt{i}) = e^{\frac{log|-\mathtt{i}|}{3}} e^{\mathtt{i}(\frac{arg(-\mathtt{i})}{3} + \frac{4\pi}{3})} = e^{\mathtt{i}(-\frac{\pi}{6} + \frac{4\pi}{3})} = e^{\frac{7\pi}{6}\mathtt{i}} = -\frac{\sqrt{3}}{2} - \frac{1}{2}\mathtt{i}$

作业4-解答

1. (1) 计算

$$\int_\gamma x dz,$$

其中γ 是从0 到$1+\mathrm{i}$的直线段.

(2) 计算

$$\int_{|z|=r} x dz,$$

其中$|z|=r$为正向圆周(后面未特别说明的闭曲线均取正向). (答案:$\mathrm{i}\pi r^2$)

解:**(1)**γ的参数方程为:$z=(1+\mathtt{i})t, t\in[0,1]$.故

$$\int_\gamma x dz = \int_0^1 t(1+\mathtt{i})dt = \left.\frac{(1+\mathtt{i})t^2}{2}\right|_0^1 = \frac{(1+\mathtt{i})}{2}$$

2. (1) 计算

$$\int_{|z|=1} \frac{e^z}{z} dz.$$

(答案:$2\pi\mathrm{i}$)

(2) 计算

$$\int_{|z|=2} \frac{1}{z^2+1} dz.$$

(答案: 0)

3. (1) 计算

$$\int_{|z|=1} |z-1||dz|.$$

(答案: 8)

(2) 计算

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2},$$

其中$|a| \neq \rho$. *Hint:* 利用$z\bar{z} = \rho^2$ 和$|dz| = -\mathrm{i}\rho\frac{dz}{z}$.

解: **(2)**

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \int_{|z|=\rho} \frac{-\mathtt{i}\rho}{z(z-a)(\bar{z}-\bar{a})}dz = \int_{|z|=\rho} \frac{-\mathtt{i}\rho}{(\rho^2-\bar{a}z)(z-a)}dz = -\mathtt{i}\rho\int_{|z|=\rho} \frac{\frac{1}{\rho^2-\bar{a}z}}{z-a}dz$$

若$|a| < \rho$,令$f(z) = \frac{1}{\rho^2-\bar{a}z}$,由柯西积分公式可得

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = -\mathtt{i}\rho\int_{|z|=\rho} \frac{\frac{1}{\rho^2-\bar{a}z}}{z-a}dz = -\mathtt{i}\rho\cdot 2\pi\mathtt{i}f(a) = \frac{2\pi\rho}{\rho^2-|a|^2}$$

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \int_{|z|=\rho} \frac{-\mathtt{i}\rho}{z(z-a)(\bar{z}-\bar{a})}dz = \int_{|z|=\rho} \frac{-\mathtt{i}\rho}{(\rho^2-\bar{a}z)(z-a)}dz = -\mathtt{i}\rho\int_{|z|=\rho} \frac{\frac{1}{\bar{a}(a-z)}}{z-\frac{\rho^2}{\bar{a}}}dz$$

若$|a| > \rho$,有$\frac{\rho^2}{|\bar{a}|} < \rho$,令$f(z) = \frac{1}{\bar{a}(a-z)}$,由柯西积分公式可得

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = -\mathtt{i}\rho\int_{|z|=\rho} \frac{\frac{1}{\bar{a}(a-z)}}{z-\frac{\rho^2}{\bar{a}}}dz = -\mathtt{i}\rho\cdot 2\pi\mathtt{i}f(\frac{\rho^2}{\bar{a}}) = \frac{2\pi\rho}{|a|^2-\rho^2}$$

4.设$f(z)$ 是包含闭曲线γ的区域上的全纯函数. 证明

$$\int_\gamma \overline{f(z)}f'(z)dz$$

是纯虚数.(这里假定$f'(z)$是连续的)

证明:设$f(z) = u + \mathtt{i}v$,则

$$\int_\gamma \overline{f(z)}f'(z)dz + \overline{\int_\gamma \overline{f(z)}f'(z)dz} = \int_\gamma \overline{f(z)}d(f(z)) + \int_\gamma f(z)\overline{d(f(z))}$$

$$= \int_\gamma (u-\mathtt{i}v)d(u+\mathtt{i}v) + \int_\gamma (u+\mathtt{i}v)d(u-\mathtt{i}v) = 2\int_\gamma udu + 2\int_\gamma vdv = 0$$

故$\int_\gamma \overline{f(z)}f'(z)dz$为纯虚数.

5.设Ω是一个区域, $f(z)\in\mathcal{H}(\Omega)$ 且满足$|f(z)-1|<1$. 证明

$$\int_\gamma \frac{f'(z)}{f(z)}dz=0$$

其中γ是Ω中的任一闭曲线.(这里假定$f'(z)$是连续的)

6. 设$P(z)$ 是一个多项式, 计算

$$\int_{|z-a|=R} P(z)d\bar{z}.$$

Answer: $-2\pi\mathrm{i}R^2P'(a)$.

解:设$\omega=z-a$,则$\bar{\omega}=\bar{z}-\bar{a}, d\bar{z}=d\bar{\omega}, dz=d\omega$.

由$\omega\bar{\omega}=R^2$可得$\bar{\omega}=\frac{R^2}{\omega}, d\bar{\omega}=-\frac{R^2}{\omega^2}d\omega$.故

$$\int_{|z-a|=R} P(z)d\bar{z}=\int_{|z-a|=R}\frac{-R^2P(z)}{(z-a)^2}dz$$

由于P(z)在$\mathbb{C}$上全纯,故有

$$P'(a)=\frac{1}{2\pi\mathtt{i}}\int_{|z-a|=R}\frac{P(z)}{(z-a)^2}dz$$

$\therefore$

$$\int_{|z-a|=R} P(z)d\bar{z}=-2\pi\mathtt{i}R^2P'(a).$$

7. 开集$\Omega\subset\mathbb{R}^2$(坐标为$x,y$)上的调和函数$u$定义为2次连续可微函数$u(x,y)$且在$\Omega$上满足

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=0.$$

(1) 若u 是开集$\Omega\subset\mathbb{C}$上的实值调和函数, 证明$\frac{\partial u}{\partial z}$ 是Ω上的全纯函数. 其中微分

算子$\frac{\partial}{\partial z}$ 定义为

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-\mathrm{i}\frac{\partial}{\partial y}\right).$$

(2) 设$\mathbb{D}=\{|z|<1\}$, $u(x,y)$是$\mathbb{D}$ 上的一个调和函数. 证明$f(z)=2\int_{C_z}\frac{\partial u}{\partial z}dz$是$\mathbb{D}$上良好定义的全纯函数, 且$u(x,y)$与$f(z)$的实部相差一个实常数. 其中$C_z$是$\mathbb{D}$中只包含水平和垂直线段的多边形路径.

证明:**(1)**$\because$

$$\frac{\partial^2 u}{\partial z\partial x}+\mathtt{i}\frac{\partial^2 u}{\partial z\partial y}=\frac{1}{2}(\frac{\partial^2 u}{\partial x^2}-\mathtt{i}\frac{\partial^2 u}{\partial x\partial y})+\mathtt{i}\cdot\frac{1}{2}(\frac{\partial^2 u}{\partial x\partial y}-\mathtt{i}\frac{\partial^2 u}{\partial y^2})=\frac{1}{2}(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})=0$$

故$\frac{\partial^2 u}{\partial z\partial x}=\frac{1}{\mathtt{i}}\frac{\partial^2 u}{\partial z\partial y}$,

$\because u$的2阶导数是连续的,

$\therefore \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right)$, $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right)$在$\Omega$上连续,

$\therefore \frac{\partial u}{\partial z}$在$\Omega$上全纯.

(2)固定某个$z_0\in\mathbb{D}$,设C_z为从z_0到z的多边形路径,$z=x+\mathtt{i}y, z_0=x_0+\mathtt{i}y_0$.定义$f(z)=2\int_{C_z}\frac{\partial u}{\partial z}dz$,则

$$f(z)=\int_{C_z}(\frac{\partial u}{\partial x}-\mathtt{i}\frac{\partial u}{\partial y})d(x+\mathtt{i}y)=\int_{C_z}(\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy)+\mathtt{i}\int_{C_z}(\frac{\partial u}{\partial x}dy-\frac{\partial u}{\partial y}dx)$$

故

$$Ref(z)=\int_{C_z}(\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy)=u(x,y)-u(x_0,y_0)=u(x,y)+C$$

其中C为常数.

作业5-解答

1. 设$g(\zeta)$是分段光滑曲线γ上的连续函数. 对于任意自然数$n \in \mathbb{N}$, 定义

$$F_n(z) \triangleq \int_\gamma \frac{g(\zeta)}{(\zeta - z)^n} d\zeta.$$

证明:对任意$n \in \mathbb{N}$, $F_n(z)$在$\mathbb{C} \setminus \gamma$上全纯, 且$F_n'(z) = nF_{n+1}(z)$. *Hint:*固定点$z_0 \in \mathbb{C} \setminus \gamma$, 先说明$F_1(z)$在点$z_0$连续,再说明$F_1(z)$ 在点z_0复可微且$F_1'(z_0) = F_2(z_0)$; 最后利用归纳假设考虑$F_n(z)$.

2. 设$f(z)$是整函数, 满足对于某个正整数n及充分大的$|z|$有$|f(z)| < |z|^n$. 证明:$f(z)$是一个多项式.

3. 设$f(z)$在$\overline{D_R(0)} = \{z \in \mathbb{C}, |z| \le R\}$上全纯, 对任意$z \in \overline{D_R(0)}$有$|f(z)| \le M$, 其中$M$是正常数. 求$|f^{(n)}(z)|$在$\overline{D_\rho(0)} \subset D_R(0)$上的上界.

4.设$\mathbb{D} = \{z \in \mathbb{C}, |z| \le 1\}$, $f(z)$在$\mathbb{D}$上全纯且满足$|f(z)| \le \frac{1}{1-|z|}$. 求$|f^{(n)}(0)|$的最佳估计.

证明:由柯西积分公式有

$$f^{(n)}(0) = \frac{n!}{2\pi \mathtt{i}} \int_{|z|=r} \frac{f(z)}{z^{n+1}} \mathrm{d}z$$

其中$0 < r < 1$,于是利用积分不等式

$$\left| f^{(n)}(0) \right| \le \frac{n!}{2\pi} \int_{|z|=r} \frac{|f(z)|}{|z|^{n+1}} |\mathrm{d}z| \le \frac{n!}{2\pi} \int_{|z|=r} \frac{|\mathrm{d}z|}{(1-|z|)|z|^{n+1}} = \frac{n!}{(1-r)r^n}$$

设$g(r) = r^n(1-r)$,由$g'(r) = r^{n-1}\left[n - (n+1)r\right] = 0$可知,
当$r = \frac{n}{n+1}$时,$g(r)$取最大值,因此

$$\left| f^{(n)}(0) \right| \le (n+1)!(1+\frac{1}{n})^n \quad (n = 1, 2, \cdots)$$

5.设U是$\mathbb{C}$中的区域, $f(z) \in U$, 固定点$z_0 \in U$. 说明$f(z)$在点z_0的导数不可能满足: 对任意正整数n均有$|f^{(n)}(z_0)| > n!n^n$.

6. 设$f(z)$ 是去心单位圆盘$\mathbb{D} \setminus \{0\} = \{z \in \mathbb{C},\ 0 < |z| < 1\}$ 上的全纯函数, 且在$\mathbb{D} \setminus \{0\}$上是平方可积的, 即满足

$$\int_{\mathbb{D}\setminus\{0\}} |f(z)|^2 dx \wedge dy < +\infty.$$

(1) 通过完成下面三步(i), (ii), (iii)的细节证明: 对任意$0 < r_0 < 1$, 有$\int_{|z|=r_0} f(z)dz = 0$.

(i)$\int_{|z|=r} f(z)dz$ 与r $(0 < r \le r_0)$选取无关.

(ii)$\int_{|z|=r_0} f(z)dz = \frac{1}{r_2 - r_1} \int_{r_1<|z|<r_2} \mathrm{i}e^{\mathrm{i}\theta} f(re^{\mathrm{i}\theta}) r dr d\theta$ for $0 < r_1 < r_2 < r_0$.

(iii)利用$\int_{\frac{r_2}{2}<|z|<r_2} |f(z)|^2 dx \wedge dy \to 0$ for $0 < r_2 < r_0$ as $r_2 \to 0$ 和下面的Hölder's 不等式: 若$1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ 且

$$\int_\Omega |f|^p dx \wedge dy < +\infty, \ \int_\Omega |g|^q dx \wedge dy < +\infty,$$

则

$$\left| \int_\Omega f g dx \wedge dy \right| \le \left(\int_\Omega |f|^p dx \wedge dy \right)^{\frac{1}{p}} \left(\int_\Omega |g|^q dx \wedge dy \right)^{\frac{1}{q}}.$$

(2) 设$n \in \mathbb{N} \cup \{0\}$, 在**(1)**中用$f(z)z^n$ 替换$f(z)$证明: $\int_{|z|=r} f(z)z^n dz = 0$ for $0 < r < 1$.

(3) 假定下面的傅里叶级数展开定理成立: 若$g(\theta)$ 是$\mathbb{R}$上的复值连续可微函数, 周期为2π (即, $g(\theta + 2\pi) = g(\theta)$ for $\theta \in \mathbb{R}$), 则

$$g(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{\mathrm{i}n\theta},$$

其中

$$c_n = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} g(\theta)e^{-\mathrm{i}n\theta}d\theta,\ n\in\mathbb{Z}.$$

利用**(2)** 证明: 若$f(z)$ 是$\mathbb{D}\setminus\{0\}$ 上的平方可积全纯函数, 则$f(z)$ 在$\mathbb{D}\setminus\{0\}$上可表示为幂级数, 因此可以全纯延拓到整个$\mathbb{D}$.

Hint: 固定$0<r<1$ 考虑$\mathbb{R}$ 上周期为2π 的复值函数$g(\theta)=f(re^{\mathrm{i}\theta})$的傅里叶级数展开.

证明:**(1)**设$0<r_1<r_2\le r_0$, 对$f(z)$在$\Omega=\{r_1\le|z|\le r_2\}$上用柯西——古萨定理得

$$\int_{\partial\Omega} f(z)dz = 0$$

从而有

$$\int_{|z|=r_2} f(z)dz + \int_{\{|z|=r_1\}^-} f(z)dz = 0.$$

即

$$\int_{|z|=r_2} f(z)dz = \int_{|z|=r_1} f(z)dz.$$

故$\int_{|z|=r_0} f(z)dz = 0$与$0<r<r_0$无关.

