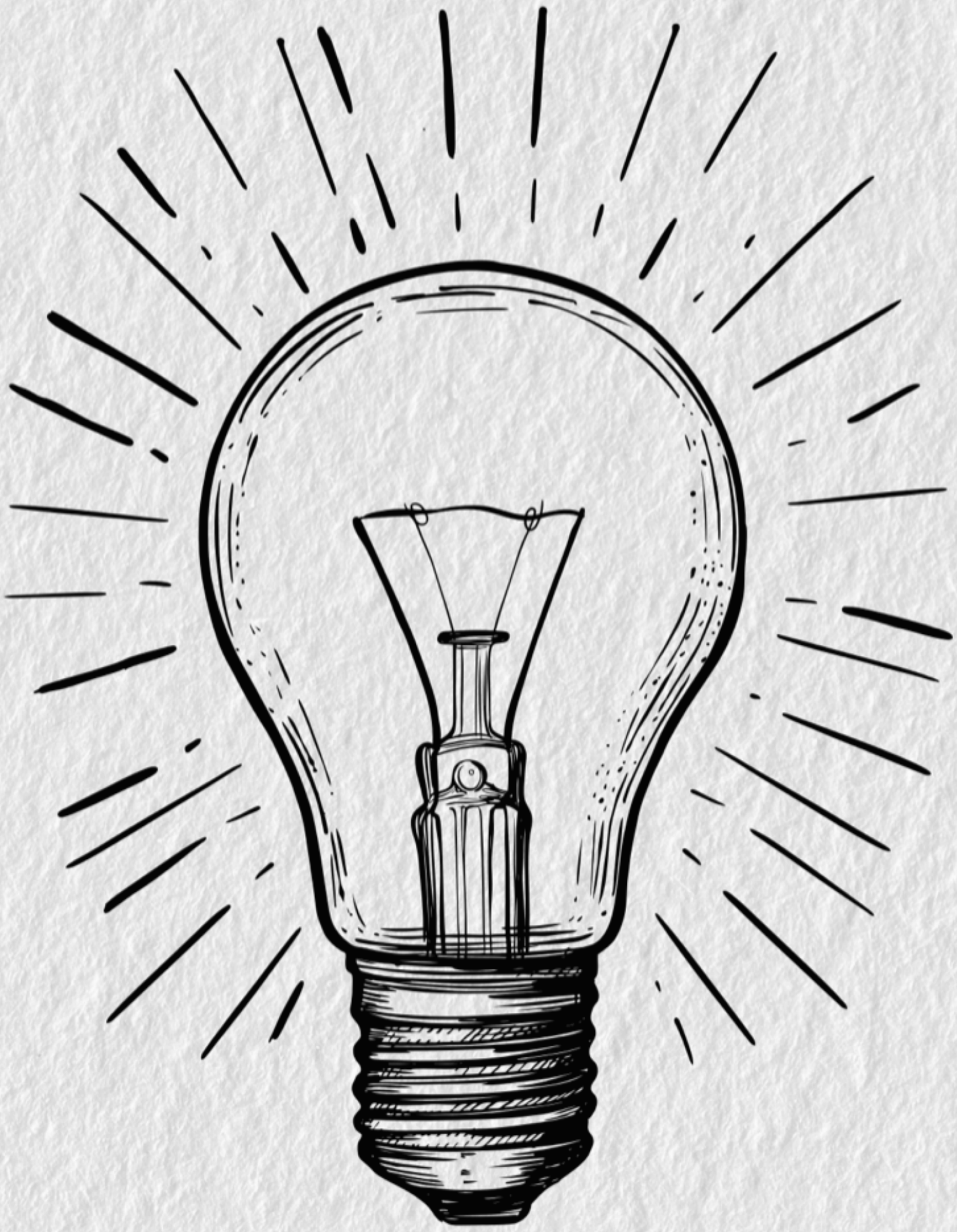
NOTEBOOK



NOTE IN

Creative notes

By reading we enrich the mind;
by writing we polish it.

1. 验证解

(1). $y=C_1e^{2x}+C_2e^{-2x}$　　$y''-4y=0.$

$y'=2C_1e^{2x}-2C_2e^{-2x}.$　代 y''. y 进去确实满足.

则 $\dfrac{D[y.y']}{D[C_1.C_2]}=\begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix}=-2-2=-4\neq 0.$

习题 2-1. 求解恰当方程.

