习题1

1.1. $X \sim b(1,p)$. $(X_1 \cdots X_5)$.

(1). $x_i = 0$ 或 1.

样本空间. $\{(x_1,x_2,x_3,x_4,x_5) \mid x_i = 0 \text{ 或 } 1\}$. 联合分布列: $p(x_1,x_2,x_3,x_4,x_5) = p^{\sum_{i=1}^{5} x_i}(1-p)^{5-\sum_{i=1}^{5} x_i}$.

(2). $T_1\ T_4$ 为统计量. T_2, T_3 非. 由 p 未知.

(3). $\bar{X} = \frac{3}{5}$ $S^2 = \frac{1}{5-1}\sum 3(\frac{3}{5}-1)^2 + 2(\frac{3}{5})^2 = \frac{3}{10}$

经验分布: $F_5(x) = \frac{V_5(x)}{5} = \begin{cases} 0 & , x<0 \\ \frac{2}{5} & , x \in [0,1) \\ 1 & , x \in [1,+\infty). \end{cases}$

1.5. $(X_1 \cdots X_n)$. $\overline{X_n}$, S_n^2 又有 X_{n+1}.

(1) $\overline{X_{n+1}} = \frac{X_1 + \cdots X_{n+1}}{n+1} = \frac{X_1 + \cdots X_n}{n+1} + \frac{X_{n+1}}{n+1} = \frac{n\overline{X_n}}{n+1} + \frac{X_{n+1}}{n+1} = \overline{X_n} - \frac{\overline{X_n}}{n+1} + \frac{X_{n+1}}{n+1} = \overline{X_n} + \frac{1}{n+1}(X_{n+1} - \overline{X_n})$

(2) $S_{n+1}^2 = \frac{1}{n}\sum_{i=1}^{n+1}(X_i - \bar{X})^2 = \cancel{\frac{1}{n}[S_n^2 (n-1) + (X_{n+1} - \bar{X})^2]}$.

$= \frac{1}{n}\left[\sum_{i=1}^{n+1}(X_i^2) - (n+1)\overline{X_{n+1}}^2\right]$.

$= \frac{1}{n}\left[X_{n+1}^2 + \sum_{i=1}^{n} X_i^2 - (n+1)\left[\overline{X_n} + \frac{1}{n+1}(X_{n+1} - \overline{X_n})\right]^2\right]$

$= \frac{1}{n}\left[X_{n+1}^2 + \sum_{i=1}^{n} X_i^2 - (n+1)\overline{X_n}^2 - 2\overline{X_n}(X_{n+1} - \overline{X_n}) - \frac{1}{n+1}(X_{n+1} - \overline{X_n})^2\right]$

$= \frac{1}{n}\left[\sum_{i=1}^{n} X_i^2 - (n-1)\overline{X_n}^2 + X_{n+1}^2 - 2\overline{X_n}X_{n+1} - \frac{(X_{n+1} - \overline{X_n})^2}{n+1}\right]$.

$= \frac{n-1}{n}S_n^2 + \frac{1}{n}\left[X_{n+1}^2 - 2\overline{X_n}X_{n+1} - \frac{(X_{n+1} - \overline{X_n})^2}{n+1}\right]$.

$= \frac{n-1}{n}S_n^2 + \frac{1}{n}\left[\frac{n}{n+1}X_{n+1}^2 - \frac{2n}{n+1}\overline{X_n}X_{n+1} + \frac{n}{n+1}\overline{X_n}^2\right]$

$= \frac{n-1}{n}S_n^2 + \frac{1}{n+1}(X_{n+1} - \overline{X_n})^2$.

$= \frac{n-1}{n}\left[S_n^2 + \frac{n}{n^2-1}(X_{n+1} - \overline{X_n})^2\right]$

1.8. $X \sim F(x)$. $(X_1 \cdots X_n)$. $\overline{X}$, $Var(X)=\sigma^2$.

(1) 求 $Var(X_i-\overline{X}) = Cov(X_i-\overline{X}, X_i-\overline{X})$.

$$= Var X_i + Var(\overline{X}) - 2Cov(X_i,\overline{X}) = \sigma^2 + \frac{\sigma^2}{n} - 2\cdot\frac{1}{n}\cdot\sigma^2 = \frac{n-1}{n}\sigma^2$$

(2). 证 $X_i-\overline{X}$, $X_j-\overline{X}$, $i\neq j$ 的 $\rho = -\frac{1}{n-1}$

$$Cov(X_i-\overline{X}, X_j-\overline{X}) = Var(\overline{X}) + Cov(X_i,X_j) - Cov(X_i,\overline{X}) - Cov(X_j,\overline{X})$$

$$= \frac{\sigma^2}{n} + 0 - \frac{1}{n}\cdot\sigma^2 - \frac{1}{n}\sigma^2 = -\frac{1}{n}\sigma^2$$

则 $\rho = \dfrac{Cov(X_i-\overline{X}, X_j-\overline{X})}{\sqrt{Var(X_i-\overline{X})Var(X_j-\overline{X})}} = \dfrac{-\frac{1}{n}\sigma^2}{\frac{n-1}{n}\sigma^2} = -\frac{1}{n-1}$

1.9. $X \sim N(\mu,\sigma^2)$. $(X_1 \cdots X_n)$. $d = \frac{1}{n}\sum_{i=1}^{n}|X_i-\mu|$.

求证: $E(d) = \sqrt{\frac{2}{\pi}}\sigma$ $\quad Var(d) = (1-\frac{2}{\pi})\frac{\sigma^2}{n}$.

$$E(d) = \frac{1}{n}\sum_{i=1}^{n}E(|X_i-\mu|)$$

$$E(|X_i-\mu|) = \int_{-\infty}^{+\infty}|x_i-\mu|\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$$

令 $t=\frac{x-\mu}{\sigma}$ 则: $= \int_{-\infty}^{+\infty}|\sigma t|\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{t^2}{2}}\cdot\sigma dt$.

$$= \sigma\int_{-\infty}^{+\infty}|t|\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt.$$

$$= 2\sigma\int_{0}^{+\infty}\frac{t}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt. = 2\sigma\cdot\frac{1}{\sqrt{2\pi}}.$$

则: $E(d) = \frac{1}{n}\cdot n\cdot 2\sigma\cdot\frac{1}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}\sigma$

$$Var(d) = \frac{1}{n^2}\sum_{i=1}^{n}Var(|X_i-\mu|) = \frac{1}{n}Var(|X_i-\mu|) = \frac{1}{n}\left[E(|X_i-\mu|^2) - E(|X_i-\mu|)^2\right]$$

$$= \frac{1}{n}\left[E(X_i^2-2\mu X_i+\mu^2) - \frac{2}{\pi}\sigma^2\right].$$

$$= \frac{1}{n}\left[E[X_i^2]-2\mu^2+\mu^2-\frac{2}{\pi}\sigma^2\right].$$

$$= \frac{1}{n}\left[\sigma^2+\mu^2-2\mu^2+\mu^2-\frac{2}{\pi}\sigma^2\right].$$

$$= \frac{1}{n}\left[\sigma^2-\frac{2}{\pi}\sigma^2\right] = (1-\frac{2}{\pi})\cdot\frac{\sigma^2}{n}$$

10. $X \sim R(a,b)$. $(X_1 \cdots X_n)$ 取自 X.

(1) $(X_1 \cdots X_n)$ 联合概率密度 $p(x_1, x_2 \cdots x_n) = \frac{1}{(b-a)^n}$

(2). 统计量. $X_{(n)} = \max\limits_{1 \le i \le n} X_i$

$$F_n(x) = P\{X_{(1)} \le x \cdots X_{(n)} \le x\} = \left(\frac{x-a}{b-a}\right)^n$$

$$f_n(x) = n\left(\frac{x-a}{b-a}\right)^{n-1} \cdot \frac{1}{b-a}$$

$$E[X_{(n)}] = \int_a^b x \cdot n \cdot \frac{1}{b-a}\left(\frac{x-a}{b-a}\right)^{n-1} dx = \frac{nb+a}{n+1}$$

$$F_1(x) = P\{X_{(1)} \le x\} = 1 - P\{X_{(1)} > x, X_{(2)} > x \cdots X_{(n)} > x\}$$

$$= 1 - \left(1 - \frac{x-a}{b-a}\right)^n$$

$$f_1(x) = n\left(\frac{b-x}{b-a}\right)^{n-1} \cdot \frac{1}{b-a}$$

$$E[X_{(1)}] = \int_a^b x f_1(x)\,dx = \frac{na+b}{n+1}$$

1.11. (1) $C = \frac{1}{3}$

(2). $\frac{1}{5} = a$. $b = \frac{1}{15}$

1.13. $X \sim R(0,1)$. $(X_1 \cdots X_n)$ 为取自 X 样本. 证明 $X_{(i)} \sim Be(i, n-i+1)$.

$$F_i(x) = P\{X_{(i)} \le x\} = P\{V_n(x) \ge i\} = \sum_{k=i}^{n}\binom{n}{k}[F(x)]^k[1-F(x)]^{n-k}$$

$$f_i(x) = \frac{n!}{(i-1)!(n-i)!}\left(\frac{x-a}{b-a}\right)^{i-1}\left(\frac{b-x}{b-a}\right)^{n-i} \cdot \frac{1}{b-a}$$

$$= \frac{\Gamma(n+1)}{\Gamma(i)\Gamma(n-i+1)} x^{i-1}(1-x)^{n-i}.$$ 则有 $X \sim Be(i, n-i+1)$.

1.14. $X \sim F(n,m)$. $0<\alpha<1$. $F_{1-\alpha}(m,n) = \frac{1}{F_\alpha(n,m)}$

— $F_{1-\alpha}(m,n) = J$. $F_\alpha(n,m) = K$. 证 $J = \frac{1}{K}$.

$$\int_{-\infty}^{J} F(m,n) = 1-\alpha. \quad \int_{-\infty}^{K} F(n,m) = \alpha. \qquad F(m,n) = \frac{1}{F(n,m)}$$

设 $F \sim F(m,n)$. 且 $P\{F \leq F_{1-\alpha}(m,n)\} = 1-\alpha$.

则 $P\{\frac{1}{F} \geq \frac{1}{F_{1-\alpha}(m,n)}\} = 1-\alpha$.

由 $F \sim F(m,n)$. 则 $\frac{1}{F} \sim F(n,m)$.

即. $P\{\frac{1}{F} \geq F_\alpha(n,m)\} = 1-\alpha$.

$\Rightarrow$. $F_{1-\alpha}(m,n) = \frac{1}{F_\alpha(n,m)}$

1.16. $(X_1 \cdots X_n)$. $\overline{X}$ 样本均值. 求 $\overline{X}$ 概率分布或分布密度

(1). $X \sim P(\lambda)$. Poisson $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

$\cancel{P(\overline{X}=k) = P(\sum_{i=1}^{n} X_i = nk). = e^{-n\lambda}}$

由 $\sum_{i=1}^{n} X_i \sim P(n\lambda)$.

则. $P(\sum_{i=1}^{n} X_i = nk) = \frac{(n\lambda)^{nk}}{(nk)!} e^{-(n\lambda)} \Rightarrow P(\frac{1}{n}\sum_{i=1}^{n} X_i = k) = \frac{(n\lambda)^{nk}}{(nk)!} e^{-(n\lambda)}$

(2). $X \sim Exp(\lambda)$. $f(x) = \lambda e^{-\lambda x}$, $x>0$; other, 0.

$\cancel{F(x) = P(\overline{X} \leq x). = P(\sum_{i=1}^{n} X_i \leq nx) = \int_{0}^{nx} \lambda^n e^{-\lambda(x_1+x_2+\cdots x_n)} d(x_1+\cdots x_n)}$.

$\cancel{f(x) = n\lambda^n e^{-\lambda n x}}$ $X \sim \Gamma(1,\lambda)$. $\sum_{i=1}^{n} X_i \sim \Gamma(n,\lambda)$. $\frac{1}{n}\sum_{i=1}^{n} X_i \sim \Gamma(n, n\lambda)$.

则: $f(\frac{1}{n}\sum_{i=1}^{n} X_i; n, n\lambda) = \begin{cases} \frac{(n\lambda)^n}{\Gamma(n)} x^{n-1} e^{-n\lambda x}, & x>0 \\ . \text{otherwise}, & x=0. \end{cases}$

(3). $X \sim \chi^2(\nu)$.