令$z=re^{\mathrm{i}\theta}$,对$0<r_1<r_2<r_0$,有

$$\begin{aligned}\int_{|z|=r_0} f(z)dz &= \int_0^{2\pi} f(re^{\mathrm{i}\theta})\mathtt{i}e^{\mathrm{i}\theta}d\theta = \int_{r_1}^{r_2}\frac{1}{r_2-r_1}dr\int_0^{2\pi} f(re^{\mathrm{i}\theta})\mathtt{i}e^{\mathrm{i}\theta}d\theta\\ &= \frac{1}{r_2-r_1}\int_{r_1<|z|<r_2}\mathtt{i}e^{\mathrm{i}\theta}f(re^{\mathrm{i}\theta})rdrd\theta.\end{aligned}$$

$$\left|\frac{1}{r_2-r_1}\int_{r_1<|z|<r_2}\mathtt{i}e^{\mathtt{i}\theta}f(re^{\mathtt{i}\theta})rdrd\theta\right| = \frac{1}{r_2-r_1}\left|\int_{r_1<|z|<r_2}\mathtt{i}e^{\mathtt{i}\theta}f(re^{\mathtt{i}\theta})dx\wedge dy\right|$$
$$\leq \frac{1}{r_2-r_1}\Big(\int_{r_1<|z|<r_2}|f(re^{\mathtt{i}\theta})|^2dx\wedge dy\Big)^{\frac{1}{2}}\Big(\int_{r_1<|z|<r_2}|\mathtt{i}e^{\mathtt{i}\theta}|^2dx\wedge dy\Big)^{\frac{1}{2}}$$
$$= \frac{\sqrt{\pi(r_2^2-r_1^2)}}{r_2-r_1}\Big(\int_{r_1<|z|<r_2}|f(re^{\mathtt{i}\theta})|^2dx\wedge dy\Big)^{\frac{1}{2}}$$

取$r_1=\frac{r_2}{2}$,得

$$\left|\int_{|z|=r_0}f(z)dz\right| \leq \sqrt{3\pi}\left(\int_{\frac{r_2}{2}<|z|<r_2}|f(re^{\mathtt{i}\theta})|^2dx\wedge dy\right)^{\frac{1}{2}}$$

令$r_2\to 0$,得$\int_{|z|=r_0}f(z)dz=0$.

(2)由$f(z)$全纯可知$f(z)z^n$全纯.又

$$\int_{\mathbb{D}\setminus\{0\}}|f(z)\cdot z^n|^2dx\wedge dy < \int_{\mathbb{D}\setminus\{0\}}|f(z)|^2dx\wedge dy < \infty$$

由(1)可知$\int_{|z|=r}f(z)z^ndz=0$对$0<r<1$和$n\in\mathbb{N}\cup\{0\}$.

(3)取$0<r<1$,令$g(\theta)=f(re^{\mathtt{i}\theta})$,当$n\leq -1$时,

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{\theta=0}^{2\pi}g(\theta)e^{-\mathtt{i}n\theta}d\theta = \frac{1}{2\pi}\int_{\theta=0}^{2\pi}f(re^{\mathtt{i}\theta})e^{-\mathtt{i}n\theta}d\theta \\ &= \frac{1}{2\pi}\int_{|z|=r}\frac{f(z)r^n}{\mathtt{i}z^{n+1}}dz = \frac{r^n}{2\pi\mathtt{i}}\int_{|z|=r}\frac{f(z)}{z^{n+1}}dz = 0 \end{aligned}$$

$\therefore$

$$f(z)=\sum_{n=0}^{\infty}c_ne^{\mathtt{i}n\theta}=\sum_{n=0}^{\infty}\frac{c_n}{r_n}z^n=\sum_{n=0}^{\infty}a_nz^n,\ z\in\mathbb{D}\setminus\{0\}$$

其中$a_n=\frac{c_n}{r_n}=\frac{1}{2\pi\mathtt{i}}\int_{|z|=r}\frac{f(z)}{z^{n+1}}dz$与$0<r<1$的选取无关.

令$f(0)=a_0$,则$f(z)=\sum_{n=0}^{\infty}a_nz^n,\ z\in\mathbb{D}$
因此,$f(z)$可延拓为$\mathbb{D}$上的全纯函数.

7. 设$\eta\in\mathbb{R}$, f 是带型区域$\Omega=\{z=x+\mathrm{i}y\in\mathbb{C},-1<y<1,\ x\in\mathbb{R}\}$上的全纯函数满足

$$|f(z)|\le A(1+|z|)^{\eta},\ \forall z\in\Omega.$$

证明:对每个整数$n\ge 0$ 存在$A_n>0$ 使得

$$|f^{(n)}(x)|\le A_n(1+|x|)^{\eta},\ \forall x\in\mathbb{R}.$$

Hint: 利用柯西不等式.

证明:对$\forall x\in\mathbb{R}$,设$C=\left\{z\big||z-x|=\frac{1}{2}\right\}$,则

$$|f^{(n)}(x)|\le\frac{n!||f||_C}{(\frac{1}{2})^n},||f||_C=\sup_{z\in C}|f(z)|$$

又$z\in C$时,$1+|z|\le 1+|x|+|z-x|=\frac{3}{2}+|x|<2(1+|x|)$,故

$$|f^{(n)}(x)|\le n!2^n||f||_C\le n!2^n\cdot A(1+|z_0|)^{\eta}<n!2^n\cdot A2^{\eta}(1+|x|)^{\eta}$$

取$A_n=n!2^{n+\eta}A$,则$|f^{(n)}(x)|\le A_n(1+|x|)^{\eta}$.

8.设Ω是$\mathbb{C}$上的有界区域, $\varphi:\Omega\to\Omega$是一个全纯函数. 证明: 若存在点$z_0\in\Omega$使得

$$\varphi(z_0)=z_0,\ \varphi'(z_0)=1,$$

则φ 是恒同映射.

Hint: 首先约化一般的情形到$z_0=0$的情形, 然后在0附近有$\varphi(z)=z+a_nz^n+O(z^{n+1})$, 考虑$k$个$\varphi$的复合映射$\varphi_k=\varphi\circ\cdots\circ\varphi$, 说明$\varphi_k(z)=z+ka_nz^n+$

$O(z^{n+1})$ $(z \to 0)$. 最后利用柯西不等式并让$k \to \infty$. 这里$f(z) = O(g(z))$ $(z \to 0)$ 表示存在正常数C使得$|f(z)| \le C|g(z)|$ $(z \to 0)$.

证明:不妨设$z_0 = 0$,否则令$h(z) = \varphi(z + z_0) - z_0$,

则$h(0) = \varphi(z_0) - z_0 = 0, h'(0) = \varphi'(z_0) = 1$,则$h$满足条件.

又在$z = 0$附近可展为$\varphi(z) = z + a_2 z^2 + \cdots$,

设a_n为第一个非0系数,则$\varphi(z) = z + a_n z^n + O(z^{n+1})$

设$\varphi_k = \varphi \circ \cdots \circ \varphi$,则$\varphi_2(z) = \varphi(z + a_n z^n + O(z^{n+1})) = z + 2a_n z^n + O(z^{n+1})$.

若$\varphi_k(z) = z + k a_n z^n + O(z^{n+1})$,

则$\varphi_{k+1}(z) = \varphi(z + k a_n z^n + O(z^{n+1})) = z + (k+1) a_n z^n + O(z^{n+1})$

故$\varphi_k(z) = z + k a_n z^n + O(z^{n+1})$

设$||\varphi_k||_\Omega = \sup\limits_{x \in \Omega} |\varphi_k(z)|$,由于$\Omega$有界, $\varphi : \Omega \to \Omega$,

故$\exists M > 0$,使得$||\varphi_k||_\Omega \le M$.

$\exists r > 0$使得$\overline{D_r(0)} \subset \Omega$.由柯西不等式得,$\left|\varphi_k^{(n)}(0)\right| \le \frac{n!M}{r^n}$

又$\varphi_k^{(n)}(0) = k a_n \cdot n!$

故$k a_n \cdot n! \le \frac{n!M}{r^n}$,从而$a_n \le \frac{M}{k r^n}$,当$k \to +\infty$时,$a_n \to 0$

从而$\varphi(z)=z$.

9. 设$R>1$, $z_0\in\mathbb{C}$且$|z_0|=1$. 设$h(z)$是$\{|z|<R\}$上的全纯函数满足$h(z_0)\neq 0$. 设m 是一个正整数且

$$f(z)=\frac{h(z)}{(z-z_0)^m}.$$

证明: 若$\sum_{n=0}^{\infty}a_nz^n$ 表示f 在$\{|z|<1\}$上的幂级数展开, 则

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=z_0.$$

Hint: $f(z)$ 可表示为如下形式

$$\sum_{k=1}^{m}\frac{A_k}{(z-z_0)^k}+g(z)$$

其中$A_1,\cdots,A_m\in\mathbb{C}$, $A_m=h(z_0)\neq 0$ 且$g(z)=\sum_{n=0}^{\infty}b_nz^n$ 收敛半径至少为R, 对任意$|z_0|<r<R$ 及非负整数n, 存在正数B 使得$|b_n|\leq\frac{B}{r^n}$. 利用b_n和$A_1,\cdots,A_m$表示a_n.

证明:设

$$f(z)=\sum_{k=1}^{m}\frac{A_k}{(z-z_0)^k}+g(z),\ \ g(z)=\sum_{n=0}^{\infty}b_nz^n$$

又

$$\frac{A_k}{(z-z_0)^k}=(-1)^kA_k\sum_{i=0}^{\infty}C_{k+i-1}^{i}\frac{z^i}{z_0^{k+i}}$$

得

$$a_n=\sum_{k=1}^{m}(-1)^kA_kC_{k+n-1}^{n}\frac{1}{z_0^{k+n}}+b_n$$

故

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=\lim_{n\to\infty}\frac{\sum_{k=1}^{m}(-1)^kA_kC_{k+n-1}^{n}\frac{1}{z_0^{k+n}}+b_n}{\sum_{k=1}^{m}(-1)^kA_kC_{k+n}^{n+1}\frac{1}{z_0^{k+n+1}}+b_{n+1}}$$

又$|b_n| \le \frac{B}{r^n}$，$|z_0| < r < R$，$|z_0| = 1$,故$\lim_{n\to\infty} b_n = 0$,因此

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

10.(1)设$u(z)$是区域Ω上的调和函数, 若存在点$z_0 \in \Omega$使得$u(z_0) = \sup\limits_{z\in\Omega} u(z)$, 则$u$是常值.

(2)(Hadamard三圆定理)设$U = \{z \in \mathbb{C}, 0 < r_1 < |z| < r_2 < +\infty\}$, $f(z)$在U上全纯, 在$\overline{U}$上连续, $M(r) = \max\limits_{|z|=r} |f(z)|$. 证明: $\ln M(r)$在$[r_1, r_2]$上是$\ln r$的凸函数, 即当$r \in [r_1, r_2]$时，不等式

$$\ln M(r) \le \frac{\ln r_2 - \ln r}{\ln r_2 - \ln r_1} \ln M(r_1) + \frac{\ln r - \ln r_1}{\ln r_2 - \ln r_1} \ln M(r_2)$$

成立.

作业6-解答

1.设$\mathbb{D}=\{|z|<1\}$, $f(z)$ 是$\mathbb{D}$ 上的全纯函数. 证明映射f的像集的直径

$$d=\sup_{z,w\in\mathbb{D}}|f(z)-f(w)|$$

满足

$$2|f'(0)|\le d.$$

证明:令$F(z)=\frac{f(z)-f(-z)}{d}$,
则$F(z)\in\mathcal{H}(\mathbb{D}),|F(z)|\le 1,F(0)=0,F'(0)=\frac{2f'(0)}{d}$.
由Schwarz引理知,$|F'(0)|\le 1$,即$2|f'(0)|\le d$.

2. 设$f(z)$ 是整函数满足对每一个点$z_0\in\mathbb{C}$, 泰勒展开式

$$f(z)=\sum_{n=0}^{\infty}c_n(z-z_0)^n$$

中至少有一个泰勒系数为零. 证明f 是一个多项式.

证明:(反证法)假设f不是多项式,则对$\forall n>0,f^{(n)}(z)\not\equiv 0$.
因为f是整函数,故对$\forall k>0,f^{(k)}(z)$也为整函数,且不恒为0.
从而$f^{(k)}(z)$至多有可数个零点.
由f的任意阶导数的零点组成的集合,是可数个可数集的并,是可数的.
又对$\forall z_0\in\mathbb{C},\exists n_0>0,c_{n_0}\cdot n_0!=f^{(n_0)}(z_0)=0$,即$z_0$为$f^{(n_0)}(z)$的零点,
从而$\mathbb{C}$为f的任意阶导数的零点集,但$\mathbb{C}$不可数,矛盾!
故f为多项式.

3.利用柯西不等式或者最大模原理解决下面问题.

(1)证明: 若$f(z)$ 是整函数满足对任意$R > 0$, 某个$k \geq 0$和常数$A, B > 0$有

$$\sup_{|z|=R} |f(z)| \leq AR^k + B,$$

则$f(z)$是一个度$\leq k$的多项式.
(2)设$w_1, \cdots, w_n$ 是复平面单位圆周上的点. 证明: 存在单位圆周上的一个点z 使得z 与所有w_j $(1 \leq j \leq n)$距离的乘积至少为1; 存在单位圆周上的一个点w 使得w 与所有w_j $(1 \leq j \leq n)$距离的乘积等于1.
(3) 证明: 若整函数f的实部是有界的, 则f是常值.

证明:**(1)**对$f(z)$在$z = 0$处作泰勒展开,有

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

对$\forall R > 0$,由柯西不等式得

$$\left| f^{(n)}(0) \right| \leq \frac{n!}{R^n} \sup_{|z|=R} |f(z)|$$

因此对某些$k \geq 0$和某些常数$A, B > 0$,有

$$\frac{\left| f^{(n)}(0) \right|}{n!} \leq \frac{\sup\limits_{|z|=R} |f(z)|}{R^n} \leq \frac{AR^k + B}{R^n}$$

当$n > k$且$R \to \infty$时,有$\frac{\left| f^{(n)}(0) \right|}{n!} \to 0$.
故$f(z)$是一个次数小于等于k的多项式.

(2)考虑全纯函数

$$f(z) = \Pi_{j=1}^{m} (z - w_j)$$

则$f(z)$是一个非常值整函数,满足

$$|f(0)| = \Pi_{j=1}^{m} |w_j| = 1$$

对$f(z)$在$\overline{\mathbb{D}}$上用最大模原理,知

$$|f(0)| \leq \max_{|z|=1} |f(z)|$$

更进一步有

$$|f(0)| < \max_{|z|=1} |f(z)|$$

($\because f(z)$非常值)

$\therefore \exists z_0, |z_0| = 1,$ 使得$|f(z_0)| > 1.$

又$\because |f(\omega_j)| = 0,$由$|f(z)|$在$\{|z| = 1\}$上连续可知

$\exists z_1, |z_1| = 1,$ 使得$|f(z_1)| = 1.$

(3)若$\mathtt{Re}(f) \leq M,\ g(z) = e^{f(z)}$,则$g(z)$为整函数且$|g(z)| = e^{\mathtt{Re}(f)} \leq e^M.$
由刘维尔定理可知,$g(z) \equiv C$,C为常数.
故$f(z) \equiv \log C$,又f连续且$\log C$的不同单值分支差$2\pi\mathtt{i}$, 因此f恒为常数.

若$\mathtt{Re}(f) \geq M_1$, 则取 $g(z) = e^{-f(z)}$, 证明同上.

4.这个问题说明全纯函数的均方收敛怎样控制它的一致收敛. 设U 是$\mathbb{C}$中的开子集. 定义函数的均方范数为

$$\| f \|_{L^2(U)} = \left(\int_U |f(z)|^2 dx \wedge dy \right)^{1/2},$$

上确界范数为

$$\| f \|_{L^\infty(U)} = \sup_{z \in U} |f(z)|.$$

(1) 设f 包含$\overline{D_r(z_0)} = \{|z - z_0| \leq r\}$的邻域上的全纯函数. 证明对任意$0 < s < r$ 存在常数$C > 0$ (依赖于s 和r) 使得

$$\| f \|_{L^\infty(D_s(z_0))} \leq C \| f \|_{L^2(D_r(z_0))} .$$

(2) 证明: 若全纯函数列$\{f_n\}$ 是均方范数$\|\cdot\|_{L^2(U)}$下的柯西列, 则$\{f_n\}$ 在U 上内闭一致收敛到某个全纯函数.

Hint: 利用全纯函数的平均值性质.

证明:**(1)**根据平均值公式,有$f^2(z) = \frac{1}{2\pi}\int_0^{2\pi} f^2(z+re^{\mathrm{i}\theta})d\theta$.因此有

$$\int_0^d |f(z)|^2 rdr \leq \frac{1}{2\pi}\int_0^d\int_0^{2\pi} |f(z+re^{\mathrm{i}\theta})|^2 rdrd\theta = \frac{1}{2\pi}\int_{D_d(z)} |f(z)|^2 dx\wedge dy$$

故

$$\frac{d^2}{2}|f(z)|^2 \leq \frac{1}{2\pi}\left[\| f \|_{L^2(D_d(z))}\right]^2$$

因此

$$|f(z)| \leq \frac{1}{\sqrt{\pi}d}\cdot \| f \|_{L^2(D_d(z))}$$

取$d = r-s$,则对$\forall z \in D_s(z_0)$有$D_d(z) \subset D_r(z_0)$.且

$$|f(z)| \leq \frac{1}{\sqrt{\pi}(r-s)}\cdot \| f \|_{L^2(D_{(r-s)}(z))} \leq \frac{1}{\sqrt{\pi}(r-s)}\cdot \| f \|_{L^2(D_r(z_0))}$$

因此

$$\underset{z\in D_s(z_0)}{sup}|f(z)| = \| f \|_{L^\infty(D_s(z_0))} \leq \frac{1}{\sqrt{\pi}(r-s)} \| f \|_{L^2(D_r(z_0))}.$$

(2)由题可知,对$\forall\varepsilon > 0,\ \exists N > 0$,使得当$m, n \geq N$时,$\| f_m - f_n \|_{L^2(U)} < \varepsilon$.

对U中任一紧集V,$\forall z \in V$,存在U中z的开邻域$B(z, r_z)$,

使得$\{B(z, r_z)|z \in V\}$为U的开覆盖,

则有有限子覆盖$\{B(z_i, r_{z_i})\}, i = 1, 2, \cdots, M$

由(1)知,$\| f_m - f_n \|_{L^\infty(D_{r_i}(z_i))} \leq C \| f_m - f_n \|_{L^2(U)}$.

因此$\| f_m - f_n \|_{L^\infty(V)} \leq \| f_m - f_n \|_{L^\infty(\bigcup_{i=1}^M B(z_i, r_{z_i}))} \leq C \| f_m - f_n \|_{L^2(U)} < C\varepsilon$.

故$\underset{z\in V}{sup}|f_m - f_n| < \varepsilon$,

故$\{f_n\}$在U的任一紧集上一致收敛到f,又$\{f_n\}$全纯,故f为全纯函数.

5.设$\mathbb{D}=\{|z|<1\}$,$f(z)$在$\mathbb{D}$上全纯, 且$f(0)=1$. 如果对每个$z\in\mathbb{D}$, $\mathrm{Re}f(z)\geq 0$成立, 利用Schwarz引理证明:
(1)不等式

$$\frac{1-|z|}{1+|z|}\leq \mathrm{Re}f(z)\leq |f(z)|\leq \frac{1+|z|}{1-|z|}$$

对每个$z\in\mathbb{D}$都成立.
(2)上述不等式中等号在z异于零时成立,当且仅当

$$f(z)=\frac{1+e^{\mathrm{i}\theta}z}{1-e^{\mathrm{i}\theta}z},\ \theta\in\mathbb{R}.$$

6. 设$f(z)$是$\{\mathrm{Im}z\geq 0\}$上的有界连续函数, 在$\{\mathrm{Im}z>0\}$上全纯. 证明: 若$f(z)$在实轴上取实值, 则$f(z)$为常值.

7.设f是开圆盘$D_{R_0}=\{|z|<R_0\}$ $(R_0>0)$上的全纯函数.
(1)证明: 对于$0<R<R_0$ 和$|z|<R$, 有

$$f(z)=\frac{1}{2\pi}\int_{\varphi=0}^{2\pi}f(Re^{\mathrm{i}\varphi})\mathrm{Re}\left(\frac{Re^{\mathrm{i}\varphi}+z}{Re^{\mathrm{i}\varphi}-z}\right)d\varphi.$$

Hint: 注意到如果$w=\frac{R^2}{\bar{z}}$, 则$\frac{f(\zeta)}{\zeta-w}$ 在$\{|\zeta|=R\}$上的积分为零. 利用这个性质和一般的柯西积分公式证明想要的恒等式.
(2) 设$z=re^{\mathrm{i}\theta}$, 证明

$$\mathrm{Re}\left(\frac{Re^{\mathrm{i}\varphi}+z}{Re^{\mathrm{i}\varphi}-z}\right)=\frac{R^2-r^2}{R^2-2Rr\cos(\varphi-\theta)+r^2}$$

(3)设u 是$\mathbb{D}=\{|z|<1\}$上的2阶连续可微函数满足

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

在$\mathbb{D}$上恒为零(即为$\mathbb{D}$上的调和函数)且连续到$\mathbb{D}$的边界. 推导下面的**泊松积分公**

式

$$u(z)=\frac{1}{2\pi}\int_{\varphi=0}^{2\pi}P_r(\varphi-\theta)u(e^{\mathrm{i}\varphi})d\varphi$$

其中$z=re^{\mathrm{i}\theta}$ $(r<1)$, $P_r(\beta)$ 是单位圆盘上的**泊松核**, 由下式给出

$$P_r(\beta)=\frac{1-r^2}{1-2r\cos\beta+r^2}.$$

Hint: 应用**(1)** 和**(2)** 到$\mathbb{D}$上实部为u的全纯函数.

(4)利用泊松积分公式解下面的Dirichlet问题: 设$u_0(e^{\mathrm{i}\varphi})$是单位圆周上的连续函数, 则

$$u(z)=\frac{1}{2\pi}\int_{\varphi=0}^{2\pi}P_r(\varphi-\theta)u_0(e^{\mathrm{i}\varphi})d\varphi$$

是单位圆盘上Dirichlet问题

$$\begin{cases}\Delta u=0,\ z\in\mathbb{D}\\ u\mid_{\partial\mathbb{D}}=u_0\end{cases}$$

的解, 且该解是唯一的.

(5)利用Dirichlet问题的解证明: 区域上具有均值性质的连续函数一定是调和函数.

证明:**(1)**设$w=\frac{R^2}{\bar{z}}$,则$\int_{|z|=R}\frac{f(\zeta)}{\zeta-w}d\zeta=0$. 故

$$\begin{aligned}
f(z)&=\frac{1}{2\pi\mathtt{i}}\int_{|z|=R}\frac{f(\zeta)}{\zeta-z}d\zeta\\
&=\frac{1}{2\pi\mathtt{i}}\left[\int_{|z|=R}\frac{f(\zeta)}{\zeta-z}d\zeta-\int_{|z|=R}\frac{f(\zeta)}{\zeta-w}d\zeta\right]\\
&=\frac{1}{2\pi\mathtt{i}}\left[\int_{\varphi=0}^{2\pi}f(Re^{\mathtt{i}\varphi})\frac{dRe^{\mathtt{i}\varphi}}{Re^{\mathtt{i}\varphi}-z}-\int_{\varphi=0}^{2\pi}f(Re^{\mathtt{i}\varphi})\frac{dRe^{\mathtt{i}\varphi}}{Re^{\mathtt{i}\varphi}-\frac{R^2}{\bar{z}}}\right]\\
&=\frac{1}{2\pi}\left[\int_{\varphi=0}^{2\pi}f(Re^{\mathtt{i}\varphi})\frac{Re^{\mathtt{i}\varphi}d\varphi}{Re^{\mathtt{i}\varphi}-z}-\int_{\varphi=0}^{2\pi}f(Re^{\mathtt{i}\varphi})\frac{\bar{z}d\varphi}{\bar{z}-Re^{-\mathtt{i}\varphi}}\right]\\
&=\frac{1}{2\pi}\int_{\varphi=0}^{2\pi}f(Re^{\mathtt{i}\varphi})\left[\frac{Re^{\mathtt{i}\varphi}}{Re^{\mathtt{i}\varphi}-z}-\frac{\bar{z}}{\bar{z}-Re^{-\mathtt{i}\varphi}}\right]d\varphi\\
&=\frac{1}{4\pi}\int_{\varphi=0}^{2\pi}f(Re^{\mathtt{i}\varphi})\left[\frac{2Re^{\mathtt{i}\varphi}}{Re^{\mathtt{i}\varphi}-z}-1+1-\frac{2\bar{z}}{\bar{z}-Re^{-\mathtt{i}\varphi}}\right]d\varphi\\
&=\frac{1}{4\pi}\int_{\varphi=0}^{2\pi}f(Re^{\mathtt{i}\varphi})\left[\frac{Re^{\mathtt{i}\varphi}+z}{Re^{\mathtt{i}\varphi}-z}+\frac{Re^{-\mathtt{i}\varphi}+\bar{z}}{Re^{-\mathtt{i}\varphi}-\bar{z}}\right]d\varphi\\
&=\frac{1}{2\pi}\int_{\varphi=0}^{2\pi}f(Re^{\mathtt{i}\varphi})\mathrm{Re}\left(\frac{Re^{\mathtt{i}\varphi}+z}{Re^{\mathtt{i}\varphi}-z}\right)d\varphi.
\end{aligned}$$

(2)

$$\begin{aligned}
\mathrm{Re}\left(\frac{Re^{\mathtt{i}\varphi}+z}{Re^{\mathtt{i}\varphi}-z}\right)&=\frac{1}{2}\left(\frac{Re^{\mathtt{i}\varphi}+z}{Re^{\mathtt{i}\varphi}-z}+\frac{Re^{-\mathtt{i}\varphi}+\bar{z}}{Re^{-\mathtt{i}\varphi}-\bar{z}}\right)\\
&=\frac{1}{2}\cdot\frac{(Re^{\mathtt{i}\varphi}+z)(Re^{-\mathtt{i}\varphi}-\bar{z})+(Re^{-\mathtt{i}\varphi}+\bar{z})(Re^{\mathtt{i}\varphi}-z)}{(Re^{\mathtt{i}\varphi}-z)(Re^{-\mathtt{i}\varphi}-\bar{z})}\\
&=\frac{1}{2}\cdot\frac{2R^2-2z\bar{z}}{R^2-\bar{z}Re^{\mathtt{i}\varphi}-zRe^{-\mathtt{i}\varphi}+z\bar{z}}\\
&=\frac{R^2-z\bar{z}}{R^2-2\mathrm{Re}(\bar{z}Re^{\mathtt{i}\varphi})+z\bar{z}}\\
&=\frac{R^2-r^2}{R^2-2\mathrm{Re}[rRe^{\mathtt{i}(\varphi-\theta)}]+r^2}\\
&=\frac{R^2-r^2}{R^2-2Rr\cos(\varphi-\theta)+r^2}
\end{aligned}$$

(3)由hw4-P7可知:$\exists\mathbb{D}$上的全纯函数$f(z)$,使得$u(z)=\mathrm{Re}f(z)$.

固定$z=re^{\mathrm{i}\theta}, r<1$,对$\forall R$满足$r<R<1$有

$$f(z)=\frac{1}{2\pi}\int_{\varphi=0}^{2\pi}f(Re^{\mathrm{i}\varphi})\frac{R^2-r^2}{R^2-2Rr\cos(\varphi-\theta)+r^2}d\varphi$$

取实部可得:

$$u(z)=\frac{1}{2\pi}\int_{\varphi=0}^{2\pi}u(Re^{\mathrm{i}\varphi})\frac{(R^2-r^2)}{R^2-2Rr\cos(\varphi-\theta)+r^2}d\varphi$$

$\because u(z)$在$\overline{\mathbb{D}}$上连续,$\therefore$令$R\to 1$得

$$u(z)=\frac{1}{2\pi}\int_{\varphi=0}^{2\pi}\frac{(1-r^2)u(e^{\mathrm{i}\varphi})}{1-2r\cos(\varphi-\theta)+r^2}d\varphi$$

(4)由$P_r(\varphi-\theta)=\mathrm{Re}\left(\frac{e^{\mathrm{i}\varphi}+z}{e^{\mathrm{i}\varphi}-z}\right)$和$\Delta=4\frac{\partial^2}{\partial z\partial\bar{z}}$知, $\Delta P_r(\varphi-\theta)=0$,从而对任意$z\in\mathbb{D}$, 有$\Delta u(z)=0$.