$\sum_{i=1}^{n} X_i \sim \chi^2(n\nu). = Ga(\frac{n\nu}{2}, \frac{1}{2})$

$\frac{1}{n}\sum_{i=1}^{n} X_i \sim Ga(\frac{n\nu}{2}, \frac{n}{2})$. $f(x;\alpha,\lambda) = \begin{cases} \frac{(\frac{n}{2})^{\frac{n\nu}{2}}}{\Gamma(\frac{n\nu}{2})} x^{\frac{n\nu}{2}-1} e^{-\frac{n}{2}x}, & x>0 \\ 0. & x=0 \end{cases}$

$X \sim N(\mu_1, \sigma^2)$. $Y \sim N(\mu_2, \sigma^2)$. $(X_1 \cdots X_n)$. $(Y_1 \cdots Y_m)$.

$Q_1 = \sum_{i=1}^{m}(X_i - \bar{X})^2 \quad Q_2 = \sum_{i=1}^{n}(Y_i - \bar{Y})^2$.

$$\frac{Q_1}{Q_1+Q_2} = \frac{Q_1/\sigma^2}{Q_1/\sigma^2 + Q_2/\sigma^2} = \frac{\chi^2(m-1)}{\chi^2(m-1)+\chi^2(n-1)} = Be\left(\frac{m-1}{2}, \frac{n-1}{2}\right).$$

$$\frac{Q_2}{Q_1+Q_2} \sim Be\left(\frac{n-1}{2}, \frac{m-1}{2}\right).$$

1.18. $X \sim N(0, \sigma^2)$. $(X_1, X_2) \in X$. $Y = \frac{(X_1+X_2)^2}{(X_1-X_2)^2} = \frac{\left(\frac{X_1+X_2}{\sqrt{2}\sigma}\right)^2}{\left(\frac{X_1-X_2}{\sqrt{2}\sigma}\right)^2}$ $\quad Y \sim \frac{\chi^2(1)}{\chi^2(1)} \sim F(1,1)$.

$$F_Y(y) = \begin{cases} \frac{1}{\pi\sqrt{y}(1+y)}, & y>0 \\ 0, & \text{otherwise} \end{cases}$$

1.22. 验证指数分布.

(1) Gamma $\{Ga(\alpha,\lambda).\ \alpha>0, \lambda>0\}$.

$$f(x_1 \cdots x_n; \alpha, \lambda) = \left[\frac{\lambda^\alpha}{\Gamma(\alpha)}\right]^n \left(\prod_{j=1}^{n} x_j\right)^{\alpha-1} e^{-\lambda(x_1+\cdots x_n)}.$$

$$= \left[\frac{\lambda^\alpha}{\Gamma(\alpha)}\right]^n e^{\alpha-1\sum_{i=1}^{n}\ln x_i} \cdot e^{-\lambda(x_1+\cdots x_n)}.$$

$a(\alpha,\lambda) = \left[\frac{\lambda^\alpha}{\Gamma(\alpha)}\right]^n$ $Q_1(\alpha,\lambda) = \alpha-1$. $T_1(\vec{x}) = \sum_{i=1}^{n}\ln x_i$. $Q_2(\alpha,\lambda) = -\lambda$. $T_2(\vec{x}) = \sum_{i=1}^{n} x_i$. $h(x_1 \cdots x_n) = 1$.

且支撑集 $\{x \mid f(x)>0\}$ 不依赖 α, λ. 则为指数分布族

(2). Beta. $\{Beta(a,b);\ a>0, b>0\}$

$$f(x_1, \cdots x_n; a, b) = \left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right]^n \left(\prod_{i=1}^{n} x_i\right)^{a-1} \left(\prod_{i=1}^{n}(1-x_i)\right)^{b-1}$$

$$= \left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right]^n e^{(a-1)\sum_{i=1}^{n}\ln x_i} e^{(b-1)\sum_{i=1}^{n}\ln(1-x_i)}$$

$a(\alpha,\lambda) = \left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right]^n$ $\quad T_1(\vec{x}) = \sum_{i=1}^{n}\ln x_i \quad Q_1(a,b) = a-1 \quad T_2(\vec{x}) = \sum_{i=1}^{n}\ln(1-x_i) \quad Q_2(a,b) = b-1$.

则为指数分布族.

1.26. $X \sim N(\mu_1, \sigma^2)$. $Y \sim N(\mu_2, \sigma^2)$ $(X_1 \cdots X_{n_1})$ $(Y_1 \cdots Y_{n_2})$. S_1^2, S_2^2

$S_w^2 = \dfrac{(n_1-1)S_1^2+(n_2-1)S_2^2}{(n_1+n_2-2)}$ 求(1) $E(S_w^2)$ (2) $Var(S_w^2)$.

(1) 由. $\dfrac{(n_1-1)S_1^2}{\sigma^2} \sim \chi^2(n_1-1)$. $\dfrac{(n_2-1)S_2^2}{\sigma^2} \sim \chi^2(n_2-1)$.

则 $E(S_w^2) = \dfrac{\sigma^2}{n_1+n_2-2} E\left(\dfrac{(n_1-1)S_1^2+(n_2-1)S_2^2}{\sigma^2}\right) = \dfrac{\sigma^2}{n_1+n_2-2} \cdot n_1+n_2-2 = \sigma^2$.

$S(S_w^2) = \dfrac{\sigma^4}{(n_1+n_2-2)^2} S\left(\dfrac{(n_1-1)S_1^2+(n_2-1)S_2^2}{\sigma^2}\right) = \dfrac{\sigma^4}{(n_1+n_2-2)^2} \cdot 2(n_1+n_2-2) = \dfrac{2\sigma^4}{(n_1+n_2-2)}$.

1.27. $X \sim N(\mu, \sigma^2)$. $(X_1 \cdots X_n)$. $\bar{X}$ S^2. X_{n+1} 证:

$T = \dfrac{X_{n+1}-\bar{X}}{S}\sqrt{\dfrac{n}{n+1}} \sim t(n-1)$.

由 $X_{n+1} \sim N(\mu, \sigma^2)$. $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ $X_{n+1} - \bar{X} \sim N(0, (1+\frac{1}{n})\sigma^2)$.

则: $\dfrac{X_{n+1}-\bar{X}}{\sqrt{\frac{n+1}{n}}\sigma} \sim N(0,1)$. 而 $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$.

则: $T = \dfrac{X_{n+1}-\bar{X}}{\sqrt{\frac{n+1}{n}}\sigma} \Big/ \sqrt{\dfrac{(n-1)S^2}{(n-1)\sigma^2}} = \dfrac{X_{n+1}-\bar{X}}{S}\sqrt{\dfrac{n}{n+1}} \sim t(n-1)$

1.28. $X \sim N(\mu, \sigma^2)$ $(\sigma>0)$. $(X_1 \cdots X_{2n})$. $(n \geq 2)$. $\bar{X} = \frac{1}{2n}\sum_{i=1}^{2n} X_i$

求: $Y = \sum_{i=1}^{n}(X_i + X_{n+i} - 2\bar{X})^2$ 的 $E[Y]$.

令 $M = X_i + X_{n+i} - 2\bar{X}$. 要求 $E[\sum_{i=1}^{n} M^2] = \sum_{i=1}^{n} E[M^2] = n[Var(M) + E[M]^2]$

$M = X_i + X_{n+i} - \frac{1}{n}(X_1 + \cdots X_{2n}) = \frac{1}{n}(n-1)[X_i + X_{n+i}] - \frac{1}{n}(X_1 + \cdots X_{2n})_{\substack{2n\neq i \\ 2n\neq j}}$

$E(M) = \mu + \mu - 2\mu = 0$. $Var(M) = \cancel{\sigma^2+\sigma^2+4\sigma^2/2n} \dfrac{(n-1)^2}{n^2}\sigma^2 + \dfrac{(n-1)^2}{n^2}\sigma^2 + \dfrac{1}{n^2}(2n-2)\sigma^2$

$= \dfrac{2\sigma^2(n-1)}{n}$.

则 $E[M^2] = Var(M) + E[M]^2 = \dfrac{2\sigma^2(n-1)}{n}$. $\Rightarrow E[Y] = nE[M^2] = 2\sigma^2(n-1)$

(1). $(\prod_{i=1}^n X_i,\ \sum_{i=1}^n X_i)$ 为 Gamma 的充分统计量.

$$f(x_1,x_2\cdots x_n;\alpha,\lambda)=\left[\frac{\lambda^\alpha}{\Gamma(\alpha)}\right]^n\left(\prod_{i=1}^n x_i\right)^{\alpha-1}e^{-\lambda\sum_{i=1}^n x_i}.$$

$$=\left[\frac{\lambda^\alpha}{\Gamma(\alpha)}\right]^n e^{\alpha-1\ \ln(\prod_{i=1}^n x_i)}\ e^{-\lambda\sum_{i=1}^n x_i}.$$

$h(\vec{x})=1.$ $a(\alpha,\lambda)=\left[\frac{\lambda^\alpha}{\Gamma(\alpha)}\right]^n.$ $\alpha-1=Q_1(\alpha,\lambda).$ $T_1(\vec{x})=\prod_{i=1}^n x_i$ $-\lambda=Q_2(\alpha,\lambda).$ $T_2(\vec{x})=\sum_{i=1}^n x_i$

则有: $(\prod_{i=1}^n X_i,\ \sum_{i=1}^n X_i)$ 为 Gamma 充分. $g(\vec{x},\alpha,\lambda)=aQ_1T_1Q_2T_2$

(2). $f(x_1\cdots x_n;\alpha,\lambda)=\left[\frac{\lambda^\alpha}{\Gamma(\alpha)}\right]^n\left(\prod_{i=1}^n x_i\right)^{\alpha-1}e^{-\lambda\sum_{i=1}^n x_i}.\ \underset{\alpha\text{已知}}{=}\ h(\vec{x})g(T(x_1\cdots x_n);\lambda)$

$h(x_1\cdots x_n)=\frac{1}{\Gamma(\alpha)^n}\left[\prod_{i=1}^n x_i\right]^{\alpha-1}.$ $Q_1(\lambda)=-\lambda.$ $T_1(\vec{x})=\sum_{i=1}^n x_i.$ $a(\lambda)=\lambda^{\alpha n}$ $g=Q_1T_1a$

(3). $f(x_1\cdots x_n;\alpha,\lambda)=\left[\frac{\lambda^\alpha}{\Gamma(\alpha)}\right]^n\left(\prod_{i=1}^n x_i\right)^{\alpha-1}e^{-\lambda\sum_{i=1}^n x_i}$ λ已知 $=h(\vec{x})g(T(x_1\cdots x_n);\alpha).$

$h(\vec{x})=e^{-\lambda\sum_{i=1}^n x_i}$ $a(\alpha)=\left[\frac{\lambda^\alpha}{\Gamma(\alpha)}\right]^n.$ $Q_1(\alpha)=\alpha-1.$ $T_1(\vec{x})=\prod_{i=1}^n x_i.$ $g=Q_1T_1a$

1.34.

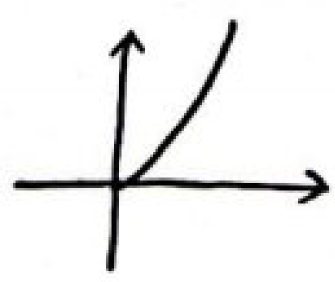
$X\sim f(x;\theta)=\frac{1}{\theta}e^{-\frac{|x|}{\theta}},\ -\infty<x<+\infty.$ $(X_1\cdots X_n)\Rightarrow T=\sum_{i=1}^n|X_i|$ 为 θ 充分.

$f(x_1\cdots x_n;\theta)=\frac{1}{\theta^n}e^{-\frac{1}{\theta}\sum_{i=1}^n|x_i|}$ $h(x_1,\cdots x_n)=1.$ $g(T,\theta)=f(x_1\cdots x_n;\theta)$

则由因子分解. T为 θ 充分.

1.35. $(X_1\cdots X_n)\in N(\sigma,\sigma^2).$ $T=(\sum_{i=1}^n X_i,\sum_{i=1}^n X_i^2)$ 为σ充分.但不完备.

$f(x_1\cdots x_n;\sigma)=\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n(x_i-\sigma)^2}$ 由因子分解: $h(\vec{x})=1.$ $g(T,\sigma)=f(\vec{x},\sigma).\Rightarrow$充分

不完备性由于不满足二维开矩形. (σ,σ^2) 代表 

习题2.

2.5. $X \sim Ga(\alpha,\theta)$. 分布

$$f(x;\alpha,\theta)=\begin{cases}\frac{\theta^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\theta x}, & x>0.\\ 0. & \text{otherwise.}\end{cases}$$

$(X_1,\cdots,X_n)\in Ga(\alpha,\theta)$. 求 α,θ 矩估计.

$$\begin{cases}\frac{1}{n}\sum_{i=1}^{n}x_i^2=\frac{\alpha}{\theta}=\bar{X}\\ \frac{1}{n}\sum_{i=1}^{n}x_i^2=\frac{\alpha}{\theta^2}+\left(\frac{\alpha}{\theta}\right)^2=\tilde{S}^2\end{cases}\Rightarrow\begin{cases}\alpha=\frac{\bar{X}^2}{\tilde{S}^2-\bar{X}^2}\\ \theta=\frac{\bar{X}}{\tilde{S}^2-\bar{X}^2}\end{cases}$$

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$$f(x;\theta)=\begin{cases}(\theta+1)x^{\theta}, & 0<x<1.\\ 0 & \text{, otherwise.}\end{cases}$$

$\theta>-1$. $(X_1\cdots X_n)\in f(x;\theta)$. 求 θ 矩估计与极大似然估计.

则 $E[X]=\int_0^1(\theta+1)x^{\theta+1}dx=\frac{\theta+1}{\theta+2}$. $\quad E[X^2]=\frac{\theta+1}{\theta+3}$

$$Var(X)=E[X^2]-E[X]^2=\frac{\theta+1}{\theta+3}-\left(\frac{\theta+1}{\theta+2}\right)^2$$

则: $\bar{X}=\frac{\theta+1}{\theta+2}\Rightarrow\theta=\frac{1-2\bar{X}}{\bar{X}-1}$

$$L(x_1\cdots x_n;\theta)=(\theta+1)^n\left(\prod_{i=1}^{n}x_i\right)^{\theta}.\quad 0<x<1.$$

$$\ln L=n\ln(\theta+1)+\sum_{i=1}^{n}\theta\ln x_i$$

由 $\frac{\partial L}{\partial\theta}=0$ 即 $\frac{n}{\theta+1}+\sum_{i=1}^{n}\ln x_i=0\Rightarrow\theta=-\frac{n}{\sum_{i=1}^{n}\ln x_i}-1.$

矩估计: $\hat{\theta}=\frac{1-2\bar{X}}{\bar{X}-1}$ 极大似然估计: $\hat{\theta}=-\frac{n}{\sum_{i=1}^{n}\ln x_i}-1.$

2.8. $X \sim Ga(\alpha,\theta)$. $f(x;\theta) = \begin{cases} \frac{\theta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta x}, & x>0. \\ 0. & otherwise. \end{cases}$

$(X_1 \cdots\cdots X_n) \in X$. 由~~α~~, θ 极大似然估计，α 已知.

$$L(x_1\cdots x_n;\theta) = \frac{\theta^{n\alpha}}{\Gamma(\alpha)^n}\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} e^{-\theta(\sum_{i=1}^{n} x_i)} \qquad \ln L = n\alpha\ln\theta - n\ln\Gamma(\alpha) + (\alpha-1)\sum_{i=1}^{n}\ln x_i - \theta(\sum_{i=1}^{n} x_i)$$

$$\begin{cases} \frac{\partial \ln L}{\partial\theta} = 0 \\ \cancel{\frac{\partial \ln L}{\partial\alpha} = 0} \end{cases} \Rightarrow \begin{cases} \frac{n\alpha}{\theta} - \sum_{i=1}^{n} x_i = 0. \\ \cancel{n\ln\theta -} \end{cases}$$

即 $\frac{\partial \ln L}{\partial\theta} = 0$

$\frac{n\alpha}{\theta} - \sum_{i=1}^{n} x_i = 0.$

$\Rightarrow \theta = \frac{n\alpha}{\sum_{i=1}^{n} x_i}.$ 即 $\hat{\theta} = \frac{\alpha}{\bar{X}}$

2.10

X	0	1	2	3
P	θ^2	$2\theta(1-\theta)$	θ^2	$(1-2\theta)$.

离散型.

其中 $\theta(0<\theta<\frac{1}{2})$ 有8个观测值.(2,1,2,0,3,1,2,3).求:

(1) $\hat{\theta}_1$ 矩.

(2) $\hat{\theta}_2$ 极大.

(1) $E[X] = 1\cdot 2\theta(1-\theta) + 2\cdot\theta^2 + 3\cdot(1-2\theta) = 3-4\theta.$

则矩估计：$\bar{X} = 3-4\theta \Rightarrow \hat{\theta} = -\frac{\bar{X}-3}{4}.$

而 $\bar{X} = \frac{1}{8}\sum_{i=1}^{8} X_i = \frac{1}{8}(2+1+2+0+3+1+2+3) = \frac{7}{4}$

则 $\hat{\theta} = -\frac{\frac{7}{4}-3}{4} = \frac{5}{16}.$

(2). $L(x_1\cdots x_8,\theta) = [\theta^2]^1[2\theta(1-\theta)]^2[\theta^2]^3[1-2\theta]^2 = 4\theta^{10}(1-\theta)^2(1-2\theta)^2$

$\ln L = \cancel{2\ln\theta + 2\ln 2\theta + 2\ln(1-\theta) + 6\ln\theta + 2\ln(1-2\theta)}\ \ln 4 + 10\ln\theta + 2\ln(1-\theta) + 2\ln(1-2\theta).$

$\frac{\partial \ln L}{\partial\theta} = \frac{10}{\theta} - \frac{2}{1-\theta} - \frac{4}{1-2\theta} = 0. \Rightarrow \hat{\theta} = \frac{9\pm\sqrt{11}}{14}$ 取 $\cancel{\frac{9+\sqrt{11}}{14}}$. $\frac{9-\sqrt{11}}{14}$

$$X \sim f(x;\theta)=\begin{cases}\theta, & 0<x<1.\\ 1-\theta, & 1\le x<2.\\ 0, & \text{otherwise.}\end{cases}\quad \theta\in(0,1).\quad (X_1\cdots X_n)\in X.$$

记 N 为 $X_1\cdots X_n$ 中小于1的个数.

(1) 求 θ 矩估计 (2) 由求 θ 极大似然估计.

(1). $E(X)=\int_0^1 x\theta\,dx+\int_1^2 x(1-\theta)\,dx$

$=\frac{1}{2}\theta+(1-\theta)\frac{1}{2}x^2\Big|_1^2$

$=\frac{1}{2}\theta+(1-\theta)\cdot\frac{3}{2}=\frac{3}{2}-\theta.=\overline{X}\ \Rightarrow\ \hat{\theta}=\frac{3}{2}-\overline{X}$

(2). $L=\theta^N(1-\theta)^{n-N}$

$\ln L=N\ln\theta+(n-N)\ln(1-\theta).$

$\frac{\partial L}{\partial\theta}=\frac{N}{\theta}-\frac{n-N}{1-\theta}=0\ \Rightarrow\ \hat{\theta}=\frac{N}{n}$

2.12 黑+白, 有放回取 n. 有 k 个白. 求黑:白 $=R$. 极大似然估计.

$P(X=k)=C_n^k\left(\frac{R}{1+R}\right)^{n-k}\left(\frac{1}{1+R}\right)^k$

$\ln P=\ln C_n^k+(n-k)\ln\frac{R}{1+R}+k\ln\frac{1}{1+R}$

$\frac{\partial \ln P}{\partial R}=(n-k)\cdot\frac{1+R}{R}\cdot\frac{1}{(1+R)^2}-k\cdot\frac{1}{1+R}=0.\ \Rightarrow\ \hat{R}=\frac{n-k}{k}$

2.15. $(X_1\cdots X_n)\in X.$

$$f(x;\mu,\sigma)=\begin{cases}\frac{1}{\sigma}e^{-\frac{x-\mu}{\sigma}}, & x\ge\mu.\\ 0. & \text{otherwise.}\end{cases}\qquad -\infty<\mu<+\infty.\ \sigma>0.$$

求 $\mu.\sigma$ 极大似然估计.

则: $L(x_1\cdots x_n;\mu,\sigma)=\begin{cases}0, & \exists x_i\in(-\infty,\mu).\\ \frac{1}{\sigma^n}e^{-\frac{\sum_{i=1}^n(x_i-\mu)}{\sigma}} & \forall x_i\ge\mu.\ i\in[1,n].\ i\in Z^*\end{cases}$

则: $\ln L=-n\ln\sigma-\frac{\sum_{i=1}^n x_i-n\mu}{\sigma}$

$\ln L = -n\ln\sigma - \frac{1}{\sigma}(\sum_{i=1}^{n} x_i - n\mu).$

$\frac{\partial \ln L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^2}(\sum_{i=1}^{n} x_i - n\mu) = 0 \Rightarrow \sigma = \frac{\sum_{i=1}^{n} x_i - n\mu}{n} = \bar{x} - \mu.$

则：令 L 为max时，即令 μ 为max，而 $\max \mu = x_{(1)}$.（$\forall x_i > \mu$）.

即 $\hat{\mu} = x_{(1)}$. $\hat{\sigma} = \bar{x} - x_{(1)}$.

2.17. $(x_1 \cdots x_n) \in X$. $(n \geq 2)$. $E(X) = \mu$, $Var(X) = \sigma^2$. 求 k，使 $\hat{\sigma}^2 = \frac{1}{k}\sum_{i=1}^{n-1}(x_{i+1} - x_i)^2$ 为 σ^2 的无偏估计量。　　　即求 $E[\hat{\sigma}^2] = C\sigma^2$，解出 C，令 $C=1$.

$E[\hat{\sigma}^2] = E[\frac{1}{k}\sum_{i=1}^{n-1}(x_{i+1}-x_i)^2] = \frac{1}{k}E[\sum_{i=1}^{n-1}(x_{i+1}-x_i)^2] = \frac{n-1}{k}E(x_{i+1}-x_i)^2$

$= \frac{n-1}{k}(E(x_{i+1}^2) - 2E(x_{i+1}x_i) + E(x_i^2))$. 　$E(x_{i+1}x_i) = E[x_{i+1}]E[x_i] = E[x]^2$

$= \frac{n-1}{k}[2E[x^2] - 2E[x]^2] = \frac{2(n-1)}{k}\sigma^2 \Rightarrow \frac{2(n-1)}{k} = 1 \Rightarrow k = 2(n-1).$

2.19. $X \sim f(x;\sigma) = \frac{1}{2\sigma}e^{-\frac{|x|}{\sigma}}$ $-\infty < x < +\infty$. $(\sigma > 0)$. $(x_1 \cdots x_n) \in X$.

(1). 求 σ 的极大似然估计 $\hat{\sigma}$. (2) 证 $\hat{\sigma}$ 为 σ 无偏（$E(\hat{\sigma}) = \sigma$）

(1). $L(x_1 \cdots x_n;\sigma) = \frac{1}{(2\sigma)^n}e^{-\frac{\sum_{i=1}^{n}|x_i|}{\sigma}}$

$\ln L = -n\ln 2\sigma - \frac{\sum_{i=1}^{n}|x_i|}{\sigma}$　$\frac{\partial \ln L}{\partial \sigma} = \frac{-2n}{2\sigma} + \frac{\sum_{i=1}^{n}|x_i|}{\sigma^2} = 0 \Rightarrow \hat{\sigma} = \frac{\sum_{i=1}^{n}|x_i|}{n}$

(2). $E(\hat{\sigma}) = \frac{1}{n}E(\sum_{i=1}^{n}|x_i|) = \frac{1}{n}\cdot n \cdot E|x_i| = E[|x_i|] = \int_{-\infty}^{+\infty}|x|\cdot\frac{1}{2\sigma}e^{-\frac{|x|}{\sigma}}dx.$

$= \int_0^{+\infty} x \cdot \frac{1}{\sigma}e^{-\frac{x}{\sigma}}dx = -\int_0^{+\infty} x\, d(e^{-\frac{x}{\sigma}}) = -xe^{-\frac{x}{\sigma}}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-\frac{x}{\sigma}}dx$

$= 0 - \sigma e^{-\frac{x}{\sigma}}\Big|_0^{+\infty} = \sigma$　则 $\hat{\sigma}$ 为 σ 无偏估计.

总体 X. μ, σ^2. $(X_1 \cdots X_m)$ $(Y_1 \cdots Y_n)$. $\overline{X_1}$ $\overline{X_2}$ 5,

试证：$a+b=1$ 的 $\forall a,b$, 有 $T=a\overline{X_1}+b\overline{X_2}$ 为 μ 无偏估计. 确定 a,b. 使 $Var(T)$ 达 min.

$$E[T]=aE[\overline{X_1}]+bE[\overline{X_2}]=a\mu+b\mu=\mu. \Rightarrow \text{无偏}.$$

$$Var(T)=a^2Var(\overline{X_1})+b^2Var(\overline{X_2})=\frac{a^2\sigma^2}{m}+\frac{b^2\sigma^2}{n}=\left(\frac{a^2}{m}+\frac{b^2}{n}\right)\sigma^2 \geq \frac{(a+b)^2}{m+n}\sigma^2=\frac{1}{m+n}\sigma^2.$$

等号成立 $\Rightarrow \frac{a}{m}=\frac{b}{n} \Rightarrow a=\frac{m}{m+n}$ $b=\frac{n}{m+n}$

2.24. X 满足 $E(X)=\mu<\infty$. $E[X^2]<+\infty$. $(X_1 \cdots X_n)\in X$. 验证：$T=\frac{2}{n(n+1)}\sum_{i=1}^{n} iX_i$ 为 μ 相合估计.