下面证明:对任意$\xi=e^{\mathrm{i}\theta_0}\in\partial\mathbb{D}$, $\lim\limits_{z\in\mathbb{D},z\to\xi}u(z)=u_0(\xi)$. 由泊松积分公式知$\frac{1}{2\pi}\int_{\varphi=0}^{2\pi}P_r(\varphi-\theta)d\varphi=1$.

由u_0在单位圆周上的连续性知, $\forall\epsilon>0,\exists\pi>\delta>0$使得$|\varphi-\theta_0|<\delta$时, $|u_0(e^{\mathrm{i}\varphi})-u_0(e^{\mathrm{i}\theta_0})|<\epsilon$.

对于$|\varphi-\theta_0|\geq\delta$,当$|\theta-\theta_0|<\frac{\delta}{2}$时, $|\varphi-\theta|\geq|\varphi-\theta_0|-|\theta-\theta_0|>\frac{\delta}{2}$, 此时有

$$1-2r\cos(\varphi-\theta)+r^2>2r(1-\cos\frac{\delta}{2}).$$

再由$\lim\limits_{r\to 1}\frac{1-r^2}{r}=0$知, 存在$\eta>0$使得$|1-r|<\eta$时,

$$\frac{1-r^2}{r}<\frac{(1-\cos\frac{\delta}{2})\epsilon}{M}$$

这里$M = \sup\limits_{\varphi\in[0,2\pi]} |u_0(e^{\mathrm{i}\varphi})|$.

$$\begin{aligned}
|u(z)-u_0(\xi)| &= |u(re^{\mathrm{i}\theta})-u_0(e^{\mathrm{i}\theta_0})| \\
&= \frac{1}{2\pi}\left|\int_{\varphi=0}^{2\pi} P_r(\varphi-\theta)[u_0(e^{\mathrm{i}\varphi})-u_0(e^{\mathrm{i}\theta_0})]d\varphi\right| \\
&\le \frac{1}{2\pi}\left(\int_{|\varphi-\theta_0|<\delta}+\int_{|\varphi-\theta_0|\ge\delta}\right) P_r(\varphi-\theta)\left|u_0(e^{\mathrm{i}\varphi})-u_0(e^{\mathrm{i}\theta_0})\right| d\varphi \\
&< \frac{\epsilon}{\pi}.
\end{aligned}$$

唯一性: 设有两个解u, v, 则调和函数$u-v$在边界上取值为零. 由调和函数最大(小)值原理知, $u-v\equiv 0$, 即$u\equiv v$.

(5)设$U\subset\mathbb{C}$是一个区域, $u(z)$是U上的连续实值函数满足, 对每一点$z_0\in U$, 存在充分小$r_0>0$, 当$0<r\le r_0$时,

$$u(z_0)=\frac{1}{2\pi}\int_{\theta=0}^{2\pi} u(z_0+re^{\mathrm{i}\theta})d\theta.$$

固定一点$z_0\in U$,下面说明$u(z)$在z_0调和.

设$v_0(e^{\mathrm{i}\theta})=u(z_0+r_0e^{\mathrm{i}\theta})$, 则通过解Dirichlet问题得到一个以$v_0(e^{\mathrm{i}\theta})$为边界值且在$D_{r_0}(z_0)$中调和的函数$v(z)$. 在$\overline{D_{r_0}(z_0)}$上考虑$u-v$, 由于$u, v$都有均值性质, 所以$u-v$也有均值性质, 从而$u-v$在边界$\partial D_{r_0}(z_0)$上取最大值与最小值. 而$u-v$在$\partial D_{r_0}(z_0)$上取值为零, 故有对任意$z\in D_{r_0}(z_0)$,$u(z)=v(z)$. 因此$u(z)$在点$z_0$调和. 由$z_0$的任意性知, $u(z)$在U上调和.

作业7-解答

1. 设$\zeta = e^{\pi \mathtt{i}/2n}$ (n是自然数). (1)证明

$$\zeta + \zeta^3 + \zeta^5 + \cdots + \zeta^{2n-1} = \frac{\mathrm{i}}{\sin(\pi/2n)}.$$

(2)设γ是以1, $1+\mathtt{i}$, $-1+\mathtt{i}$, -1为顶点的矩形边界(取正向), 计算下面积分的值

$$\int_\gamma \frac{dz}{z^{2n}+1}.$$

证明:(1) 由于$\zeta = e^{\pi \mathtt{i}/2n} \neq 0$故

$$\begin{aligned}
&\zeta + \zeta^3 + \zeta^5 + \cdots + \zeta^{2n-1} \\
&= \frac{\zeta(1-\zeta^{2n})}{1-\zeta^2} \\
&= \frac{1-e^{\pi \mathrm{i}}}{e^{-\pi \mathrm{i}/2n} - e^{\pi \mathrm{i}/2n}} \\
&= \frac{2}{e^{-\pi \mathtt{i}/2n} - e^{\pi \mathrm{i}/2n}} \\
&= \frac{\mathrm{i}}{\frac{e^{\pi \mathtt{i}/2n} - e^{-\pi \mathtt{i}/2n}}{2\mathrm{i}}} \\
&= \frac{\mathrm{i}}{\sin(\pi/2n)}.
\end{aligned}$$

(2)$f(z) = \frac{1}{z^{2n}+1}$在闭矩形区域内有$n$个一阶极点$\zeta, \zeta^3, \zeta^5, \cdots, \zeta^{2n-1}$,其中$\zeta = e^{\pi \mathtt{i}/2n}$.

若n是偶数,则这n个极点均在内部, 此时

$$\begin{aligned}
\int_\gamma \frac{dz}{z^{2n}+1} &= 2\pi\mathrm{i}\sum_{k=1}^{n}\operatorname*{Res}_{z=\zeta^{2k-1}} f(z) \\
&= 2\pi\mathrm{i}\sum_{k=1}^{n}\frac{1}{2n(\zeta^{2k-1})^{2n-1}} \\
&= 2\pi\mathrm{i}\sum_{k=1}^{n}\frac{1}{2n(-\zeta^{1-2k})} \\
&= -\frac{\pi\mathrm{i}}{n}\sum_{k=1}^{n}\zeta^{2k-1} \\
&= -\frac{\pi\mathrm{i}}{n}\cdot\frac{\mathrm{i}}{\sin(\pi/2n)} \\
&= \frac{\pi}{n\sin(\pi/2n)}.
\end{aligned}$$

若n是奇数,则$\zeta^{2\cdot\frac{n+1}{2}-1}=\zeta^n=e^{\frac{\pi}{2}\mathrm{i}}=\mathrm{i}$在矩形边界$D$上,此时

$$\begin{aligned}
\int_\gamma \frac{dz}{z^{2n}+1} &= 2\pi\mathrm{i}\sum_{k=1}^{n}\operatorname*{Res}_{z=\zeta^{2k-1}} f(z) - \pi\mathrm{i}\operatorname*{Res}_{z=\mathrm{i}} f(z) \\
&= \frac{\pi}{n\sin(\pi/2n)} - \pi\mathrm{i}\operatorname*{Res}_{z=\mathrm{i}} f(z) \\
&= \frac{\pi}{n\sin(\pi/2n)} - \pi\mathrm{i}\cdot\frac{1}{2n\mathrm{i}^{2n-1}} \\
&= \frac{\pi}{n\sin(\pi/2n)} - \frac{\pi}{2n}
\end{aligned}$$

2. 计算下列积分:

(1)$\int_{x=0}^{\frac{\pi}{2}}\frac{1}{a+\sin^2 x}dx$ $(a>0)$; (答案:$\frac{\pi}{2\sqrt{a(a+1)}}$)

(2)$\int_{x=-\infty}^{\infty}\frac{1}{(1+x^2)^{n+1}}dx$ (n是正整数); (答案:$\frac{(2n)!\pi}{4^n(n!)^2}$)

(3)$\int_{x=-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx$; (答案:$\frac{5\pi}{12}$)

(4)$\int_{x=0}^{\infty}\frac{x\sin x}{x^2+1}dx$. (答案:$\frac{\pi}{2e}$)

3. 计算下列积分:

(1)$\int_{x=0}^{\infty}\frac{x^{1/3}}{1+x^2}dx$;

(2)$\int_{x=0}^{\infty}\frac{\log(x^2+1)}{x^2+1}dx$ (提示: 考虑$f(z)=\frac{\log(z+\mathrm{i})}{z^2+1}$).

解:(1)设$f(z)=\frac{z^{1/3}}{1+z^2}$, $z=re^{\mathrm{i}\theta}$, $-\frac{\pi}{2}<\theta<\frac{3\pi}{2}$, $z^{\frac{1}{3}}=r^{\frac{1}{3}}e^{\mathrm{i}\frac{\theta}{3}}$
积分路线取$\Omega=\{z|\varepsilon<|z|<R且\mathrm{Im}z>0\}$的边界,由留数定理得:

$$\int_{C_R}f(z)\mathrm{d}z-\int_{C_\varepsilon}f(z)\mathrm{d}z+\int_{-R}^{-\varepsilon}f(z)\mathrm{d}z+\int_{\varepsilon}^{R}f(z)\mathrm{d}z=2\pi\mathrm{i}\operatorname*{Res}_{z=\mathrm{i}}f(z)$$

当$R\to\infty$时,

$$\int_{C_R}f(z)\mathrm{d}z=0$$

当$\varepsilon\to 0$时,

$$\int_{C_\varepsilon}f(z)\mathrm{d}z=0$$

$$\int_{-R}^{-\varepsilon}f(z)\mathrm{d}z=\int_{-R}^{-\varepsilon}\frac{(-x)^{\frac{1}{3}}e^{\mathrm{i}\frac{\pi}{3}}}{1+x^2}\mathrm{d}x=e^{\mathrm{i}\frac{\pi}{3}}\int_{\varepsilon}^{R}\frac{x^{\frac{1}{3}}}{1+x^2}\mathrm{d}x,\ \int_{\varepsilon}^{R}f(z)\mathrm{d}z=\int_{\varepsilon}^{R}\frac{x^{\frac{1}{3}}}{1+x^2}\mathrm{d}x$$

又

$$2\pi\mathrm{i}\operatorname*{Res}_{z=\mathrm{i}}f(z)=2\pi\mathrm{i}\lim_{z\to\mathrm{i}}(z-\mathrm{i})\cdot\frac{z^{1/3}}{1+z^2}=2\pi\mathrm{i}\cdot\frac{e^{\mathrm{i}\frac{\pi}{6}}}{2i}=\pi\cdot e^{\mathrm{i}\frac{\pi}{6}}$$

故

$$(e^{\mathrm{i}\frac{\pi}{3}}+1)\int_{0}^{\infty}\frac{x^{\frac{1}{3}}}{1+x^2}\mathrm{d}x=2\pi\mathrm{i}\operatorname*{Res}_{z=\mathrm{i}}f(z)=\pi\cdot e^{\mathrm{i}\frac{\pi}{6}}$$

因此

$$\int_{0}^{\infty}\frac{x^{\frac{1}{3}}}{1+x^2}\mathrm{d}x=\frac{\pi\cdot e^{\mathrm{i}\frac{\pi}{6}}}{e^{\mathrm{i}\frac{\pi}{3}}+1}=\frac{\pi}{e^{\mathrm{i}\frac{\pi}{6}}+e^{-\mathrm{i}\frac{\pi}{6}}}=\frac{\pi}{2\cos\frac{\pi}{6}}=\frac{\pi}{\sqrt{3}}$$

(2)考虑$f(z)=\frac{\log(z+\mathrm{i})}{z^2+1}$
取$\Omega=\{z||z|<R且\mathrm{Im}z>0\}$,
选择$\log(z+\mathrm{i})$的全纯分支满足$\arg(z+\mathrm{i})\in\left(-\frac{\pi}{2},\frac{3\pi}{2}\right)$, 由留数定理得:

$$\int_{C_R}f(z)\mathrm{d}z+\int_{x=-R}^{R}f(x)\mathrm{d}x=2\pi\mathrm{i}\operatorname*{Res}_{z=\mathrm{i}}f(z)$$

其中

$$
\begin{aligned}
2\pi \mathrm{i} \operatorname*{Res}_{z=\mathrm{i}} f(z) &= 2\pi \mathrm{i} \lim_{z\to \mathrm{i}} (z-\mathrm{i}) \cdot \frac{\log(z+\mathtt{i})}{z^2+1} = 2\pi \mathrm{i} \cdot \frac{\log 2\mathrm{i}}{2\mathrm{i}} \\
&= \pi \log 2\mathrm{i} = \pi \log 2 + \mathrm{i} \cdot \frac{\pi^2}{2}
\end{aligned}
$$

$$
\begin{aligned}
\int_{x=-R}^{0} f(x)\mathrm{d}x &= \int_{-R}^{0} \frac{\log(x+\mathtt{i})}{x^2+1}\mathrm{d}x \\
&= \int_{x=R}^{0} \frac{\log(-x+\mathtt{i})}{x^2+1}\mathrm{d}(-x) \\
&= \int_{x=0}^{R} \frac{\log(-x+\mathtt{i})}{x^2+1}\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
\int_{x=-R}^{R} \frac{\log(x+\mathtt{i})}{x^2+1}\mathrm{d}x &= \int_{x=0}^{R} \frac{\log(-x+\mathtt{i}) + \log(x+\mathtt{i})}{x^2+1}\mathrm{d}x \\
&= \int_{x=0}^{R} \frac{\log|-x+\mathtt{i}| + \mathrm{i}\arg(-x+\mathrm{i}) + \log(x+\mathrm{i}) + \mathrm{i}\arg(x+\mathrm{i})}{x^2+1}\mathrm{d}x \\
&= \int_{x=0}^{R} \frac{\log(x^2+1) + \mathrm{i}\pi}{x^2+1}\mathrm{d}x
\end{aligned}
$$

$$
\left| \int_{C_R} \frac{\log(z+\mathtt{i})}{z^2+1}\mathrm{d}z \right| \le \frac{\log(R+1)+\pi}{R^2-1} \cdot \pi R \to 0 \ (R\to\infty)
$$

故取$R\to\infty$得

$$
\int_{x=0}^{\infty} \frac{\log(x^2+1)+\mathrm{i}\pi}{x^2+1}\mathrm{d}x = \pi \log 2 + \mathrm{i} \cdot \frac{\pi^2}{2}
$$

取实部得

$$
\int_{x=0}^{\infty} \frac{\log(x^2+1)}{x^2+1}\mathrm{d}x = \pi \log 2
$$

4. 证明: 设$f(z)$是单位圆盘$\mathbb{D} = \{|z| < 1\}$上的全纯函数。若$\zeta \in \mathbb{D}$, 则

$$
f(\zeta) = \frac{1}{\pi} \int\int_{|z|<1} \frac{f(z)}{(1-\bar{z}\zeta)^2} dx \wedge dy
$$

(提示: 用极坐标表示面积元积分,其中关于角度的定积分部分可以转化为线积分, 从而可以利用留数定理.)