判断可估性.

$$E[T]=E\left[\frac{2}{n(n+1)}\sum_{i=1}^{n} iX_i\right]=\frac{2}{n(n+1)}\sum_{i=1}^{n}E[iX_i]=\frac{2}{n(n+1)}\left(\sum_{i=1}^{n} i\right)E[X_i]=\frac{2}{n(n+1)}\cdot\frac{n(n+1)}{2}\mu=\mu.$$

且 $\lim_{n\to\infty}E[T]=\theta$.

$$\lim_{n\to\infty}D[T]=\lim_{n\to\infty}\frac{4}{n^2(n+1)^2}D\left(\sum_{i=1}^{n} iX_i\right).$$

$$=\lim_{n\to\infty}\frac{4}{n^2(n+1)^2}\left(\sum_{i=1}^{n} i^2\right)D(X_i)$$

$$=\lim_{n\to\infty}\frac{4}{n^2(n+1)^2}\cdot\frac{n(n+1)(2n+1)}{6}\left(E[X^2]-E[X]^2\right)\to 0.$$

则有：T 为 μ 相合估计.

2.25. $X\sim(\theta,\theta+1)$. 均匀 $R(\theta,\theta+1)$. $(X_1 \cdots X_n)\in X$.

(1). 求 θ 矩估计. 极大似然估计.

(2). 证明：$\hat{\theta}_1=\overline{X}-\frac{1}{2}$ $\hat{\theta}_2=X_{(n)}-\frac{n}{n+1}$ $\hat{\theta}_3=X_{(1)}-\frac{1}{n+1}$ 为 θ 无偏.

(3). 说明 $\hat{\theta}_1$. $\hat{\theta}_2$. $\hat{\theta}_3$ 方差 min.

(4). $\hat{\theta}_1$. $\hat{\theta}_2$. $\hat{\theta}_3$ 为 θ 相合估计?

(1). $E(X)=\int_{\theta}^{\theta+1}x\,dx=\frac{(\theta+1)^2}{2}-\frac{\theta^2}{2}=\frac{2\theta+1}{2}=\overline{X} \Rightarrow \hat{\theta}=\frac{2\overline{X}-1}{2}$

$$L(x_1,\cdots x_n;\theta)=\begin{cases}1. & \theta\leq x_i<\theta+1.\ \forall i\in[1,n].\ i\in Z^*\\ 0. & \text{otherwise}\end{cases}$$

则. 使 L 为 max. 则 $\forall\theta\in[X_{(n)}-1, X_{(1)}]$ 均可.

(2).

$E[\hat{\theta}_1] = E[\bar{X}-\frac{1}{2}] = E[\bar{X}] - \frac{1}{2} = \frac{2\theta+1}{2} - \frac{1}{2} = \theta.$

$E[\hat{\theta}_2] = E[X_{(n)}] - \frac{n}{n+1}$

而 $X_{(n)}$ 概率 $P(X_{(n)} \le x) = P(X_{(1)} \le x)^n = (x-\theta)^n.$

则: $E[X_{(n)}] = n\int_\theta^{\theta+1} x(x-\theta)^{n-1}dx = \int_\theta^{\theta+1} x\, d(x-\theta)^n = x(x-\theta)^n\Big|_\theta^{\theta+1} - \int_\theta^{\theta+1}(x-\theta)^n dx.$

$= \theta + \frac{n}{n+1}.$

则 $E[\hat{\theta}_2] = E[X_{(n)}] - \frac{n}{n+1} = \theta.$

$E[\hat{\theta}_3] = E[X_{(1)}] - \frac{1}{n+1}.$

而 $X_{(1)}$. $P(X_{(1)} \le x) = 1 - P(X_{(1)} \ge x) = 1 - P(X_{(1)} \ge x)^n = 1 - (1 - P(X_{(1)} \le x))^n$

$= 1 - (1-x+\theta)^n$ 则 $f(x;\theta) = n(1-x+\theta)^{n-1}$

则 $E[X_{(1)}] = \int_\theta^{\theta+1} x\, d[-(\theta+1-x)^n] = -x(\theta+1-x)^n\Big|_\theta^{\theta+1} + \int_\theta^{\theta+1}(\theta+1-x)^n dx = \theta + \frac{1}{n+1}$

则 $E[\hat{\theta}_3] = \theta + \frac{1}{n+1} - \frac{1}{n+1} = \theta.$

(3). $Var(\hat{\theta}_1) = Var(\bar{X}) = \frac{1}{n}Var(X) = \frac{1}{12n}$

$Var(\hat{\theta}_2) = Var(X_{(n)}) = E[X_{(n)}^2] - E[X_{(n)}]^2$

$= \int_\theta^{\theta+1} nx^2(x-\theta)^{n-1}dx - (\theta + \frac{n}{n+1})^2$

$= (\theta+1)^2 - 2(\theta+1)\cdot\frac{1}{n+1} + 2\cdot\frac{1}{n+1}\cdot\frac{1}{n+2} - \theta^2 - \frac{2n\theta}{n+1} - (\frac{n}{n+1})^2$

$= \frac{n}{(n+2)(n+1)^2}$

$Var(\hat{\theta}_3) = Var(X_{(1)}) = \int_\theta^{\theta+1} nx^2(\theta+1-x)^{n-1}dx - (\theta + \frac{1}{n+1})^2.$

$= \frac{n}{(n+2)(n+1)^2}$

$n \le 7$ 时, $\hat{\theta}_1$ 方差min. $n \ge 8$. $\hat{\theta}_2$, $\hat{\theta}_3$ 方差max.

(4) $\lim_{n\to\infty} V(\hat{\theta}_i) = 0$ 方差 $\to 0$, 均为相合估计.

$X \sim Ga(\alpha_0, \lambda)$

$$f(x;\lambda) = \begin{cases} \frac{\lambda^{\alpha_0}}{\Gamma(\alpha_0)} x^{\alpha_0 - 1} e^{-\lambda x}, & x \in (0, +\infty). \\ 0. & , \text{other} \end{cases}$$

α_0 已知，$\lambda > 0$. 求 λ 的无偏估计方差下界.

由 C-R：$Var_\lambda(T) \geq \frac{[\lambda']^2}{nI(\lambda)} = \frac{1}{nI(\lambda)}$.

下计算 fisher 信息函数. $I(\lambda) = \int_0^{+\infty} \left[\frac{\partial \ln f(x;\lambda)}{\partial \lambda}\right]^2 f(x;\lambda)\,dx$.

$$\ln f(x;\lambda) = \alpha_0 \ln\lambda - \ln\Gamma(\alpha_0) + (\alpha_0 - 1)\ln x - \lambda x.$$

$$\frac{\partial \ln f}{\partial \lambda} = \frac{\alpha_0}{\lambda} - x \quad I(\lambda) = \int_0^{+\infty} \left(\frac{\alpha_0}{\lambda} - x\right)^2 f(x;\lambda)\,dx.$$

$$I(\lambda) = E\left[\left(\frac{\alpha_0}{\lambda} - X\right)^2\right] = \frac{\alpha_0^2}{\lambda^2} - \frac{2\alpha_0}{\lambda}E(X) + E[X^2].$$

$$= \frac{\alpha_0^2}{\lambda^2} - \frac{2\alpha_0}{\lambda}\cdot\frac{\alpha_0}{\lambda} + \frac{\alpha_0}{\lambda^2} + \left(\frac{\alpha_0}{\lambda}\right)^2$$

$$= \frac{\alpha_0}{\lambda^2}.$$

则 $Var_\lambda(T) \geq \frac{\lambda^2}{n\alpha_0}$. 则 C-R 下界：$\frac{\lambda^2}{n\alpha_0}$.

2.34. $X \sim N(\mu, \sigma^2)$. $-\infty < \mu < +\infty$. $\sigma^2 > 0$. $(X_1 \cdots X_n) \in X$.

(1) $3\mu + 4\sigma^2$ 一致最小方差无偏估计. (2) $\mu^2 - 4\sigma^2$ 一致最小方差无偏估计.

引理：$L(x_1 \cdots x_n; \mu, \sigma^2) = \frac{1}{(\sqrt{2\pi}\sigma)^n} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu)^2\right]$.

$$= \frac{1}{(\sqrt{2\pi}\sigma)^n} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2 + \frac{\mu}{\sigma^2}\sum_{i=1}^n x_i\right] \cdot \exp\left[-\frac{n\mu}{2\sigma^2}\right]$$

由正态分布为指数分布族. 进而：$(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)$ 为 (μ, σ^2) 的充分完备统计量.

(1). $T_1=\sum_{i=1}^{n}X_i$、 $T_2=\sum_{i=1}^{n}X_i^2$.

则: $\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2=\frac{1}{n-1}\sum_{i=1}^{n}X_i^2-n\bar{X}^2\cdot\frac{1}{n-1}$

$=\frac{1}{n-1}T_2-\frac{1}{n-1}\cdot\frac{T_1^2}{n}$

则. $E\left[\frac{3T_1}{n}+4\left(\frac{T_2}{n-1}-\frac{1}{n-1}\cdot\frac{T_1^2}{n}\right)\right]=3\mu+4\sigma^2$. 无偏. 则为 $3\mu+4\sigma^2$ 一致最小方差无偏估计.

(2) $T_1=\sum_{i=1}^{n}X_i$. $T_2=\sum_{i=1}^{n}X_i^2$

$E[\bar{X}^2]=E[\bar{X}]+Var(\bar{X}).=\mu^2+\frac{\sigma^2}{n}\Rightarrow\mu^2\Rightarrow\frac{T_1^2}{n^2}-\frac{1}{n}\left(\frac{T_2}{n-1}-\frac{1}{n-1}\cdot\frac{T_1^2}{n}\right)$.

则 $\mu^2-4\sigma^2$ 估计为: $\frac{T_1^2}{n^2}-\frac{1}{n}\left(\frac{T_2}{n-1}-\frac{1}{n-1}\cdot\frac{T_1^2}{n}\right)-4\left(\frac{T_2}{n-1}-\frac{1}{n-1}\cdot\frac{T_1^2}{n}\right)$.

2.35. $X\sim b(1,p)$. $0<p<1$. $(X_1\cdots X_n)\in X$. 求 p^2 的一致最小方差无偏估计.

$L(x_1\cdots x_n;p)=p^{\sum_{i=1}^{n}x_i}(1-p)^{n-\sum_{i=1}^{n}x_i}$ $x_i=0.1$. $0<p<1$.

则: $L(x_1\cdots x_n;p)=(1-p)^n\left(\frac{p}{1-p}\right)^{\sum_{i=1}^{n}x_i}$.

$=(1-p)^n\,e\left\{\sum_{i=1}^{n}x_i\ln\left(\frac{p}{1-p}\right)\right\}$ 易证 $T=\sum_{i=1}^{n}x_i$ 为 p 充分完备统计量

$E[T]=nE[X_i]=np$. ~~$E[T^2]=Var(T)+E[T]^2$~~ ~~$=nE[X_i^2]+n(n-1)E[X_iX_j]_{i\neq j}$~~

~~$=np(1-p)+p^2$~~ ~~$=np+n(n-1)p^2$~~

~~$=n^2p^2$~~

$E[T^2]=Var(T)+E[T]^2=np(1-p)+n^2p^2$.

则. $\frac{T^2-T}{n^2-n}$ 为 p^2 一致最小方差无偏估计.

$E\left[\frac{T^2-T}{n^2-n}\right]=\frac{1}{n^2-n}\left(np(1-p)+n^2p^2-np\right)=p^2$.

~~习~~题 3.

3.1. $X_1 \cdots X_{25}$. 取自 $N(\mu, 9)$. μ 未知. 检验问题.

$$H_0: \mu = \mu_0 \longleftrightarrow H_1: \mu \neq \mu_0.$$

其中 μ_0 已知. 取拒绝域 $W = \{|\bar{X} - \mu_0| \geqslant C\}$. $\bar{X}$ 样本均值.

(1) 确定 C. 使检验显著性 0.05

(2). $\alpha = 0.05$ 下. 犯第二类错误概率.

(1) $P\{|\bar{X} - \mu_0| \geqslant C\} = 0.05$. $\quad \bar{X} \sim N(\mu, \frac{9}{25})$.

而 $\frac{(\bar{X} - \mu_0) \cdot 5}{3} \overset{H_0}{\sim} N(0,1)$. [即当 $\mu = \mu_0$ 时. 会有: $\frac{(\bar{X} - \mu_0) \times 5}{3} \sim N(0,1)$]

则: $P\left\{\frac{|\bar{X} - \mu_0| \times 5}{3} \geqslant \frac{5}{3}C\right\} = 0.05$.

则. 有 $u_{0.975} = \frac{5}{3}C \Rightarrow \frac{5}{3}C = 1.96 \Rightarrow C = 1.176$.

(2). $\beta = P\{$接受 $H_0 \mid H_1$ 正确$\}$. $\quad \bar{X} \sim N(\mu, \frac{9}{25})$. $\quad t = \frac{5}{3}(\bar{X} - \mu) \quad t \sim N(0,1)$

$= P\{|\bar{X} - \mu_0| \leqslant C \mid \mu \neq \mu_0\}$ $\quad \bar{X} = \frac{3}{5}t + \mu$.

$= P\left\{\frac{|\bar{X} - \mu_0| \times 5}{3} \leqslant \frac{5}{3}C\right\}$.

$= P\{-C \leqslant \frac{3}{5}t + \mu - \mu_0 \leqslant C\}$.

$= P\{(\mu_0 - \mu - C) \times \frac{5}{3} \leqslant t \leqslant (\mu_0 - \mu + C) \times \frac{5}{3}\}$.

$= \Phi[(\mu_0 - \mu + C) \times \frac{5}{3}] - \Phi[(\mu_0 - \mu - C) \times \frac{5}{3}]$

$= \Phi[(\mu_0 - \mu + 1.176) \times \frac{5}{3}] - \Phi[(\mu_0 - \mu - 1.176) \times \frac{5}{3}]$.

3.2. $X \sim R(0,\theta)$. $\theta>0$. $(X_1 \cdots X_n) \in X$. $X_{(n)} = \max\{X_1 \cdots X_n\}$.

$H_0: \theta \geq 2 \longleftrightarrow H_1: \theta < 2$.

求: $\varphi(x_1 \cdots x_n) = \begin{cases} 1. & x_{(n)} \leq \frac{3}{2} \quad 拒绝域. \\ 0. & x_{(n)} > \frac{3}{2} \end{cases}$

犯第一类错误概率。

则: $\alpha = P\{拒绝H_0 | H_0正确\}$.

$= P\{X_{(n)} \leq \frac{3}{2} | \theta \geq 2\}$.

由 $P\{X_{(n)} \leq \frac{3}{2}\} = P\{X_{(i)} \leq \frac{3}{2}\}^n = (\frac{3}{2\theta})^n$

3.5. 电子元件寿命要求不低于1000h. 抽25件. 测得平均950. 服从 $N(\mu, 100^2)$. 确定合格性

$H_0: \mu = 1000 \longleftrightarrow H_1: \mu < 1000$.

则: ~~$\bar{X} \sim (950, \frac{100^2}{25})$~~. $X \sim N(\mu, 100^2)$.

$\frac{\bar{X} - 1000}{\sqrt{\frac{\sigma^2}{n}}} \overset{H_0}{\sim} N(0,1)$. 拒绝域 $W = \{U < U_\alpha\}$

则. 查表. $u_{0.05} = -1.645$. 而 $U = \frac{950-1000}{\sqrt{\frac{100^2}{25}}} = -2.5 < -1.645$. 落入拒绝域,

则 $\mu < 1000$. 产品不合格.

(本题为 σ^2 已知, 检 μ. 用正态 U 检).

3.6. X 表示人注射疫苗后抗体强度: $X \sim N(\mu, \sigma^2)$. 另一家平均抗体强度 1.9. 若甲为证其有更强抗体, 则.

(1) H_0. H_1

(2). 甲厂取16样本. $\bar{X} = 2.225$. $S^2 = 0.2687$. 检(1)假设. ($\alpha = 0.05$).

3.

(1) $H_0: \mu=1.9 \longleftrightarrow H_1: \mu>1.9$.

(2) 选取: $\dfrac{(\bar{X}-\mu)/\frac{\sigma}{\sqrt{n}}}{\sqrt{\frac{S^2(n-1)}{(n-1)\sigma^2}}} \sim t(n-1)$. $P\{t>t_{1-\alpha}\}=\alpha$.

则拒绝域 $W=\{t>t_{1-\alpha}\}$.

而查表得 $t_{0.95}(15)=1.753$. 而计算: $t=\dfrac{2.225-1.9}{\sqrt{0.2681}}\sqrt{16}=2.51$.

由 $t>t_{0.95}(15)$. 落入拒绝域,则有:拒绝 H_0. H_1成立. 甲有更高抗体。

(本题为 σ^2未知,检μ题型,用t检验).

3.7. 以经验,其服从 $\sigma=7.5$ 正态. 抽 25个. 其 $S=9.5h$. 是否同往年有显著变化.

则 μ不已知. $\chi^2=\dfrac{\sum_{i=1}^{25}(X_i-\bar{X})^2}{\sigma^2} \overset{H_0}{\sim} \chi^2(25-1)$. 即 $\chi^2=\dfrac{(n-1)S^2}{\sigma^2}$

$H_0: \sigma=7.5 \longleftrightarrow H_1: \sigma\neq 7.5$.

则拒绝域 $W=\{|\chi^2|\geq \chi^2_{0.975}(24)\}$. 查表 $\chi^2_{0.975}(24)=39.304$.

而: 计算 $\chi^2=\dfrac{(n-1)S^2}{\sigma^2}=\dfrac{24\times 9.5^2}{7.5^2}=38.507$.

未落入拒绝域,接受H_0. 故 $\sigma=7.5$. 无显著变化。

(本题为μ未知,检σ^2情形,用$\chi^2(n-1)$).

3.10. 选两中学各25人. 第一中学均分74($\bar{X}_1$). 标准差8分(S_1). 第二中学均分78($\bar{X}_2$). 标准差7分(S_2)

且正态分布. 试 $\alpha=0.05$下. 是否有显著差异.

$H_0: \mu_1=\mu_2 \longleftrightarrow H_1: \mu_1\neq\mu_2$.

仅知 $n_1=n_2=n$.

题目问题:未知σ_1^2与σ_2^2的大小关系. 若学校总体的标准差$S_1=8$

$S_2=7$.

本题为（比较 μ_1,μ_2，~~$n_1=n_2=n$~~ $\sigma_1^2\neq\sigma_2^2$.情形）.

$H_0:\mu_1=\mu_2 \longleftrightarrow H_1:\mu_1\neq\mu_2$. $n_1=n_2=25$.

则：$E(\overline{X}-\overline{Y})=\mu_1-\mu_2$. $D(\overline{X}-\overline{Y})=\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}$.

且：$U=\frac{(\overline{X}-\overline{Y})-0}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}\overset{H_0}{\sim}(0,1)$.

拒绝域 $W=\{|U|\geqslant u_{1-\frac{\alpha}{2}}\}$. 查表 $u_{0.975}=1.96$

计算：$|U|=\frac{|74-78|}{\sqrt{\frac{8^2}{25}+\frac{7^2}{25}}}=1.826<1.96$. 接受假设.无显著差异.

3.21. 已知元件寿命 $X\sim N(\mu,\sigma^2)$.现抽11元件.平均值 $\overline{x}=800h$. 标准差 $s=11h$.求 μ 和 σ^2 的90%置信区间。

(1) 由 μ 的90%置信区间. [σ^2未知]

$\frac{\overline{X}-\mu}{S}\sqrt{n}\sim t(n-1)$. $\alpha=0.1$.

则：$P\left\{\left|\frac{\overline{X}-\mu}{S}\sqrt{n}\right|\geqslant t_{\frac{\alpha}{2}}(n-1)\right\}=\alpha$. 此 $t_{\frac{\alpha}{2}}(n-1)=t_{1-\frac{\alpha}{2}}(n-1)$ ppt定义. 分位数.

$=t_{0.95}(10)$.

$=1.812$

解：$P\left\{\left|\frac{\overline{X}-\mu}{S}\sqrt{n}\right|\leqslant t_{\frac{\alpha}{2}}(n-1)\right\}=1-\alpha$.

$\left|\frac{\overline{X}-\mu}{S}\sqrt{n}\right|\leqslant t_{0.95}(10)\Rightarrow \overline{X}-\frac{t\cdot S}{\sqrt{n}}\leqslant\mu\leqslant\overline{X}+\frac{t\cdot S}{\sqrt{n}}\Rightarrow \mu\in[793.99,806.01]$

(2) 求 σ^2 的90%置信区间. [μ未知]

$\chi^2=\frac{(n-1)S^2}{\sigma^2}$ 枢轴量.

$P\left\{\left|\frac{(n-1)S^2}{\sigma^2}\right|\geqslant\chi^2_{\frac{\alpha}{2}}(n-1)\right\}=\alpha$. $\alpha=0.1$. $\chi^2_{0.95}(10)=18.307$. $\chi^2_{0.05}(10)=3.940$.

解：$\left|\frac{(n-1)S^2}{\sigma^2}\right|\leqslant\chi^2_{0.95}(10)\Rightarrow \sigma^2\in\left[\frac{(n-1)S^2}{\chi^2_{0.95}(10)},\frac{(n-1)S^2}{\chi^2_{0.05}(10)}\right]$ $\sigma^2\in[66.095,307.11]$

$y_2=2P$

y. 23. 0.50, 1.25, 0.80, 2.00. 来自X, 已知 $Y=\ln X\sim N(\mu,1)$. 5.

(1) 求 $E(X)$. 记 $b=E[X]$

(2). 求 μ 的 0.95 置信区间

(3). 利用上述, 求 b 的 0.95 置信区间.

(1). ~~$E[X]=\int_{-\infty}^{+\infty}$~~

$$f(y)=\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-\mu)^2}{2}}$$

则 $E(X)=E[e^Y]=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^y e^{-\frac{(y-\mu)^2}{2}}dy \xlongequal{t=y-\mu} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\frac{1}{2}(t-1)^2}e^{\mu+\frac{1}{2}}dt.$

$$=e^{\mu+\frac{1}{2}}=b.$$

(2). $\alpha=0.05$.

$\overline{Y}\sim N(\mu,\frac{1}{4})$. $n=4$.

则 $P\{|\frac{\overline{Y}-\mu}{\sqrt{\frac{1}{4}}}|\geq u_{1-\frac{\alpha}{2}}\}=\alpha$. $u_{1-\frac{\alpha}{2}}=u_{0.975}=1.96$.

解: $|\frac{\overline{Y}-\mu}{\sqrt{\frac{1}{4}}}|\leq u_{1-\frac{\alpha}{2}} \Rightarrow \overline{Y}-\frac{1}{2}\underset{u_{0.975}}{\leq}\mu\leq\overline{Y}+\frac{1}{2}0.975.$

则由 $\overline{Y}=\frac{1}{4}\sum_{i=1}^{4}y_i=\frac{1}{4}[\ln\frac{1}{2}+\ln\frac{5}{4}+\ln\frac{4}{5}+\ln 2]=0.$

则 $\mu\in[-0.98,+0.98]$.

(3). $P(-0.98<\mu<+0.98)=P(-0.48<\mu+\frac{1}{2}<+0.48)=0.95.$

$\Rightarrow e^{\mu+\frac{1}{2}}\in[e^{-0.48},e^{+0.48}]$

3.25. $(X_1\cdots X_{n_1})$ $(Y_1\cdots Y_{n_2})$. $\in N(\mu_1,\sigma_1^2)$ $N(\mu_2,\sigma_2^2)$. 独立.

(1) 若 σ_1^2,σ_2^2 已知. 求 $\mu_1-\mu_2$ 有固定长度 L 的置信水平 $1-\alpha$ 区间

$$\bar X\sim N\left(\mu_1,\frac{\sigma_1^2}{n_1}\right).\quad \bar Y\sim N\left(\mu_2,\frac{\sigma_2^2}{n_2}\right).$$

则. $$\frac{(\bar X-\bar Y)-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}\sim N(0,1).$$

则 $$\cancel{P\left(\left|\frac{(\bar X-\bar Y)-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}\right|\right)}\quad P\left(u_\beta\leqslant\frac{(\bar X-\bar Y)-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}\leqslant u_{\beta+1-\alpha}\right)=1-\alpha.$$

解出 $\mu_1-\mu_2\in\left((\bar X-\bar Y)-u_{\beta+1-\alpha}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}\;(\bar X-\bar Y)-u_\beta\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}\right)$

(2) 若 $\sigma_1^2=\sigma_2^2=\sigma^2$ 已知. 为使水平0.95的 $\mu_1-\mu_2$ 区间长 $\frac{2}{5}\sigma$. 样本大小 $n_1=n_2$ 取多大?