作业8-解答

1.设z_0是全纯函数的孤立奇点, 则z_0是$f(z)$的可去奇点当且仅当

$$\lim_{z\to z_0}(z-z_0)f(z)=0.$$

证明:"$\Rightarrow$"由于z_0是$f(z)$的可去奇点,故$f(z)$在z_0的去心邻域中有界, 从而有

$$\lim_{z\to z_0}(z-z_0)f(z)=0$$

"$\Leftarrow$" 设

$$f(z)=\sum_{n=-\infty}^{\infty}c_n(z-z_0)^n$$

则

$$(z-z_0)f(z)=\sum_{n=-\infty}^{\infty}c_n(z-z_0)^{n+1}.$$

由 $\lim\limits_{z\to z_0}(z-z_0)f(z)=0$知,$z_0$是$(z-z_0)f(z)$的可去奇点, 从而有$\forall n<-1, c_n=0$且$c_{-1}=0$,
即对$\forall n<0, c_n=0$,因此z_0是$f(z)$的可去奇点.

2. 设$f(z)$ 是去心开圆盘

$$D_r(z_0)\setminus\{z_0\}=\{0<|z-z_0|<r\}$$

上的全纯函数. 若存在$A>0, 0<\epsilon<1$ 和$0<r'<r$ 使得

$$|f(z)|\le\frac{A}{|z-z_0|^{1-\epsilon}}$$

对任意$0<|z-z_0|<r'$ 成立, 则z_0 是$f(z)$ 的可去奇点.

证明:由已知可得$|(z-z_0)f(z)|\le A|z-z_0|^\epsilon$,则当$z\to z_0$时,有

$$\lim_{z\to z_0}|(z-z_0)f(z)|=0$$

故

$$\lim_{z\to z_0}(z-z_0)f(z)=0$$

由Problem 1可知,z_0是$f(z)$的可去奇点.

3. (1)证明: 若z_0是$f(z)$的孤立奇点, 则z_0不可能是$e^{f(z)}$的极点.
(2)证明: 若z_0是$f(z)$的孤立奇点且在z_0的某个去心邻域内$\mathrm{Re}f(z)$或$\mathrm{Im}f(z)$ 有上界或下界, 则z_0是$f(z)$的可去奇点.
(3)证明: 若z_0是$f(z)$的孤立奇点且在z_0的某个去心邻域内有$\mathrm{Re}f(z)\le -c\log|z-z_0|$, 其中$c$是一个正常数, 则$z_0$是$f(z)$的可去奇点.

证明:**(1)** (i)若z_0是$f(z)$的可去奇点,则当$z\to z_0$时,有

$$\lim_{z\to z_0}f(z)=a$$

a为一个有限数,因此

$$\lim_{z\to z_0}e^{f(z)}=e^a$$

也为有限数,故z_0是$e^{f(z)}$的可去奇点.

(ii)若z_0是$f(z)$的本质奇点,则由Casorati-Weierstrass定理知,

$$\begin{aligned}\exists\{z_n\}\to z_0, s.t.\ f(z_n)\to 0 \quad &\Rightarrow \quad e^{f(z_n)}\to 1\\ \{z_n'\}\to z_0, s.t.\ f(z_n')\to 1 \quad &\Rightarrow \quad e^{f(z_n')}\to e\end{aligned}$$

故z_0是$e^{f(z)}$的本质奇点.

(iii)若z_0是$f(z)$的k阶极点,则存在z_0的小邻域$D_r(z_0)$,使得

$$f(z)=\frac{g(z)}{(z-z_0)^k}$$

其中$g(z)$在$D_r(z_0)$内全纯且不取0,则

$$F(z)=\frac{1}{f(z)}=\frac{(z-z_0)^k}{g(z)}$$

在$D_r(z_0)$内全纯,且$F(z_0)=0$.
由开映射定理知,$F(D_r(z_0))$是开集,即$\exists\delta>0,\text{s.t. } D_\delta(0)\subset F(D_r(z_0))$.
从而有$\mathbb{C}\setminus\overline{D_{\frac{1}{\delta}}(0)}\subset f(D_r(z_0)\setminus\{z_0\})$,故

$$\begin{aligned}\exists\{z_n\}\to z_0,\text{s.t. } f(z_n)\to-\infty &\quad\Rightarrow\quad e^{f(z_n)}\to 0\\ \{z_n'\}\to z_0,\text{s.t. } f(z_n')\to+\infty &\quad\Rightarrow\quad e^{f(z_n')}\to+\infty\end{aligned}$$

故z_0是$e^{f(z)}$的本质奇点.
综上,若z_0是$f(z)$的孤立奇点,则z_0不可能是$e^{f(z)}$的极点.

(2) $\because\ \left|e^{f(z)}\right|=e^{\mathrm{Re}f(z)},\left|e^{-f(z)}\right|=e^{-\mathrm{Re}f(z)},\left|e^{-\mathrm{i}f(z)}\right|=e^{\mathrm{Im}f(z)},\left|e^{\mathrm{i}f(z)}\right|=e^{-\mathrm{Im}f(z)}$,
$\therefore$Re$f(z)$或Im$f(z)$ 有上界或下界分别意味着$\left|e^{f(z)}\right|,\left|e^{-f(z)}\right|,\left|e^{-\mathrm{i}f(z)}\right|,\left|e^{\mathrm{i}f(z)}\right|$有界,
$\therefore z_0$分别为$e^{f(z)},e^{-f(z)},e^{-\mathrm{i}f(z)},e^{\mathrm{i}f(z)}$的可去奇点,
由(1)知z_0为$f(z)$的可去奇点.

(3) $\because\ \mathrm{Re}f(z)\le -c\log|z-z_0|$
$\therefore\ \left|e^{f(z)}\right|=e^{\mathrm{Re}f(z)}\le e^{-c\log|z-z_0|}=|z-z_0|^{-c}$
$\therefore\ \exists$充分大$n>c$,使得

$$\lim_{z\to z_0}(z-z_0)^n e^{f(z)}=0$$

$\therefore\ z_0$是$e^{f(z)}$的可去奇点或极点.
由(1)可知z_0不可能是$e^{f(z)}$的极点.

故z_0是$e^{f(z)}$的可去奇点且z_0是$f(z)$的可去奇点.

4.证明: 扩充复平面$\mathbb{C}_\infty$上的亚纯函数是有理函数.

证明:设$f(z)$为$\mathbb{C}_\infty$上的亚纯函数,若$f(z)$在$\mathbb{C}_\infty$上有无穷多个极点z_j,若点列$\{z_j\}$有界,则存在收敛子列$\{z_{j_k}\}$极限为有限复数z_c,此时z_c不是孤立奇点,矛盾.若点列$\{z_j\}$无界,则存在子列极限为无穷远点,此时无穷远点不是孤立奇点,矛盾.因此,$f(z)$在$\mathbb{C}_\infty$上只有有限个极点,故在$\mathbb{C}$上只有有限个极点,设为$z_1,\cdots,z_t$,阶分别为$m_1,\cdots,m_t$,则$f(z)$在z_i的某个去心邻域的洛朗展式的主部为

$$B_i(z)=\sum_{n=-m_i}^{-1}c_n(z-z_i)^n$$

若$z=\infty$是$f(z)$的极点,则$f(z)$在$z=\infty$附近的洛朗展式的主部$B(z)$是一个多项式,令$g(z)=f(z)-B_1(z)-\cdots-B_t(z)-B(z)$, 则$g(z)$在$\mathbb{C}_\infty$上全纯,故$g(z)$为常数,设为$C$,则$f(z)=C+B(z)+B_1(z)+\cdots+B_t(z)$为有理函数.

5.求扩充复平面$\mathbb{C}_\infty$的亚纯自同构群$\mathrm{Aut}(\mathbb{C}_\infty)$, 即$\mathbb{C}_\infty$上所有亚纯自同构映射构成的集合.

6.利用$\log(1+\frac{z}{n})$的单值全纯分支的泰勒展开证明

$$\lim_{n\to\infty}(1+\frac{z}{n})^n=e^z$$

在任一紧集上一致收敛.

证明:

$$\log(1+\omega)=\sum_{j=0}^{+\infty}(-1)^j\frac{\omega^{j+1}}{j+1},\ |\omega|<1$$

对任一紧集$K\subset\mathbb{C}$,$\exists R(=\max_{z\in K}|z|+1)>0$,使得$K\subset\overline{B(0,R)}$.

考虑充分大$n > R$,$\forall z \in K, |z| \le R < n$,有$\left|\frac{z}{n}\right| \le \frac{R}{n} < 1$,设

$$f_n(z) = n\log(1+\frac{z}{n}),\ f(z) = z$$

则

$$f_n(z) = n\sum_{j=0}^{+\infty}(-1)^j\frac{\left(\frac{z}{n}\right)^{j+1}}{j+1} = z - \frac{1}{2}\cdot\frac{z^2}{n} + \frac{1}{3}\cdot\frac{z^3}{n^2} - \cdots$$

因此有

$$\begin{aligned}
|f_n(z) - f(z)| &= \left|n\log(1+\frac{z}{n}) - z\right| \\
&= \left|z\left[-\frac{1}{2}\cdot\left(\frac{z}{n}\right) + \frac{1}{3}\cdot\left(\frac{z}{n}\right)^2 - \cdots\right]\right| \\
&\le |z|\sum_{k=2}^{\infty}\frac{1}{k}\left|\frac{z}{n}\right|^{k-1} \\
&\le |z|\sum_{k=1}^{\infty}\left|\frac{z}{n}\right|^{k} \\
&= |z|\cdot\frac{\left|\frac{z}{n}\right|}{1-\left|\frac{z}{n}\right|} \\
&\le \frac{R^2}{n-R} \to 0\ (n\to\infty)
\end{aligned}$$

故当$n\to\infty$时,$f_n(z)$在K上一致收敛到$f(z)$.

$$|e^{f_n(z)} - e^{f(z)}| = |e^{f(z)}||e^{f_n(z)-f(z)} - 1| \le e^R|e^{f_n(z)-f(z)} - 1|$$

由e^z在$z=0$的连续性知,$\forall\varepsilon>0,\exists\delta>0,\text{s.t.}|e^z-1|<\varepsilon$对$|z|<\delta$.
由$f_n(z)$在K上一致收敛到$f(z)$知,$\exists\delta>0,\text{s.t.}|f_n(z)-f(z)|<\delta$对$n>N,\forall z\in K$.
故$|e^{f_n(z)}-e^{f(z)}|<\varepsilon e^R$对$n>N,\forall z\in K$.
因此,$\{e^{f_n(z)}\}$在K上一致收敛到$e^{f(z)}$,
故$\{(1+\frac{z}{n})^n\}$在K上一致收敛到e^z.

7.

(1) 证明$(e^z-1)^{-1}$ 在原点$z_0=0$的洛朗展开式是如下形式:

$$\frac{1}{z}-\frac{1}{2}+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}z^{2k-1}$$

这里B_k 称为伯努利数.
(2) 计算B_1,B_2,B_3.
(3)利用伯努利数表示$\tan z=\frac{\sin z}{\cos z}$ 在原点$z_0=0$的泰勒展开式和$\cot z=\frac{\cos z}{\sin z}$ 在原点$z_0=0$的洛朗展开式.

证明:(1)由于

$$\lim_{z\to 0}\frac{z}{e^z-1}=1$$

故0为$(e^z-1)^{-1}$的一阶极点,设

$$\frac{1}{e^z-1}=\frac{1}{z}+c_0+\sum_{n=1}^{+\infty}c_nz^n$$

则

$$c_0=\lim_{z\to 0}\left(\frac{1}{e^z-1}-\frac{1}{z}\right)=-\frac{1}{2}$$

故

$$\frac{1}{e^z-1}=\frac{1}{z}-\frac{1}{2}+\sum_{n=1}^{+\infty}c_nz^n$$

因为

$$\left(\frac{1}{e^z-1}-\frac{1}{z}+\frac{1}{2}\right)+\left(\frac{1}{e^{-z}-1}+\frac{1}{z}+\frac{1}{2}\right)=\frac{e^z+e^{-z}-2}{2-e^z-e^{-z}}+1=0$$

所以$\frac{1}{e^z-1}-\frac{1}{z}+\frac{1}{2}$是奇函数,因此$c_{2k}=0,k\geq 1$, 记$c_{2k-1}=(-1)^{k-1}\frac{B_k}{(2k)!}$,则

$$\frac{1}{e^z-1}=\frac{1}{z}-\frac{1}{2}+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}z^{2k-1}$$

(2)

$$(e^z-1)\cdot\frac{1}{e^z-1}=\left(\sum_{k=1}^{\infty}\frac{z^k}{k!}\right)\left[\frac{1}{z}-\frac{1}{2}+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}z^{2k-1}\right]=1$$

由左右两边对应系数相等可得:$B_1=\frac{1}{6}, B_2=\frac{1}{30}, B_3=\frac{1}{42}$.

(3)

$$\cot z=\frac{\mathtt{i}(e^{\mathtt{i}z}+e^{-\mathtt{i}z})}{e^{\mathtt{i}z}-e^{-\mathtt{i}z}}=\mathtt{i}\left(1+\frac{2}{e^{2\mathtt{i}z}-1}\right)=\frac{1}{z}-\sum_{k=1}^{\infty}\frac{2^{2k}B_k}{(2k)!}z^{2k-1}$$

$$\begin{aligned}\tan z&=\cot z-2\cot 2z\\&=\frac{1}{z}-\sum_{k=1}^{\infty}\frac{2^{2k}B_k}{(2k)!}z^{2k-1}-\frac{1}{z}+\sum_{k=1}^{\infty}\frac{2\cdot 2^{2k}B_k}{(2k)!}2^{2k-1}\cdot z^{2k-1}\\&=\sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)B_k}{(2k)!}z^{2k-1}\end{aligned}$$

作业9-解答

1. 假设u不是一个整数, 利用留数定理对

$$f(z)=\frac{\pi\cot\pi z}{(u+z)^2}$$

在圆周$|z|=N+\frac{1}{2}$(N是整数, $N\geq|u|$)上积分, 并取$N\to\infty$, 证明

$$\sum_{n=-\infty}^{\infty}\frac{1}{(u+n)^2}=\frac{\pi^2}{(\sin\pi u)^2}.$$

证明:设$C_N=\{z||z|=N+\frac{1}{2}\}$,根据留数定理,

$$\int_{C_N}f(z)\mathrm{d}z=2\pi\mathrm{i}\sum_{j=-n}^{n}\operatorname*{Res}_{z=j}f(z)+2\pi\mathrm{i}\operatorname*{Res}_{z=-u}f(z)$$

设$z=x+\mathrm{i}y$,当$|y|>\frac{1}{2\pi}$时,$|\cot\pi z|\leq M_1$;当$z=\frac{1}{2}+\mathrm{i}y$, $-\frac{1}{2}\leq y\leq\frac{1}{2}$时,$|\cot\pi z|\leq M_2$,又$\cot\pi(z+1)=\cot\pi z$,故$\cot\pi z$在$C_N$上一致有界,故

$$\lim_{n\to\infty}\int_{C_N}f(z)\mathrm{d}z=0$$

又

$$\operatorname*{Res}_{z=j}\frac{\pi\cot\pi z}{(u+z)^2}=\frac{1}{(u+j)^2},\ \operatorname*{Res}_{z=-u}\frac{\pi\cot\pi z}{(u+z)^2}=-\frac{\pi^2}{\sin^2\pi u}$$

所以当$N\to\infty$时,有

$$2\pi\mathrm{i}\cdot\sum_{n=-\infty}^{\infty}\frac{1}{(u+n)^2}-2\pi\mathrm{i}\cdot\frac{\pi^2}{(\sin\pi u)^2}=0.$$

即

$$\sum_{n=-\infty}^{\infty}\frac{1}{(u+n)^2}=\frac{\pi^2}{(\sin\pi u)^2}.$$

2. 设$P(x), Q(x)$是多项式满足$\deg Q(x) - \deg P(x) \geq 2$且$Q(n) \neq 0$ $(\forall n \in \mathbb{Z})$, 证明

$$\sum_{n=-\infty}^{\infty} (-1)^n \frac{P(n)}{Q(n)} = -\sum_{j=1}^{k} \mathrm{Res}_{z=a_j} \left(\frac{P(z)}{Q(z)} \pi \csc \pi z \right),$$

其中$a_1, \cdots, a_k$是$Q(z)$的互异零点.

证明:设$f(z) = \frac{P(z)}{Q(z)} \frac{\pi}{\sin \pi z}$,取$C_n$是以$(n+\frac{1}{2}) \cdot (\pm 1 \pm \mathrm{i})$为顶点的正方形,根据留数定理,

$$\int_{C_n} f(z) \mathrm{d}z = 2\pi \mathrm{i} \sum_{j=-n}^{n} \operatorname*{Res}_{z=j} f(z) + 2\pi \mathrm{i} \sum_{j=1}^{k} \operatorname*{Res}_{z=a_j} f(z)$$

设$z = x + \mathrm{i}y$,当$|y| \geq \frac{1}{2\pi}$时,不妨设$y \leq -\frac{1}{2\pi}$,则

$$\left| \frac{1}{\sin \pi z} \right| = \left| \frac{2\mathrm{i}}{e^{\mathrm{i}\pi z} - e^{-\mathrm{i}\pi z}} \right| = 2 \left| \frac{e^{\mathrm{i}\pi z}}{e^{2\mathrm{i}\pi z} - 1} \right| \leq \frac{2e^{\pi y}}{e^{-2\pi y} - 1} \leq \frac{2e^{-\frac{1}{2}}}{e - 1}$$

当$|y| \leq \frac{1}{2\pi}$时,考虑$z = \frac{1}{2} + \mathrm{i}y, -\frac{1}{2} \leq y \leq \frac{1}{2}$,则

$$\left| \frac{1}{\sin \pi z} \right| = 2 \left| \frac{e^{\mathrm{i}\pi z}}{e^{2\mathrm{i}\pi z} - 1} \right| = 2 \left| \frac{e^{-\pi y}}{e^{-2\pi y} + 1} \right| \leq \frac{2e^{\frac{\pi}{2}}}{e^{-\pi} + 1}$$

因此,$\frac{1}{\sin \pi z}$在C_n上一致有界,故

$$\lim_{n \to \infty} \int_{C_n} f(z) \mathrm{d}z = 0$$

又

$$\operatorname*{Res}_{z=n} f(z) = \lim_{z \to n} \frac{P(z)}{Q(z)} \frac{\pi(z-n)}{\sin \pi z} = (-1)^n \frac{P(n)}{Q(n)},$$

所以当$n \to \infty$时,有

$$2\pi \mathrm{i} \sum_{n=-\infty}^{\infty} (-1)^n \frac{P(n)}{Q(n)} + 2\pi \mathrm{i} \sum_{j=1}^{k} \operatorname*{Res}_{z=a_j} \left(\frac{P(z)}{Q(z)} \frac{\pi}{\sin \pi z} \right) = 0$$

即

$$\sum_{n=-\infty}^{\infty} (-1)^n \frac{P(n)}{Q(n)} = -\sum_{j=1}^{k} \operatorname*{Res}_{z=a_j} \left(\frac{P(z)}{Q(z)} \pi \csc \pi z \right).$$

3. 证明
$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots=\frac{\pi^3}{32}.$$
提示：无穷和可表示为
$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots=\frac{1}{2^4}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(n+\frac{1}{2})^3}.$$

证明:
$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}=\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(2n+1)^3}=\frac{1}{2^4}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(n+\frac{1}{2})^3}.$$
设$f(z)=\frac{\pi\csc\pi z}{(z+\frac{1}{2})^3}$,由Problem2知:
$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(n+\frac{1}{2})^3}=-\operatorname*{Res}_{z=-\frac{1}{2}}f(z)=-\lim_{z\to-\frac{1}{2}}\frac{1}{2}(\pi\csc\pi z)''=\frac{\pi^3}{2}$$
所以
$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots=\frac{\pi^3}{32}.$$

4. 利用亚纯函数的部分分式展开定理证明
$$\csc z=\frac{1}{z}+\sum_{n\in\mathbb{Z}\setminus\{0\}}(-1)^n\left(\frac{1}{z-n\pi}+\frac{1}{n\pi}\right).$$

证明:设$f(z)=\csc z-\frac{1}{z}$,极点$\{n\pi\}$, $(n\in\mathbb{Z}\setminus\{0\})$为单极点,
取闭曲线$C_n=\{z||z|=(n+\frac{1}{2})\pi\}$,则$R_n=\operatorname{dist}(0,C_n)=n\pi+\frac{1}{2}\to\infty(n\to\infty)$,且$C_n$长度$L_n=2\pi R_n=O(R_n)$,由Problem2知
$$|f(z)|\le|\csc z|+\left|\frac{1}{z}\right|\le|\csc z|+1$$

在C_n上一致有界,故$f(z)=o(R_n)$,又

$$\operatorname*{Res}_{z=n\pi} f(z) = (-1)^n$$

由亚纯函数的部分分式展开定理得

$$f(z) = \sum_{n\in\mathbb{Z}\setminus\{0\}} (-1)^n \left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right)$$

即

$$\csc z = \frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}} (-1)^n \left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right).$$

5. 利用亚纯函数的部分分式展开定理证明

$$\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + 2z\sum_{n=1}^{\infty} \frac{1}{z^2+4n^2\pi^2}.$$

证明:设$f(z)=\frac{1}{e^z-1}-\frac{1}{z}$,极点$\{2n\pi\mathrm{i}\},(n\in\mathbb{Z}\setminus\{0\})$为单极点,且

$$f(0) = \lim_{z\to 0} f(z) = -\frac{1}{2}$$

取闭曲线$C_n=\{z||z|=(2n+\frac{1}{2})\pi\}$,
则$R_n=\mathrm{dist}(0,C_n)=2n\pi+\frac{1}{2}\to\infty(n\to\infty)$,
且C_n长度$L_n=2\pi R_n=O(R_n)$,又

$$\lim_{n\to\infty} \frac{\left|\frac{1}{e^z-1}-\frac{1}{z}\right|}{R_n} = 0$$

故$f(z)=o(R_n)$,又

$$\operatorname*{Res}_{z=2n\pi\mathrm{i}} f(z) = 1$$

由亚纯函数的部分分式展开定理得

$$\begin{aligned}f(z)&=-\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{z+2n\pi\mathrm{i}}-\frac{1}{2n\pi\mathrm{i}}+\frac{1}{z-2n\pi\mathrm{i}}+\frac{1}{2n\pi\mathrm{i}}\right)\\&=-\frac{1}{2}+2z\sum_{n=1}^{\infty}\frac{1}{z^2+4n^2\pi^2}\end{aligned}$$

即

$$\frac{1}{e^z-1}=\frac{1}{z}-\frac{1}{2}+2z\sum_{n=1}^{\infty}\frac{1}{z^2+4n^2\pi^2}.$$

6. 利用$\sin z$在$z=\frac{\pi}{2}$处的无穷乘积公式证明下面的Wallis's乘积公式

$$\frac{\pi}{2}=\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdots\frac{2m\cdot 2m}{(2m-1)\cdot(2m+1)}\cdots.$$

证明:$\because$

$$\sin\pi z=\pi z\prod_{n\in\mathbb{Z}\setminus\{0\}}(1-\frac{z}{n})e^{\frac{z}{n}}$$

$\therefore$令$z=\frac{1}{2}$,有

$$\begin{aligned}1&=\frac{\pi}{2}\prod_{n\in\mathbb{Z}\setminus\{0\}}(1-\frac{1}{2n})e^{\frac{1}{2n}}\\&=\frac{\pi}{2}\prod_{n=1}^{\infty}\frac{2n-1}{2n}\cdot\frac{2n+1}{2n}e^{\frac{1}{2n}-\frac{1}{2n}}\\&=\frac{\pi}{2}\prod_{n=1}^{\infty}\frac{2n-1}{2n}\cdot\frac{2n+1}{2n}\end{aligned}$$

$\therefore$

$$\frac{\pi}{2}=\prod_{n=1}^{\infty}\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}=\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdots\frac{2m\cdot 2m}{(2m-1)\cdot(2m+1)}\cdots.$$

7. (1)(Poisson-Jensen公式) 设$f(z)$是闭圆盘$\{|z|\leq R\}$ $(0<R<\infty)$上的不恒为

零的亚纯函数,在$\{|z|=R\}$上无零点或极点, $a_1,\cdots,a_p$和$b_1,\cdots,b_q$分别是$f(z)$在开圆盘$\{|z|<R\}$上的零点和极点(均重复计数). 则对于开圆盘$\{|z|<R\}$内任一异于a_i $(i=1,\cdots,p)$ 与b_j $(j=1,\cdots,q)$的点z, 有

$$\begin{aligned}\log|f(z)| &= \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\log|f(Re^{\mathrm{i}\theta})|\mathrm{Re}\frac{Re^{\mathrm{i}\theta}+z}{Re^{\mathrm{i}\theta}-z}d\theta \\ &\quad -\sum_{i=1}^{p}\log\left|\frac{R^2-\overline{a_i}z}{R(z-a_i)}\right|+\sum_{j=1}^{q}\log\left|\frac{R^2-\overline{b_j}z}{R(z-b_j)}\right|.\end{aligned}$$

(2)(Jensen公式) 设$f(z)$是闭圆盘$\{|z|\le R\}$ $(0<R<\infty)$上的不恒为零的亚纯函数, $a_1,\cdots,a_p$和$b_1,\cdots,b_q$分别是$f(z)$在开圆盘$\{|z|<R\}$上非零的零点和极点(均重复计数). 设$f(z)=c_f z^{\mathrm{ord}_0 f}+\cdots$, $\mathrm{ord}_0 f\in\mathbb{Z}$, and c_f 是首项非零项系数. 则

$$\log|c_f|=\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\log|f(Re^{\mathrm{i}\theta})|d\theta-\sum_{i=1}^{p}\log\left|\frac{R}{a_i}\right|+\sum_{j=1}^{q}\log\left|\frac{R}{b_j}\right|-(\mathrm{ord}_0 f)\log R.$$

作业10-解答

1. (1)求方程

$$z^8 - 4z^5 + z^2 - 1 = 0$$

在单位圆盘$\{|z| < 1\}$内的根的个数(计重数).

(2)用辐角原理证明, 设α, β是正实数, 考虑方程

$$z^{2n} + \alpha^2 z^{2n-1} + \beta^2 = 0.$$

若n是奇数, 则该方程有$n-1$个根(计重数)具有正实部; 若n是偶数, 则该方程有n个根(计重数)具有正实部.

证明: (1).设$f(z) = -4z^5$,$g(z) = z^8 + z^2 - 1$,则在单位圆周上,$|g(z)| < |f(z)|$,
由儒歇定理得$\sharp_{zeros}(f+g, \Omega) = \sharp_{zeros}(f, \Omega) = 5$,
即方程$z^8 - 4z^5 + z^2 - 1 = 0$在单位圆盘$\{|z| < 1\}$内有5个根(计重数) .
(2).设$f(z) = z^{2n} + \alpha^2 z^{2n-1} + \beta^2, \Omega = \{z||z| < R, \mathrm{Re}z > 0\}$
当$z = \mathrm{i}y$时,$f(\mathrm{i}y) = (-1)^n y^{2n} + (-1)^{n-1}\alpha^2 y^{2n-1}\mathrm{i} + \beta^2$
当$C_R = \{z||z| = R, \mathrm{Re}z > 0\}$时,

$$\begin{aligned}
\Delta_{C_R}\arg f &= \Delta_{C_R}\arg\left[z^{2n}(1 + \frac{\alpha^2}{z} + \frac{\beta^2}{z^{2n}})\right] \\
&= \Delta_{C_R}\arg(z^{2n}) + \Delta_{C_R}\arg\left(1 + \frac{\alpha^2}{z} + \frac{\beta^2}{z^{2n}}\right) \\
&= 2n\pi + \Delta_{C_R}\arg\left(1 + \frac{\alpha^2}{z} + \frac{\beta^2}{z^{2n}}\right)
\end{aligned}$$

故当$R \to \infty$时,$\Delta_{C_R}\arg f = 2n\pi$.
当n为偶数时,$f(\mathrm{i}y) = y^{2n} + \beta^2 - \alpha^2 y^{2n-1}\mathrm{i}$,
故$\frac{1}{2\pi}\Delta_{\partial\Omega}\arg f = \frac{1}{2\pi}(2n\pi - 0) = n = \sharp_{zeros}(f, \Omega)$.
当n为奇数时,$f(\mathrm{i}y) = -y^{2n} + \beta^2 + \alpha^2 y^{2n-1}\mathrm{i}$,
故$\frac{1}{2\pi}\Delta_{\partial\Omega}\arg f = \frac{1}{2\pi}(2n\pi - 2\pi) = n - 1 = \sharp_{zeros}(f, \Omega)$.
即n为偶数时,方程有n个根(计重数)具有正实部; n为奇数时,方程有$n-1$个根(计

重数)具有正实部.