$\alpha=0.05.$ $$P\left\{\left|\frac{(\bar X-\bar Y)-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}\right|\geqslant u_{1-\frac{\alpha}{2}}\right\}=\alpha.$$

解 $$\left|\frac{(\bar X-\bar Y)-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}\right|\leqslant u_{1-\frac{\alpha}{2}}\Rightarrow(\mu_1-\mu_2)\in[a,b].$$

且 $(a-b)=2u_{1-\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}=\frac{2}{5}\sigma.\Rightarrow n_1=50u_{0.975}^2=50\times1.96^2\approx192$

(3). $\mu_1\mu_2$, σ_1^2. σ_2^2 未知. 求 $\frac{\sigma_1^2}{\sigma_2^2}$ 置信水平 $1-\alpha$ 区间.

$$F=\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}\sim F(n_1-1,n_2-1).\quad S_1^2=\frac{1}{n_1-1}\sum_{i=1}^{n_1}(X_i-\bar X)^2\quad S_2^2=\frac{1}{n_2-1}\sum_{i=1}^{n_2}(X_i-\bar X)^2$$

则: $$P\left\{\left|\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}\right|\geqslant F_{1-\frac{\alpha}{2}}(n_1-1,n_2-1)\right\}=\alpha.$$

解: $$\left|\frac{\frac{S_1^2}{S_2^2}}{\frac{\sigma_1^2}{\sigma_2^2}}\right|\leqslant F_{1-\frac{\alpha}{2}}(n_1-1,n_2-1).\Rightarrow\frac{\sigma_1^2}{\sigma_2^2}\in\left[\frac{S_1^2}{S_2^2}\cdot\frac{1}{F_{1-\frac{\alpha}{2}}(n_1-1,n_2-1)},\frac{S_1^2}{S_2^2}\frac{1}{F_{\frac{\alpha}{2}}(n_1-1,n_2-1)}\right]$$

$$\Rightarrow\frac{\sigma_1^2}{\sigma_2^2}\in\left[\frac{S_1^2}{S_2^2}F_{\frac{\alpha}{2}}(n_2-1,n_1-1),\frac{S_1^2}{S_2^2}F_{1-\frac{\alpha}{2}}(n_2-1,n_1-1)\right]$$

Y_1 3.11. $X \sim b(1,p)$. $0<p<1$. $(X_1 \cdots X_3) \in X$. 7,

$H_0: p=\frac{1}{2} \longleftrightarrow H_1: p=\frac{3}{4}$ $\quad \varphi(x_1,x_2,x_3)=\begin{cases}1, & \nu\geq 1\\ 0, & \nu<1.\end{cases}$ ν 表示取值为1的频数

(1) 求犯第一,二类错误概率

(2). $p=\frac{3}{4}$. 功效函数。

(1). $P\{\nu \geq 1 \mid p=\frac{1}{2}\}$

$= P\{1,0,0\} + P\{1,1,0\} + P\{1,1,1\}$.

$= C_3^1 \frac{1}{2}(\frac{1}{2})^2 + C_3^2 (\frac{1}{2})^3 + C_3^3 (\frac{1}{2})^3 = \frac{7}{8}$

$P\{\nu<1 \mid p=\frac{3}{4}\}$.

$= P\{0,0,0\} = C_3^3 (\frac{1}{4})^3 = \frac{1}{64}$

(2). $\beta(\theta) = E_\theta[\varphi(X)]$.

$\beta(\frac{3}{4}) = E_{\frac{3}{4}}[\varphi(X)] = E_{\frac{3}{4}}[\nu\geq 1] = 3\cdot\frac{3}{4}(\frac{1}{4})^2 + 3(\frac{3}{4})^2\cdot\frac{1}{4} + 1(\frac{1}{4})^3 = \frac{63}{64}$

$= P\{\nu \geq 1 \mid p=\frac{3}{4}\}$.

3.13. $(X_1 \cdots X_n) \in P(\lambda)$.

(1) $H_0: \lambda=\lambda_0 \leftrightarrow H_1: \lambda=\lambda_1 (<\lambda_0)$. (2) $H_0: \lambda=\lambda_0 \leftrightarrow H_1: \lambda=\lambda_1 (>\lambda_0)$.

水平α的最大功效检验。

(1). $L(x_1 \cdots x_n; \lambda_0) = \frac{\lambda_0^{x_1+\cdots+x_n}}{x_1!\, x_2! \cdots x_n!} e^{-n\lambda_0}$. $L(x_1 \cdots x_n; \lambda_1) = \frac{\lambda_1^{x_1+\cdots+x_n}}{x_1! \cdots x_n!} e^{-n\lambda_1}$ $(\lambda_0 > \lambda_1)$.

$\lambda(x) = \frac{L(x_1 \cdots ; \lambda_1)}{L(x_1 \cdots ; \lambda_0)} = \cancel{\frac{\lambda_1^{n\bar{x}}}{\lambda_0^{n\bar{x}}} e^{n(\lambda_0-\lambda_1)}}$ $\left(\frac{\lambda_1}{\lambda_0}\right)^{n\bar{x}} e^{n(\lambda_0-\lambda_1)}$.

由 N-P 引理: $P\{\lambda x \geq k\} = P\{\sum_{i=1}^{n} x_i < C\}$.

拒绝域：$W=\{\lambda(x)\geqslant k\}\Rightarrow \cancel{\{(\frac{\lambda_0}{\lambda_1})^{n\bar{x}}e^{\lambda_1-\lambda_0}>k\}}=\{\sum_{i=1}^{n}x_i<C_1\}$

在 $H_0(\lambda=\lambda_0)$ 成立，$n\bar{X}\sim P(n\lambda_0)$，则有.

$$E_{\lambda_0}[\varphi(X)]=P\{n\bar{X}<C_1\mid\lambda=\lambda_0\}=\alpha.$$

$$\cancel{即\ P\{n\bar{X}>C\mid\lambda=\lambda_0\}=1-\alpha.}$$

$$C_1=P_\alpha(n\lambda_0).$$

$$\Rightarrow\varphi(x_1\cdots x_n)=\begin{cases}1, & \sum_{i=1}^{n}x_i<C_1.\\ \delta_1, & \sum_{i=1}^{n}x_i=C_1\\ 0, & \sum_{i=1}^{n}x_i>C_1.\end{cases}$$

其中 $\delta_1=\dfrac{\alpha-\alpha_1}{P(\sum_{i=1}^{n}x_i=C_1)}=\dfrac{(\alpha-\alpha_1)C_1!}{(n\lambda_0)^{C_1}e^{-n\lambda_0}}$

$$\alpha_1=P\{(\sum_{i=1}^{n}X_i)<C_1\}.$$

(2) 同理

$$\Rightarrow\varphi(x_1\cdots x_n)=\begin{cases}1, & \sum_{i=1}^{n}x_i>C_1.\\ \delta_2, & \sum_{i=1}^{n}x_i=C_1.\\ 0, & \sum_{i=1}^{n}x_i<C_1.\end{cases}$$

$$\delta_2=\frac{(\alpha-\alpha_2)C_1!}{(n\lambda_0)^{C}e^{-n\lambda_0}}$$

$$\alpha_2=P\{(\sum_{i=1}^{n}X_i)>C_1\}.$$

3.15. $(X_1\cdots X_n)\in N(\mu_0,\sigma^2)$. μ_0 已知.

(1). $H_0:\sigma^2=\sigma_0^2\longleftrightarrow H_1:\sigma^2=\sigma_1^2\ (>\sigma_0^2)$ 的水平 α MP φ^*

(2) 证明：$\varphi^*(x_1\cdots x_n)$ 也为 $H_0:\sigma^2=\sigma_0^2\leftrightarrow H_1:\sigma^2>\sigma_0^2$ 的 αUMP.

(1) $L(x_1\cdots x_n;\sigma_0^2)=\left(\frac{1}{\sqrt{2\pi}\sigma_0}\right)^n e^{-\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{2\sigma_0^2}}$ $L(x_1\cdots x_n;\sigma_1^2)=\left(\frac{1}{\sqrt{2\pi}\sigma_1}\right)^n e^{-\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{2\sigma_1^2}}$

$$\lambda=\frac{L(\sigma_1^2)}{L(\sigma_0^2)}=\left(\frac{\sigma_0}{\sigma_1}\right)^n e^{\sum_{i=1}^{n}(x_i-\mu_0)^2\left(\frac{1}{2\sigma_0^2}-\frac{1}{2\sigma_1^2}\right)}\quad 且\quad \frac{1}{2\sigma_0^2}-\frac{1}{2\sigma_1^2}>0,\ \uparrow$$

拒：$W=\{\lambda(x_1\cdots x_n)>k\}=\{\sum_{i=1}^{n}(x_i-\mu_0)^2>C\}$.

$\alpha=P_{\sigma_0}\{\sum_{i=1}^{n}(x_i-\mu_0)^2>C\}=1-P_{\sigma_0}\{\sum_{i=1}^{n}(x_i-\mu_0)^2\leq C\}$. 则 $C=\sigma_0^2\chi^2_{1-\alpha}(n)$

$$\varphi^*(x_1\cdots x_n)=\begin{cases}1 & \sum_{i=1}^{n}(x_i-\mu_0)^2>\sigma_0^2\chi^2_{1-\alpha}(n).\\ 0. & \text{other}\end{cases}$$

(2). φ^* 与 σ_1 无关，且 $\varphi^*(x_1\cdots x_n)$ 为简单假设对备择假设 $H_0:\sigma^2=\sigma_0^2\leftrightarrow H_1:\sigma^2>\sigma_1^2$. $\forall$ 水平 α 的检验，且 $\beta_{\varphi^*}(\sigma_1)>\cancel{\beta_{\varphi^*}(\sigma_0)}$. $\beta_\varphi(\sigma_1)$. 只需 $\sigma_1>\sigma_0$. 即成立.

3.16. $X \sim Exp(\frac{1}{\theta})$. $f(x;\theta)=\begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & x>0 \\ 0, & x\leq 0 \end{cases}$ $\theta>0$ $(X_1 \cdots X_n) \in X$.

(1) $H_0: \theta=\theta_0 \leftrightarrow H_1: \theta=\theta_1$. 的水平$\alpha$的MP检 $\varphi^*(x_1 \cdots x_n)$

(2). 证明φ^*为 $H_0: \theta=\theta_0 \leftrightarrow H_1: \theta>\theta_0$ 的水平α的UMP

(1) $L(x_1 \cdots x_n; \theta_0) = \frac{1}{\theta_0^n} e^{-\frac{\sum_{i=1}^n x_i}{\theta_0}}$ $L(x_1 \cdots x_n; \theta_1) = \frac{1}{\theta_1^n} e^{-\frac{\sum_{i=1}^n x_i}{\theta_1}}$

$\lambda = \frac{L(x_1 \cdots x_n; \theta_1)}{L(x_1 \cdots x_n; \theta_0)} = (\frac{\theta_0}{\theta_1})^n e^{\sum_{i=1}^n x_i (\frac{1}{\theta_0} - \frac{1}{\theta_1})}$. $\frac{1}{\theta_0} - \frac{1}{\theta_1} > 0$. 则 λ 对 x_i. ↑.

$X \sim \Gamma(1, \frac{1}{\theta})$.

拒: $W = \{\lambda(x_1 \cdots x_n) > k\} = \{\sum_{i=1}^n x_i > C\}$ 且 $\sum_{i=1}^n x_i \sim \Gamma(n, \frac{1}{\theta})$.

则 $\alpha = P_{\theta_0}\{\sum_{i=1}^n x_i > C\} = 1 - P_{\theta_0}\{\sum_{i=1}^n x_i < C\}$ $\Rightarrow$ $C = \Gamma_{1-\alpha}(n, \frac{1}{\theta})$.

$\Rightarrow \varphi^* = \begin{cases} 1, & \sum_{i=1}^n x_i > \Gamma_{1-\alpha}(n, \frac{1}{\theta}) \\ 0 & \text{other} \end{cases}$

(逆回归).

(2) φ^* 与 θ_1 无关，仅与 α 有关. 且 φ^* 为简单假设的 MP.