2. (1)用辐角原理证明, 四次方程

$$z^4+z^3+4z^2+2z+3=0$$

在开的第一象限$\Omega=\{\mathrm{Re}z>0,\ \mathrm{Im}z>0\}$内没有根.
(2)利用多项式$z^4+z^3+4z^2+2z+3$的系数都是实数的事实, 证明方程

$$z^4+z^3+4z^2+2z+3=0$$

在开的第四象限内没有根, 在开的第二,三象限内各有两个根.

证明:(1).设$f(z)=z^4+z^3+4z^2+2z+3$, $\Omega_R=\{z||z|<R,\mathrm{Re}z>0,\mathrm{Im}z>0\}$
$L_y=\{z=\mathrm{i}y,y>0\}$, $L_x=\{z=x>0\}$.

当$z=\mathrm{i}y$时,$f(\mathrm{i}y)=y^4-y^3\mathrm{i}-4y^2+2y\mathrm{i}+3=(y^4-4y^2+3)+(2y-y^3)\mathrm{i}$
$\arg f(\mathrm{i}y)=\arctan\frac{y(2-y^2)}{(y^2-1)(y^2-3)}$. 可以看到, 当$y\to\infty$时,$\Delta_{L_y^-}\arg f=-2\pi$.

当$z=x$时,$f(x)=x^4+x^3+4x^2+2x+3>0$, 可以看到, 当$x\to\infty$时,$\Delta_{L_x}\arg f=0$.

当$C_R=\{z||z|=R,\mathrm{Re}z>0,\mathrm{Im}z>0\}$时,

$$\begin{aligned}\Delta_{C_R}\arg f&=\Delta_{C_R}\arg(z^4)+\Delta_{C_R}\arg\left(1+\frac{1}{z}+\frac{4}{z^2}+\frac{2}{z^3}+\frac{3}{z^4}\right)\\&=2\pi+\Delta_{C_R}\arg\left(1+\frac{1}{z}+\frac{4}{z^2}+\frac{2}{z^3}+\frac{3}{z^4}\right)\end{aligned}$$

故当$R\to\infty$时,$\Delta_{C_R}\arg f=2\pi$.
因此$\frac{1}{2\pi}\Delta_{\partial\Omega_R}\arg f=\frac{1}{2\pi}(2\pi-2\pi+0)=0=\sharp_{zeros}(f,\Omega_R)$.

即$f(z)$在开的第一象限内没有根.

(2).由于$f(z)$的系数都是实数,故$f(z)$的零点是共轭出现的.

由(1)知$f(z)$在开的第四象限内没有根.

由于$f(z)$在开的第二,三象限内的零点是成对出现的,故只需说明$f(z)$在负实轴上无零点即可.

设$z=-x,(x>0)$,则$f(z)=x^4-x^3+4x^2-2x+3=x^2(x^2-x+4)+(-2x+3)$

当$0<x<1$时,两项均为正,故$f(z)>0$.

又$f(z)=x^3(x-1)+[2x(2x-1)+3]$

当$x>1$时,两项均为正,故$f(z)>0$.

又$f(1)=5>0$,因此$f(z)$在负实轴上取值为正,无零点.

3. (1)证明方程$z^4-6z+3=0$在$\{|z|<1\}$内有一个根, 在$\{1<|z|<2\}$内有三个根(计重数).

(2)求方程$z^4-8z+10=0$在$\{|z|<1\}$内及$\{1<|z|<3\}$内根的个数(计重数).

证明:(1).设$f(z)=z^4-6z+3,\ g(z)=-6z,$

在$\{|z|=1\}$上,有$|f-g|=|z^4+3|\leq 4<6=|g|$

由儒歇定理可得$\sharp_{zeros}(f,\{|z|<1\})=\sharp_{zeros}(g,\{|z|<1\})=1$

即$f(z)$在$\{|z|<1\}$内有一个根.

设$h(z)=z^4,$

在$\{|z|=2\}$上,有$|f-h|=|-6z+3|\leq 15<16=|h|$

由儒歇定理可得$\sharp_{zeros}(f,\{|z|<2\})=\sharp_{zeros}(h,\{|z|<2\})=4$

故$f(z)$在$\{|z|<2\}$内有四个根(计重数).

又$f(z)$在$\{|z|=1\}$上无根,

因此$f(z)$在$\{1<|z|<2\}$内有三个根(计重数).

(2).设$f(z)=z^4-8z+10,\ g(z)=10,$

在$\{|z|=1\}$上,有$|f-g|=|z^4-8z|\leq 9<10=|g|$

由儒歇定理可得$\sharp_{zeros}(f,\{|z|<1\})=\sharp_{zeros}(g,\{|z|<1\})=0$

即$f(z)$在$\{|z|<1\}$内没有根.

设$h(z)=z^4,$

在$\{|z|=3\}$上,有$|f-h|=|-8z+10|\leq 34<81=|h|$

由儒歇定理可得$\sharp_{zeros}(f,\{|z|<3\})=\sharp_{zeros}(h,\{|z|<3\})=4$
故$f(z)$在$\{|z|<3\}$内有四个根(计重数).
又$f(z)$在$\{|z|=1\}$上无根,
因此$f(z)$在$\{1<|z|<3\}$内有四个根(计重数).

4. 设Ω是$\mathbb{C}$中的区域, $f_n(z)\in\mathcal{H}(\Omega)$在$\Omega$上无零点, 若$\{f_n(z)\}$在$\Omega$上内闭一致收敛到$f(z)$, 则$f(z)$在$\Omega$上恒为零, 或者$f(z)$在$\Omega$上无零点.

证明:若$f(z)\not\equiv 0$且$\exists a\in\Omega$使得$f(a)=0$,
取充分小$r>0$,使得$\overline{B(a,r)}\in\Omega$, $f(z)$在$\overline{B(a,r)}$上无其他零点.
取$\epsilon=\min_{|z-a|=r}|f(z)|>0$,在$\{|z-a|=r\}$上,$|f(z)|\geq\epsilon>0$.
因为$\{f_n(z)\}$在Ω上内闭一致收敛到$f(z)$,
则$\exists$充分大的n,使得对$\forall z\in\overline{B(a,r)}$,有$|f_n(z)-f(z)|<\frac{\epsilon}{2}<|f(z)|$
由儒歇定理可得$\sharp_{zeros}(f,B(a,r))=\sharp_{zeros}(f_n,B(a,r))=0$
与a是$f(z)$的零点矛盾!
故$f(z)$在Ω上恒为零,或$f(z)$在Ω上无零点.

5. (1)设$\mathbb{D}=\{|z|<1\}$, 若$f(z)\in\mathcal{H}(\mathbb{D})$, 且$f(0)=0$, $f'(0)=1$, 对任意$z\in\mathbb{D}$有$|f(z)|\leq M$, 则$M\geq 1$且

$$f(\mathbb{D})\supset D_{\frac{1}{6M}}(0)=\left\{|z|<\frac{1}{6M}\right\}.$$

(2)设$D_R(z_0)=\{|z-z_0|<R\}$, 若$f(z)\in\mathcal{H}(D_R(z_0))$, 且$f(z_0)=0$, $|f'(z_0)|=\mu>0$, 对任意$z\in D_R(z_0)$有$|f(z)|\leq M$, 则

$$f(D_R(z_0))\supset D_{\frac{R^2\mu^2}{6M}}(0)=\left\{|z|<\frac{R^2\mu^2}{6M}\right\}.$$

证明:(1).$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,\ |z|<1,$
由柯西不等式,$|a_n|\le \frac{M}{r^n},\ 0<r<1$
取$r\to 1$,得$|a_n|\le M$,特别$1=|a_1|\le M.$
当$|z|=r(0<r<1)$时,

$$|f(z)|\ge |z|-|\sum_{n=2}^{\infty}a_nz^n|\ge r-\sum_{n=2}^{\infty}Mr^n=r-Mr^2\cdot\sum_{n=0}^{\infty}r^n=r-Mr^2\cdot\frac{1}{1-r}$$

令$\varphi(r)=r-\frac{Mr^2}{1-r},\ 0<r<1$,有

$$\varphi(\frac{1}{4M})=\frac{1}{4M}-M\cdot(\frac{1}{4M})^2\cdot\frac{1}{1-\frac{1}{4M}}=\frac{1}{4M}-\frac{1}{4(4M-1)}\ge\frac{1}{4M}-\frac{1}{4\cdot 3M}=\frac{1}{6M}$$

当$|z|=\frac{1}{4M}$时,$|f(z)|\ge\frac{1}{6M}>0$
$\forall a\in D_{\frac{1}{6M}}(0)$,当$|z|=\frac{1}{4M}$时,$|f(z)|\ge\frac{1}{6M}>|a|$
由儒歇定理,$f(z)-a$在$D_{\frac{1}{4M}}(0)$内零点个数与$f(z)$相同.(至少为1)
故$D_{\frac{1}{6M}}(0)\subset f(D_{\frac{1}{4M}}(0))\subset f(\mathbb{D})$.
(2).设$g(z)=\frac{f(Rz+z_0)}{\mu R},\ z\in\mathbb{D}$,
则$g(z)\in\mathcal{H}(\mathbb{D}),g(0)=0,g'(0)=1,|g(z)|\le\frac{M}{\mu R}$.
由(1)得$g(\mathbb{D})=\frac{1}{\mu R}f(D_R(z_0))\supset D_{\frac{\mu R}{6M}}(0)$,
故$f(D_R(z_0))\supset D_{\frac{R^2\mu^2}{6M}}(0)$.

6. (1)确定以原点为心的最大圆盘使得映射$w=z+z^2$限制在该圆盘上是一一映射.
答案:$\{|z|<\frac{1}{2}\}$
(2)确定以原点为心的最大圆盘使得映射$w=e^z$限制在该圆盘上是一一映射.
答案:$\{|z|<\pi\}$

作业11-解答

1. 设Aut($\mathbb{H}$)表示上半平面$\mathbb{H}$的全纯自同构群, 试证明: (1)

$$\mathrm{Aut}(\mathbb{H}) = \left\{ f(z) = \frac{az+b}{cz+d}, a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}.$$

(2)

$$\mathrm{Aut}(\mathbb{H}) \cong SL(2, \mathbb{R})/\{\pm I_2\}.$$

证明:(1).考虑映射

$$g : \mathbb{D} \xrightarrow{\varphi_a} \mathbb{H} \xrightarrow{f} \mathbb{H} \xrightarrow{\varphi_b^{-1}} \mathbb{D}$$

$$0 \mapsto a \mapsto b \mapsto 0$$

其中$a, b \in \mathbb{H}$,则$g \in \mathrm{Aut}(\mathbb{D})$且$g(0) = 0$,故$g = e^{\mathrm{i}\theta} z$. 又易知

$$\varphi_a^{-1} = \frac{z-a}{z-\overline{a}},\ \varphi_b^{-1} = \frac{z-b}{z-\overline{b}}$$

则

$$\varphi_a = \frac{\overline{a}z - a}{z-1},\ \varphi_b = \frac{\overline{b}z - b}{z-1}$$

则由$g = \varphi_b^{-1} \circ f \circ \varphi_a$可得

$$f = \varphi_b \circ g \circ \varphi_a^{-1} = \frac{(e^{\mathrm{i}\theta}\overline{b} - b)z + b\bar{a} - e^{\mathrm{i}\theta} a\bar{b}}{(e^{\mathrm{i}\theta} - 1)z + \bar{a} - e^{\mathrm{i}\theta} a} = \frac{\mathrm{i}(e^{\mathrm{i}\frac{\theta}{2}}\overline{b} - e^{-\mathrm{i}\frac{\theta}{2}} b)z + \mathrm{i}(e^{-\mathrm{i}\frac{\theta}{2}} b\bar{a} - e^{\mathrm{i}\frac{\theta}{2}} a\bar{b})}{\mathrm{i}(e^{\mathrm{i}\frac{\theta}{2}} - e^{-\mathrm{i}\frac{\theta}{2}})z + \mathrm{i}(e^{-\mathrm{i}\frac{\theta}{2}}\bar{a} - e^{\mathrm{i}\frac{\theta}{2}} a)}$$

令

$$A = \mathrm{i}(e^{\mathrm{i}\frac{\theta}{2}}\bar{b} - e^{-\mathrm{i}\frac{\theta}{2}} b), B = \mathrm{i}(e^{-\mathrm{i}\frac{\theta}{2}} b\bar{a} - e^{\mathrm{i}\frac{\theta}{2}} a\bar{b}), C = \mathrm{i}(e^{\mathrm{i}\frac{\theta}{2}} - e^{-\mathrm{i}\frac{\theta}{2}}), D = \mathrm{i}(e^{-\mathrm{i}\frac{\theta}{2}}\bar{a} - e^{\mathrm{i}\frac{\theta}{2}} a)$$

则有$A,B,C,D\in\mathbb{R}$且

$$AD-BC=-(a-\bar{a})(b-\bar{b})=4\text{Im}a\cdot\text{Im}b>0$$

令$A'=\frac{A}{\sqrt{AD-BC}},B'=\frac{B}{\sqrt{AD-BC}},C'=\frac{C}{\sqrt{AD-BC}},D'=\frac{D}{\sqrt{AD-BC}}$,则

$$f=\frac{A'z+B'}{C'z+D'},\ (A',B',C',D'\in\mathbb{R})$$

且

$$A'D'-B'C'=1$$

因此

$$\text{Aut}(\mathbb{H})=\left\{\frac{az+b}{cz+d},a,b,c,d\in\mathbb{R},ad-bc=1\right\}.$$

(2).$\forall F\in SL(2,\mathbb{R}),F=\begin{pmatrix}a&b\\c&d\end{pmatrix},ad-bc=1$

$$\varphi:SL(2,\mathbb{R})\to\text{Aut}(\mathbb{H}),F\mapsto\frac{az+b}{cz+d}$$

是满射,又

$$\begin{aligned}\varphi(F_1\cdot F_2)&=\varphi\begin{pmatrix}a_1a_2+b_1c_2&a_1b_2+b_1d_2\\c_1a_2+d_1c_2&c_1b_2+d_1d_2\end{pmatrix}\\&=\frac{(a_1a_2+b_1c_2)z+(a_1b_2+b_1d_2)}{(c_1a_2+d_1c_2)z+(c_1b_2+d_1d_2)}\\&=\frac{a_1\cdot\frac{a_2z+b_2}{c_2z+d_2}+b_1}{c_1\cdot\frac{a_2z+b_2}{c_2z+d_2}+d_1}\\&=\varphi(F_1)\circ\varphi(F_2)\end{aligned}$$

故φ是满同态,由群同态基本定理可知

$$SL(2,\mathbb{R})/\ker\varphi\cong\text{Aut}(\mathbb{H})$$

若$\varphi(F)=\mathrm{id}$,则$F=\pm I_2$,故$\ker\varphi=\pm I_2$,因此

$$\mathrm{Aut}(\mathbb{H})\cong SL(2,\mathbb{R})/\{\pm I_2\}.$$

2. 找一个共形映射将区域$\Omega=\{|z|<1,|z-1|<1\}$ 映成单位圆盘$\mathbb{D}=\{|w|<1\}$.