$H_0: \theta=\theta_0 \leftrightarrow H_1: \theta>\theta_0$ 的水平 α 检. $\beta_{\varphi^*}(\theta_1) \geqslant \beta_{\varphi}(\theta_1)$. $\forall \theta_1>\theta_0$

故为 UMPT.

习题4

（多元线性回归）

4.3. $y_1=\theta_1+e_1$

$y_2=2\theta_1-\theta_2+e_2$

$y_3=\theta_1+2\theta_2+e_3$

$$\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}1&0\\2&-1\\1&2\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}+\begin{bmatrix}e_1\\e_2\\e_3\end{bmatrix}.$$

$E(\theta_i)=0$ $E(e_ie_j)=0.\ (i\neq j)$.

(1). 求θ_1、θ_2 最小二乘估计 $\hat\theta_1$, $\hat\theta_2$

(2). 求 $\hat\theta=(\hat\theta_1,\hat\theta_2)^T$ 的协方差阵. 假设 $Var(e_i)=\sigma^2$. $i=1,2,3$.

(3). 求σ^2 无偏估计.

(1)

偏离平方和 $Q(\theta_1,\theta_2)=(y_1-\theta_1)^2+(y_2-2\theta_1+\theta_2)^2+(y_3-\theta_1-2\theta_2)^2$.

$$\begin{cases}\dfrac{\partial Q}{\partial\theta_1}=-2(y_1-\theta_1)-4(y_2-2\theta_1+\theta_2)-2(y_3-\theta_1-2\theta_2)=0\\[2mm] \dfrac{\partial Q}{\partial\theta_2}=2(y_2-2\theta_1+\theta_2)-4(y_3-\theta_1-2\theta_2)=0\end{cases}$$

$$\Rightarrow \hat\theta_1=\frac{y_1+2y_2+y_3}{6}\qquad \hat\theta_2=\frac{2y_3-y_2}{5}$$

(2).

$$\hat\theta=\begin{bmatrix}\frac{y_1+2y_2+y_3}{6}\\ \frac{2y_3-y_2}{5}\end{bmatrix}$$ $Var(e_i)=\sigma^2$. $E(e_i)=0$. $E(e_ie_j)=0.\ i\neq j$.

$$S=(X^TX)=\begin{bmatrix}6&0\\0&5\end{bmatrix}$$

$$S^{-1}=(X^TX)^{-1}=\begin{bmatrix}\frac16&0\\0&\frac15\end{bmatrix}.$$

$$Cov(\hat\theta,\hat\theta)=\sigma^2(X^TX)^{-1}=\sigma^2\begin{bmatrix}\frac16&0\\0&\frac15\end{bmatrix}.$$

(3). $\hat\sigma^2=\dfrac{Q_e}{n-k-1}$ 其中$k+1=2$, $n=3$. $\hat\sigma^2=Q_e=\hat\varepsilon^T\hat\varepsilon$ 而 $\hat\varepsilon=Y-X\hat\beta=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}-\begin{bmatrix}1&0\\2&-1\\1&2\end{bmatrix}\begin{bmatrix}\hat\theta_1\\ \hat\theta_2\end{bmatrix}$

则 $$\hat\varepsilon=\begin{bmatrix}\frac{5y_1-2y_2-y_3}{6}\\ \frac{-5y_1+2y_2+y_3}{15}\\ \frac{-5y_1+2y_2+y_3}{30}\end{bmatrix}$$

则 $Q_e=\hat\varepsilon^T\hat\varepsilon=\frac56y_1^2+\frac{2}{15}y_2^2+\frac{1}{30}y_3^2-\frac23y_1y_2-\frac13y_1y_3+\frac{2}{15}y_2y_3$.

（已算y_1^2、y_1y_2系数正确，其它抄了答案）.

4.4. 线性回归模型 $(Y, X\beta, \sigma^2 I_n)$. X 为 $n\times m\,(n>m)$ 列满秩. X, β 分块.

$$X\beta = (X_1, X_2)\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}$$

证明: β_2 最小二乘估计: $\hat{\beta}_2 = [X_2^TX_2 - X_2^TX_1(X_1^TX_1)^{-1}X_1^TX_2]^{-1}[X_2^TY - X_2^TX_1(X_1^TX_1)^{-1}X_1^TY]$

$$\begin{bmatrix}\hat{\beta}_1\\ \hat{\beta}_2\end{bmatrix} = (X^TX)^{-1}X^TY = \left\{\begin{bmatrix}X_1^T\\ X_2^T\end{bmatrix}[X_1, X_2]\right\}^{-1}\begin{bmatrix}X_1^T\\ X_2^T\end{bmatrix}Y = \begin{bmatrix}X_1^TX_1 & X_1^TX_2\\ X_2^TX_1 & X_2^TX_2\end{bmatrix}^{-1}\begin{bmatrix}X_1^T\\ X_2^T\end{bmatrix}Y$$

令 $S^{-1} = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \Rightarrow \begin{cases}cX_1^TX_1 + dX_2^TX_1 = 0\\ cX_1^TX_2 + dX_2^TX_2 = 1\\ aX_1^TX_1 + bX_2^TX_1 = 1\\ aX_1^TX_2 + bX_2^TX_2 = 0\end{cases} \Rightarrow \begin{cases}c = -dX_2^TX_1(X_1^TX_1)^{-1}\\ d = (X_2^TX_2 - X_2^TX_1(X_1^TX_1)^{-1}X_1^TX_2)^{-1}\\ a = -bX_2^TX_2(X_1^TX_2)^{-1}\\ b = (X_2^TX_1 - X_2^TX_2(X_1^TX_2)X_1^TX_1)^{-1}\end{cases}$

$$\begin{bmatrix}\hat{\beta}_1\\ \hat{\beta}_2\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}X_1^T\\ X_2^T\end{bmatrix}Y$$

$\hat{\beta}_1 = (aX_1^T + bX_2^T)Y = b[X_2^T - X_2^TX_2(X_1^TX_2)^{-1}X_1^T]Y$

$\hat{\beta}_2 = (cX_1^T + dX_2^T)Y = [X_2^TX_2 - X_2^TX_1(X_1^TX_1)^{-1}X_1^TX_2]^{-1}[X_2^TY - X_2^TX_1(X_1^TX_1)^{-1}X_1^TY]$

A

4.5. 线性回归模型.

$$Y_i=\theta+\varepsilon_i. \quad i=1,2\cdots m.$$

$$Y_{m+i}=\theta+\phi+\varepsilon_{m+i}. \quad i=1\cdots m$$

$$Y_{2m+i}=\theta-2\phi+\varepsilon_{2m+i}. \quad i=1\cdots n.$$ ε_i 之间互不相关.

$$E(\varepsilon_i)=0. \quad Var(\varepsilon_i)=\sigma^2. \quad i=1,2\cdots,2m+n.$$

求 θ 与 ϕ 最小二乘估计. 证 $m=2n$ 时. $\hat\theta$ 与 $\hat\phi$ 互不相关。

$$Q(\theta,\phi)=\sum_{i=1}^{m}(Y_i-\theta)^2+\sum_{i=m+1}^{2m}(Y_i-\theta-\phi)^2+\sum_{j=2m+1}^{2m+n}(Y_j-\theta+2\phi)^2$$

$$\frac{\partial Q}{\partial\theta}=-2\sum_{i=1}^{m}(Y_i-\theta)-2\sum_{i=m+1}^{2m}(Y_i-\theta-\phi)-2\sum_{j=2m+1}^{2m+n}(Y_j-\theta+2\phi).$$

$$\Rightarrow \sum_{i=1}^{2m+n}y_i=(2m+n)\theta+(m-2n)\phi \quad ①$$

$$\frac{\partial Q}{\partial\phi}=-2\sum_{i=m+1}^{2m}(Y_i-\theta-\phi)+4\sum_{j=2m+1}^{2m+n}(Y_j-\theta+2\phi)=0.$$

$$\Rightarrow \sum_{i=1}^{m}y_{m+i}-2\sum_{i=1}^{n}y_{2m+i}=(m-2n)\theta+(m+4n)\phi. \quad ②$$

① 与 ② $\Rightarrow$

$$\begin{cases}\hat\theta=\dfrac{(m+4n)\sum_{i=1}^{2m+n}y_i-(m-2n)\sum_{i=1}^{m}y_{m+i}+2(m-2n)\sum_{i=1}^{n}y_{2m+i}}{m^2+13mn}\\ \hat\phi=\dfrac{(2m+n)\sum_{i=1}^{m}y_{n+i}-2(2m+n)\sum_{i=1}^{n}y_{2m+i}-(m-2n)\sum_{i=1}^{2m+n}y_i}{m^2+13mn}\end{cases}$$

当 $m=2n$ 时.

$$Cov(\hat\beta,\hat\beta)=\sigma^2(X^TX)^{-1}=\sigma^2\begin{bmatrix}\frac{m+4n}{|S|} & \frac{2n-m}{|S|}\\ \frac{2n-m}{|S|} & \frac{n+2m}{|S|}\end{bmatrix}\triangleq\sigma^2(C_{ij})_{2\times2}.$$

$$Cov(\theta,\phi)=\sigma^2C_{12} \text{ 或 } \sigma^2C_{21}=\sigma^2(2n-m)/|S|=0.$$

则不相关。

4.9. 悬挂不同质量弹簧长.

重.	5	10	15	20	25	30
长.	7.25	8.12	8.95	9.90	10.9	11.8.

在一元正态模型下.

(1) 散点是否可以为重量与长度存在线性关系

(2) 求出经验回归函数

(3). 在 $\alpha=0.05$ 下. 检 $H_0: \beta_1=0$.

(4). 求 $x_0=16$ 时. y_0 的双侧 95% 预测区间.

(1). 散点图略. 线性关系

(2). $\bar{x}=17.5$ $\bar{y}=9.487$. $S_{xx}=437.5$ $S_{xy}=\cancel{79.6125}$ 80.1 $S_{yy}=14.678334$.

$\hat{y}=\hat{\beta_0}+\hat{\beta_1}x$.

$\hat{\beta_1}=\frac{S_{xy}}{S_{xx}}$ $\hat{\beta_0}=\bar{y}-\hat{\beta_1}\bar{x}$

则: $\hat{\beta_0}=.6.2845$. $\hat{\beta_1}=.0.183$.

$\hat{y}=.0.183x+6.2845$.

(3). $1-\alpha=0.95$. $H_0: \beta_1=0 \Leftrightarrow H_1: \beta_1\neq 0$.

$n=6$. $F=4\times\frac{\hat{\beta_1}^2 S_{xx}}{S_{yy}-\hat{\beta_1}^2 S_{xx}}=.2178.9359$. $F_{0.95}(1,4)=4.54$. 落入拒绝域.

有显著线性性.

$\hat{\sigma}=0.082$. $t_{0.975}(4)=2.7764$.

$\sqrt{1+\frac{1}{6}+\frac{(16-17.5)^2}{437.5}}=1.0825015$

$t_{0.975}(4)$

$\cancel{t_{0.95}(4)=.1.8132}$

(4). $\hat{y}=.0.183\times 16+6.2845=9.2125$.

$\hat{\sigma}^2=\frac{Q_e}{n-2}=.\frac{0.0268965}{4}=.0.006724125$.

$\delta(x_0)=.\hat{\sigma}\, t_{0.975}(4)\sqrt{1+\frac{1}{6}+\frac{(16-17.5)^2}{437.5}}=.0.246$. 区间: [8.967, 9.457].

4.10. 物体降落距离S与时间t关系

$$S_i=\beta_0+\beta_1 t_i+\beta_2 t_i^2+\varepsilon_i,\quad i=1,2\cdots n.$$

$$\varepsilon_i\sim N(0,\sigma^2).\quad i=1,2\cdots n.$$

t_i(s)	1/30	2/30	3/30	4/30	5/30	6/30	7/30	8/30	9/30	10/30	11/30	12/30	13/30	14/30	15/30
S_i(cm).	11.86	15.67	20.60	26.69	33.71	41.93	51.13	61.49	72.90.	85.44.	99.08.	113.77.	129.54	146.48	165.0

(1) 求 $\beta_0.\beta_1.\beta_2$ 最小二乘估计及σ^2估计.

(2). 检 $H_0:\beta_2=0$.