解:分式线性映射$f(z)=\frac{z-\frac{1+\sqrt{3}\mathrm{i}}{2}}{z-\frac{1-\sqrt{3}\mathrm{i}}{2}}=\frac{z-e^{\mathrm{i}\frac{\pi}{3}}}{z-e^{-\mathrm{i}\frac{\pi}{3}}}$ 将Ω映成角形区域$\left\{\frac{2\pi}{3}\le\arg f\le\frac{4\pi}{3}\right\}$.
旋转映射$g(z)=e^{-\mathrm{i}\frac{2\pi}{3}}f(z)$将$\Omega$映成角形区域$\left\{0\le\arg g\le\frac{2\pi}{3}\right\}$.
映射$h(z)=g(z)^{\frac{3}{2}}$将Ω映成上半平面.
映射$l(z)=\frac{h(z)-\mathrm{i}}{h(z)+\mathrm{i}}$将$\Omega$映成单位圆盘.
因此共形映射为

$$l(z)=\frac{\left[\left(\frac{z-e^{\mathrm{i}\frac{\pi}{3}}}{z-e^{-\mathrm{i}\frac{\pi}{3}}}\right)e^{-\mathrm{i}\frac{2\pi}{3}}\right]^{\frac{3}{2}}-\mathrm{i}}{\left[\left(\frac{z-e^{\mathrm{i}\frac{\pi}{3}}}{z-e^{-\mathrm{i}\frac{\pi}{3}}}\right)e^{-\mathrm{i}\frac{2\pi}{3}}\right]^{\frac{3}{2}}+\mathrm{i}}$$

3. 求共形映射将抛物线$y^2=2px\ (p>0)$的外部映成单位圆盘$\mathbb{D}=\{|w|<1\}$使得$z=0,z=-\frac{p}{2}$分别映到$w=1,w=0$.

解:取$f(z)=\sqrt{z-\frac{p}{2}}$,有$f(0)=\sqrt{\frac{p}{2}}\mathrm{i},\ f(-\frac{p}{2})=\sqrt{p}\mathrm{i}$,
取$g(z)=\frac{\sqrt{p}\mathrm{i}-f(z)}{f(z)-(\sqrt{2}-1)\sqrt{p}\mathrm{i}}$,有$g(0)=1,\ g(-\frac{p}{2})=0$.
因此共形映射为

$$g(z)=\frac{\sqrt{p}\mathrm{i}-\sqrt{z-\frac{p}{2}}}{\sqrt{z-\frac{p}{2}}-(\sqrt{2}-1)\sqrt{p}\mathrm{i}}$$

4. 求共形映射将双曲线$x^2-y^2=a^2\ (a>0)$右边分支的内部映成单位圆盘$\mathbb{D}=\{|w|<1\}$使得焦点、顶点分别映到$w=0,w=-1$.

解: 右边分支的焦点、顶点分别为$\left(\sqrt{2}a,0\right)$、$(a,0)$,
设

$$f=z^2-a^2,\ g=\mathrm{i}f,\ w=e^{\mathrm{i}\theta}\cdot\frac{g-\mathrm{i}a^2}{g+\mathrm{i}a^2}$$

又$w(a)=-1$,故$e^{\mathrm{i}\theta}=1$,综上,

$$w=\frac{z^2-2a^2}{z^2}=1-\frac{2a^2}{z^2}$$

5(丘赛-2011). 找一个具体的共形映射将开集$U=\{|z|>1\}\setminus(-\infty,-1]$映成单位圆盘$\mathbb{D}=\{|w|<1\}$.

解:映射$w_1=-\frac{1}{z}$将U映成$\mathbb{D}\setminus[0,1)$.
映射$w_2=\sqrt{w_1}$将$\mathbb{D}\setminus[0,1]$映成$\{z||z|<1,\mathrm{Im}(z)>0\}$.
映射$w_3=-\frac{w_2+1}{w_2-1}$将$\{z||z|<1,\mathrm{Im}(z)>0\}$映成$\{z|\mathrm{Re}(z)>0,\mathrm{Im}(z)>0\}$.
$w_4=w_3^2$将$\{z|\mathrm{Re}(z)>0,\mathrm{Im}(z)>0\}$映成$\{z|\mathrm{Im}(z)>0\}$.
$w=\frac{w_4-\mathrm{i}}{w_4+\mathrm{i}}$将上半平面映成单位圆盘.
因此共形映射为

$$w=\frac{\left(\frac{\sqrt{-\frac{1}{z}}+1}{\sqrt{-\frac{1}{z}}-1}\right)^2-\mathrm{i}}{\left(\frac{\sqrt{-\frac{1}{z}}+1}{\sqrt{-\frac{1}{z}}-1}\right)^2+\mathrm{i}}$$

6(丘赛-2012). 构造一个共形映射将区域

$$U=\left\{|z-\frac{\mathtt{i}}{2}|<\frac{1}{2}\right\}\setminus\left\{|z-\frac{\mathtt{i}}{4}|\le\frac{1}{4}\right\}$$

映到单位圆盘$\mathbb{D}=\{|w|<1\}$.

解:映射$w_1=\frac{z-\frac{\mathrm{i}}{2}}{z}$, $w_2=e^{\mathrm{i}2\pi w_1}$将$U$映成上半平面,
映射$w=\frac{w_2-\mathrm{i}}{w_2+\mathrm{i}}$将上半平面映成单位圆盘.
因此共形映射为

$$w=\frac{e^{\mathrm{i}\pi\frac{2z-\mathrm{i}}{z}}-\mathrm{i}}{e^{\mathrm{i}\pi\frac{2z-\mathrm{i}}{z}}+\mathrm{i}}$$

7(丘赛-2014). 证明: 若存在共形映射将圆环区域$\{r_1<|z|<r_2\}$映成圆环区域$\{\rho_1<|z|<\rho_2\}$, 则$\frac{r_2}{r_1}=\frac{\rho_2}{\rho_1}$.

8(丘赛-2016). 证明: 映射$w = f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$ 将$\{z \in S^2, |z| > 1\}$映成$S^2 \setminus [-1, 1]$.

证明:设$w = u + \mathrm{i}v$, $z = re^{\mathrm{i}\theta}$,则$u + \mathrm{i}v = \frac{1}{2}(re^{\mathrm{i}\theta} + \frac{1}{r}e^{-\mathrm{i}\theta})$,因此

$$\begin{cases} u = \frac{1}{2}(r + \frac{1}{r})\cos\theta \\ v = \frac{1}{2}(r - \frac{1}{r})\sin\theta \end{cases}$$

当$|z| = \rho(\rho > 1)$时,

$$\frac{u^2}{\frac{1}{4}(\rho + \frac{1}{\rho})^2} + \frac{v^2}{\frac{1}{4}(\rho - \frac{1}{\rho})^2} = 1$$

为平面上一个椭圆,

当$|z| = 1$时,$w(\{|z| = 1\}) = [-1, 1]$.

当$\rho > 1$逐渐增大时, 椭圆的长轴短轴逐渐增大而且越来越圆, 直至扫遍整个复平面.

9. (1)设单位圆盘$\mathbb{D} = \{|z| < 1\}$, $f(z) \in \mathcal{H}(\mathbb{D}) \cap C^0(\overline{\mathbb{D}})$, 在$\mathbb{D}$上不取零值, 且当$|z| = 1$时, $|f(z)| = 1$. 证明: $f(z)$ 是常值.

(2)设单位圆盘$\mathbb{D} = \{|z| < 1\}$, $f(z) \in \mathcal{H}(\mathbb{D}) \cap C^0(\overline{\mathbb{D}})$, 且当$|z| = 1$时, $|f(z)| = 1$. 证明: $f(z)$ 是有理函数.

证明:(1)定义

$$F(z) = \begin{cases} f(z), & |z| \le 1 \\ \frac{1}{\overline{f(\frac{1}{\bar{z}})}}, & |z| > 1 \end{cases}$$

由题意可知,$F(z)$在$|z| \le 1, |z| > 1$上连续,

对$\forall z_0 \in \{z \big| |z| = 1\}$, 当$|z| > 1$时,有

$$\lim_{z \to z_0, |z| > 1} F(z) = \lim_{z \to z_0, |z| > 1} \frac{1}{\overline{f(\frac{1}{\bar{z}})}} = \lim_{\frac{1}{\bar{z}} \to \frac{1}{\bar{z}_0} = z_0, \left|\frac{1}{\bar{z}}\right| < 1} \frac{1}{\overline{f\left(\frac{1}{\bar{z}}\right)}} = \frac{1}{\overline{f(z_0)}} = f(z_0) = F(z_0).$$

故$F(z)$在$\mathbb{C}$上连续.

又$F(z)$在$|z| < 1$上全纯,由Schwarz反射原理可知,$F(z)$在$\mathbb{C}$上全纯.

因为$f(0)\neq 0$, 所以$\lim\limits_{z\to\infty}F(z)=\frac{1}{f(0)}$是有限复数. 故$F(z)$为有界整函数,由刘维尔定理可知$F(z)$为常数,因此$f(z)$为常数.

(2)考虑

$$F(z)=\begin{cases} f(z) & |z|\leq 1 \\ \frac{1}{f(\frac{1}{\bar{z}})} & |z|>1 \end{cases}$$

由(1)知全纯函数$F(z)$在$\mathbb{C}$上可能的奇点为$f(z)$的零点. 若$f(z)$在$\overline{\mathbb{D}}$上无零点,则由(1)知$f(z)\equiv C$.

若$f(z)$在$\overline{\mathbb{D}}$上有零点,由全纯函数零点的孤立性及$\overline{\mathbb{D}}$是紧集知$f(z)$在$\overline{\mathbb{D}}$上只有有限个零点,且均在$\mathbb{D}=\{|z|<1\}$内,设为$z_1,\cdots,z_k$,则$\frac{1}{\bar{z}_1},\cdots,\frac{1}{\bar{z}_k}$ (可能含∞)为$F(z)$的极点. 故$F(z)$为$\mathbb{C}_\infty$上的亚纯函数, 从而为有理函数,因此$f(z)$为有理函数.

10. 设单位圆盘$\mathbb{D}=\{|z|<1\}$, 单位圆周$C=\{|z|=1\}$, $f(z)\in\mathcal{H}(\mathbb{D})$. 对于$C$上的点$w$, 若存在$w$的开邻域$U$ 和U上的全纯函数g使得在$\mathbb{D}\cap U$上$f=g$, 则称w是f的**正则点**. 如果C上没有f的正则点, 那么我们说$\mathbb{D}$上的全纯函数f不能全纯延拓穿过C. 设

$$f(z)=\sum_{n=0}^{\infty}z^{2^n}\ \text{for}\ |z|<1.$$

注意到该级数的收敛半径是1. 证明: f不能全纯延拓穿过C.

Hint: 考虑$\theta=\frac{2\pi p}{2^k}$, 其中$p,k\in\mathbb{Z}^+$. 设$z=re^{\mathrm{i}\theta}$, 则$|f(re^{\mathrm{i}\theta})|\to\infty\ (r\to 1^-)$.

证明:设$\theta=\frac{2\pi p}{2^k},\ p,k\in\mathbb{N}^*,\ z=re^{\mathrm{i}\theta}$,

则

$$|f(re^{\mathrm{i}\theta})|=\left|\sum_{n=0}^{\infty}r^{2^n}e^{\mathrm{i}2^n\theta}\right|\leq\left|\sum_{n=0}^{k}r^{2^n}e^{\mathrm{i}\frac{2\pi p}{2^{k-n}}}\right|+\sum_{n=k+1}^{\infty}r^{2^n}$$

对$\forall M>0,\exists\delta$,使得$(1-\delta)^{2^{M+1}}>1-\frac{1}{M+1}=\frac{M}{M+1}$.

当$1 > r > 1-\delta$时,

$$\sum_{n=0}^{\infty} r^{2^n} > \sum_{n=0}^{M+1} (1-\delta)^{2^n} > (M+1)\cdot(1-\delta)^{2^{M+1}} > (M+1)\cdot\frac{M}{M+1} = M.$$

故

$$\lim_{r\to 1^-} \sum_{n=0}^{\infty} r^{2^n} = +\infty$$

因此

$$\lim_{r\to 1^-} \left|f(re^{\mathtt{i}\theta})\right| = +\infty$$

假设ω是一个正则点，则存在ω的一个开邻域U, $U \subset \mathbb{C}$, g为U的上一个全纯函数,使得在$\mathbb{D}\cap U$上, $f = g$.由于$E = \left\{e^{\mathtt{i}\theta} | \theta = \frac{2\pi p}{2^k},\ p, k \in \mathbb{N}^*\right\} \subset \{z||z| = 1\}$稠密,存在$e^{\mathtt{i}\theta} \in E$,使得$e^{\mathtt{i}\theta} \in U$,则有

$$\lim_{r\to 1^-} \left|g(re^{\mathtt{i}\theta})\right| = \lim_{r\to 1^-} \left|f(re^{\mathtt{i}\theta})\right| = +\infty$$

与g全纯矛盾！故C上不存在正则点,命题得证.

11. 设分式线性变换$Tz = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$, $ad - bc = 1$. 证明：若$-2 < a+d < 2$, 则T是椭圆型；若$a+d = \pm 2$, 则T是抛物型；若$a+d < -2$, or, > 2, 则T是双曲型.

12. 设分式线性变换T满足:存在某个$n \in \mathbb{Z}$使得$T^n z = z$, 则T是椭圆型.