(3). $t=\frac{1}{10}$时. 给出S置信0.95区间.

$$X=\begin{bmatrix}1 & \frac{1}{30} & (\frac{1}{30})^2\\ \vdots & & \vdots\\ 1 & \frac{15}{30} & (\frac{15}{30})^2\end{bmatrix}\begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\end{bmatrix}$$

(1). $t_{i2}=t_i^2$. 则 $S_i=\beta_0+\beta_1 t_{i1}+\beta_2 t_{i2}+\varepsilon_i.\quad i=1\cdots n$

$$\overline{t_1}=\frac{(\frac{1}{30}+\frac{15}{30})\times 15}{2\times 15}=\frac{4}{15}\qquad \overline{t_2}=\frac{62}{675}\qquad \overline{S}=71.69.$$

$S_{yy}=34577.5376.\quad Q_e=Y^TAY.\quad A=I_n-X(X^TX)^{-1}X^T\quad \hat\beta=(X^TX)^{-1}X^TY$

$Q_e=0.2463548354.\quad U=34577.291.\quad \hat\beta_0=9.2646.\ \hat\beta_1=64.059\ \hat\beta_2=493.6532.$

(2) $$F=\frac{U/k}{Q_e/(n-k-1)}=\frac{U/2}{Q_e/12}\sim F(2,12).\qquad F=842133.86.$$

$F_{0.95}(2,12)=3.89.\quad F>3.89.\Rightarrow$ 拒绝H_0.

???

(3). $$\hat S_i=\hat\beta_0+\hat\beta_1\times\frac{1}{10}+\hat\beta_2\times\frac{1}{100}=20.6071.$$

$$\delta(t)=\hat\sigma\, t_{0.975}(12)\times\sqrt{x_0^T(X^TX)^{-1}x_0}\qquad x_0=[1,\frac{1}{10},\frac{1}{100}].$$

$=0.3365.$

$[20.2706,\ 20.9436]$.

4.11. 12份订单. 2km. 5km. 8km. 15km 各3份.

距离(km)	2	2	2	5	5	5	8	8	8	15	15	15
时间(min).	10.2	14.6	18.2	20.1	22.4	30.6	30.8	35.4	50.6	60.1	68.4	72.1

假定送货时间 $\sim N(\mu, 6^2)$.

(1) 求距离关于时间最小二乘回归直线

(2) 送 8km, 时间?

(3). 6^2 估计

(4). 求. $\hat{a}$. $\hat{b}$ 标准误.

(5). 利用直线,预测 4.8km 时间. 求90%预测区间

(6) $\alpha = 0.05$. 检: $H_0: b=0 \longleftrightarrow H_1: b>0$.

(1). $\overline{x} = 7.5$. $\overline{y} = 36.125$. $S_{xx} = 279$. $S_{xy} = 1143.65$ S_y

$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = .4.0991$. $\hat{\beta}_0 = \overline{y} - \hat{\beta}_1\overline{x} = 5.3817$.

$\hat{y} = 4.0991x + 5.3817$.

(2). $x_0 = 8 \Rightarrow \hat{y}_0 = 38.1745$.

(3). $\hat{6}^2 = \frac{Q_e}{n-2}$ $\Rightarrow \hat{6}$

(4). $\hat{\beta}_1$ 标准差 $\sqrt{\frac{\hat{6}^2}{S_{xx}}}$ $\hat{\beta}_0$ 标准差. $\sqrt{(\frac{1}{n} + \frac{\overline{x}^2}{S_{xx}})\hat{6}^2}$

(5). $x_0 = 4.8$. $\hat{y} = 25.05738$. $\hat{y} \pm \delta(x)$. $\delta(x) = \hat{6}\, t_{0.95}(4) \sqrt{1 + \frac{1}{n} + \frac{(x_0 - \overline{x})^2}{S_{xx}}}$

$t_{0.95}(10) = \cancel{2.23}\ 1.8125$. $[13.0806, 37.0343]$.

(6). $H_0: \beta_0 = 0 \longleftrightarrow H_1: \beta_0 > 0$.

$F: \frac{U(n-2)}{Q_e} \sim F(1, n-2)$. $F_{0.95}(1, 10) = 3.28$.

$F > F_{0.95}(1, 10)$. 拒 H_0.

$y_i = \{x$

$\text{次}'$

4.13 黏虫生长过程数据.

平均温度 T.	11.8	14.7.	15.4	16.5	17.1	18.1	19.8	20.3.
历期 N.	30.4	15.0	13.8	12.7	10.7	7.5	6.8	5.7

历期为卵孵化成虫天数. 经研究 N 与 T 有如下关系: $N = k/(T-C)+\varepsilon$. 求 k, C 估计.

$$\frac{1}{N} = \frac{(T-C)}{K} + \varepsilon_0.$$

$$y = \frac{1}{N}.\quad a = \frac{1}{k}.\quad b = -\frac{C}{K}.\quad x = T. \Rightarrow y = ax + b + \varepsilon_0.$$

计算: $\hat{a} = \frac{S_{xy}}{S_{xx}}$. $\hat{b} = \bar{y} - \hat{a}\bar{x}$ $\bar{x} = \bar{T} = 16.7125.$ $\bar{y} = \overline{\frac{1}{N}} = 0.100067533.$

且 $\sum_{i=1}^{8} x_i y_i = 14.26794491.$ $\sum_{i=1}^{8} x_i^2 = 2288.89.$

则 $\hat{a} = \frac{S_{xy}}{S_{xx}} = \frac{\sum_{i=1}^{n} x_i y_i - 8\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - 8\bar{x}^2} = 0.01648103243.$ $\hat{b} = \bar{y} - \hat{a}\bar{x} = -0.175432$

而: $\hat{k} = \frac{1}{\hat{a}} = 60.67581047.$

而 $\hat{b} \Rightarrow$. $\hat{c} = -\hat{b}\hat{k} = 10.64450919.$

4.14. 肥料对作物产量影响. 三种肥1作物产量.

Ⅰ.	794	1800	576	411	897	$\sim N(\mu_1, 6^2)$
Ⅱ	2012	2477	3498	2092	1808	$\sim N(\mu_2, 6^2)$
Ⅲ.	2118	1947.	3361	2117	19	$\sim N(\mu_3, 6^2)$.

不同浓度肥料对作物产量是否有显著差异.

$H_0: \mu_1 = \mu_2 = \mu_3$.

$\overline{X_Ⅰ} = 895.6$. $\overline{X_Ⅱ} = 2377.4$. $\overline{X_Ⅲ} = 2300$ $\overline{X} = \frac{1}{3}(\overline{X_Ⅰ} + \overline{X_Ⅱ} + \overline{X_Ⅲ}) = 1857.67$.

组间平方和: $S_A = 5 \times [(\overline{X_Ⅰ} - \overline{X})^2 + (\overline{X_Ⅱ} - \overline{X})^2 + (\overline{X_Ⅲ} - \overline{X})^2] =$.

组内平方和: $S_E = \sum_{i=1}^{3}\sum_{j=1}^{5}(X_{ij} - \overline{X_i})^2 =$.

$$F = \frac{S_A/p-1}{S_E/n-p} = \frac{S_A/2}{S_E/(15-3)} = \frac{6S_A}{S_E} \sim F(2, 12).$$

则 $W = \{F > F_{1-\alpha}\}$ 而 $F_{0.9}(2, 12) = 2.81$.

计算: $F = \frac{6S_A}{S_E} = 9.476 > F_{0.9}(2, 12)$. 拒绝 H_0: 有显著差异.

)/2

4.15. 小白鼠接种3种菌型存活天数.

菌型.	天数.											
1	2	4	3	2	4	7	7	2	2	5	4	(11个)
2	5	6	8	5	10	7	12	12	6	6	(10个)	
3.	7	11	6	6	7	9	5	5.	10	6	3	10. (12个)

(1) 判断小白鼠被注射后平均存活天数有无显著差异.(0.05).

(2). 被注射不同菌型后平均存活天数差的95%置信区间.

单因子试验方差分析.

(1). $\overline{X_1}=3.8187$、$\overline{X_2}=7.7$. $\overline{X_3}=7.084$. $\overline{X}=\frac{1}{n}\sum_{i=1}^{p}\sum_{j=1}^{n_i}y_{ij}=\frac{1}{33}(11\times\overline{X_1}+10\times\overline{X_2}+12\times\overline{X_3})$

$=6.182.$

组内平方和: $S_E=\sum_{i=1}^{p}\sum_{j=1}^{n_i}(X_{ij}-\overline{X_i})^2.$

组间平方和 $S_A=\sum_{i=1}^{p}n_i(\overline{X_i}-\overline{X})^2$

$$F=\frac{S_A/(p-1)}{S_E/(n-p)}=\frac{15S_A}{S_E}\qquad F\sim F(2,30)\qquad W=\{F>F_{0.95}(2,30)\}$$

$F_{0.95}(2,30)=3.32.$ $\quad F=\frac{15S_A}{S_E}=$

(2). $y_i\sim N(\mu_i,\sigma^2)$ $y_j\sim N(\mu_j,\sigma^2)$. $1-\alpha$置信区间. $[\overline{y_i}-\overline{y_j}\pm\sqrt{\frac{1}{n_i}+\frac{1}{n_j}}\,\hat{\sigma}\,t_{1-\frac{\alpha}{2}}(n-p).]$

①. $\mu_1-\mu_2$的$1-\alpha$区间 [ ] $\quad\hat{\sigma}=\frac{S_E}{n-p.}=\frac{S_E}{30.}=$

②. $\mu_2-\mu_3$ $1-\alpha$区间 [ ] $\quad t_{0.975}(n-p).=t_{0.975}(30).=$

③. $\mu_1-\mu_3$ $1-\alpha$区间. [ ]

习题5.

5.1. $A_i=\{x: (i-1)/2 < x \le i/2\}$. $i=1,2,3$. $A_4=\{x: 3/2 < x < 2\}$.

X ~ 100 次观测. 落入 $A_i (i=1,2,3,4)$ 频数 30, 20, 36, 14. X 是否服从 (0,2) 上均匀分布. $\alpha=0.05$.

H_0: $X \sim R(0,2)$.

则. $P(X\in A_i)=\frac{1}{4}$. $\sum_{i=1}^{r}\frac{(n_i-np_i)^2}{np_i} \sim \chi^2(r-1)$.

$$\chi^2=\frac{(30-25)^2}{25}+\frac{(20-25)^2}{25}+\frac{(36-25)^2}{25}+\frac{(14-25)^2}{25}=11.68$$

由 $\chi^2_{0.95}(3)=7.815$. $\chi^2>\chi^2_{0.95}(3)$ 拒 H_0. $X \not\sim R(0,2)$.

5.6. 一台设备进行寿命检验，10次无故障工作时间.

420 500 920 1330. 1510. 1650 1760 2100 2300 2350

此设备的无故障工作时间是否服从指数分布.

$H_0: X \sim$ Exp$(\hat{\lambda})$.

$f(x;\lambda) = \lambda^{10} e^{-\lambda \sum_{i=1}^{n} x_i}$

$\cancel{E(X) = \frac{1}{\lambda} = \sum_{i=1}^{n} x_i}$

$\cancel{2P(\lambda\bar{x} > 1489) =}$

$\ln f = 10\ln\lambda - \lambda \sum_{i=1}^{n} x_i$.

$\frac{\partial \ln f}{\partial \lambda} = \frac{10}{\lambda} - \sum_{i=1}^{n} x_i = 0 \Rightarrow \hat{\lambda} = \frac{10}{14890}$ $\hat{\lambda} = \frac{1}{1489}$

则 $f_0(x) = \begin{cases} \frac{1}{1489} e^{-\frac{1}{1489}x}, & x > 0 \\ 0, & x \le 0. \end{cases}$

$0 < x \le 500$. 2

$500 < x \le 1000$ 1

$1000 < x \le 1500$ 1

$1500 < x \le 2000$. 3

$2000 < x \le 2500$ 3.

$2500 < x < +\infty$ 0.

$\sum_{i=1}^{6} \frac{(n_i - np_i)^2}{np_i} = \chi^2$. 再比 $\chi^2_{0.95}(6-1-1)$.

5.9、父母行为对孩子影响，看电影为例.

孩子 \ 父母	每周一次	每月一次.	难得一次.	
每周一次	74	67	27	168
每月一次	11	14	33.	58
难得一次.	5	10	17.	32
	90	91	77	共258

检验父母与孩子看电影情况是否独立 $\alpha=0.05$

$$H_0: \forall i,j.\ P_{ij}=P_{i\cdot}\cdot P_{\cdot j} \longleftrightarrow H_1: \exists i,j.\ P_{ij}\neq P_{i\cdot}\cdot P_{\cdot j}.$$

$$\chi^2 = n\left(\sum_{i=1}^{p}\sum_{j=1}^{q}\frac{n_{ij}^2}{n_{i\cdot}n_{\cdot j}}-1\right).$$

$$= 258\left(\frac{74^2}{90\times168}-1+\frac{67^2}{168\times91}\cdots\cdots\right).$$

再比较 χ^2 与 $\chi^2_{0.95}(2\times2)$ 大小.

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