1. $(ax+by)dx+(bx+cy)dy=0.$

由 $\dfrac{\partial P}{\partial y}=b=\dfrac{\partial Q}{\partial x}$ 为恰当方程.

$\phi(x)=\int P\,dx=\frac{1}{2}ax^2+bxy+m(y)$

$\phi(x)=\int Q\,dy=bxy+\frac{1}{2}cy^2+n(x)$

$\Rightarrow$　$\frac{1}{2}ax^2+bxy+\frac{1}{2}cy^2=C$

2. $(t^2+1)\cos u\,du+2t\sin u\,dt=0.$

$\dfrac{\partial P}{\partial t}=2t\cos u=\dfrac{\partial Q}{\partial u}.$

则 $\phi=\int P\,du=(t^2+1)\sin u+m(V).$

$\phi=\int Q\,dt=t^2\sin u+n(u).$

$\Rightarrow \phi=t^2\sin u+\sin u=C$

3. $\dfrac{2s-1}{t}ds+\dfrac{s-s^2}{t^2}dt=0.$

$\dfrac{\partial P}{\partial t}=-\dfrac{2s-1}{t^2}=\dfrac{\partial Q}{\partial s}.$

则 $\phi=\int P\,ds=\dfrac{s^2-s}{t}+m(t)$

$\dfrac{\partial\phi}{\partial t}=\dfrac{s-s^2}{t^2}+m'(t)=\dfrac{s-s^2}{t^2}$　$m'(t)=0$

$\Rightarrow$　$\dfrac{s^2-s}{t}=C.$

4. $xf(x^2+y^2)dx+yf(x^2+y^2)dy=0.$

则 $\dfrac{\partial P}{\partial y}=2xyf'(x^2+y^2).$　$\dfrac{\partial Q}{\partial x}=2xyf'(x^2+y^2).$

则有: $\phi=\int P\,dx=\frac{1}{2}F(x^2+y^2)+m(y).$

$\dfrac{\partial\phi}{\partial y}=yf(x^2+y^2)+m'(y)=yf(x^2+y^2).\Rightarrow m'(y)=0.\Rightarrow m(y)=C'$　　$\frac{1}{2}F(x^2+y^2)=C.$

习题2.2.

1. (3). $\frac{dy}{dx}+y^2\sin x=0$.

$\frac{dy}{dx}=-y^2\sin x$. $\frac{dy}{-y^2}=dx\sin x$ $(y=0)$

$\Rightarrow \frac{1}{y}=-\cos x+C$.

(5). $\frac{dy}{dx}=(\cos x\cos 2y)^2$.

$2\cos^2 x-1=\cos 2x$.

则有: $\frac{dy}{(\cos 2y)^2}=\cos^2 x\,dx$ 且 $2y=\frac{\pi}{2}+k\pi$. $k\in Z$.

$\frac{1}{2}\tan 2y=\int\frac{\cos 2x+1}{2}dx$

$\frac{1}{2}\tan 2y=\frac{\sin 2x}{4}+\frac{1}{2}x+C$.

(6) $x\frac{dy}{dx}=\sqrt{1-y^2}$.

$\Rightarrow \frac{dy}{\sqrt{1-y^2}}=\frac{1}{x}dx$. 且 $y=\pm 1$

$\arcsin y=\ln|x|+C$.

2. (4). $\frac{dy}{dx}=\frac{\ln|x|}{1+y^2}$. $y(1)=0$.

$(1+y^2)dy=\ln|x|\,dx$.

$\Rightarrow y+\frac{1}{3}y^3=x\ln x-x+C$. 代 $y(1)=0\Rightarrow C=1$.

(5). $\sqrt{1+x^2}\frac{dy}{dx}=xy^3$ $y(0)=1$

由: $y=0$. $\frac{dy}{y^3}=\frac{x}{\sqrt{1+x^2}}dx$.

则: $-\frac{1}{2}\cdot\frac{1}{y^2}=\sqrt{1+x^2}+C$. 代 $y(0)=1$. $C=-\frac{3}{2}$

3. (4) $\frac{dy}{dx}=y^n$ $(n=\frac{1}{3},1,2)$.

① $\frac{dy}{dx}=y^{\frac{1}{3}}\Rightarrow\frac{dy}{\sqrt[3]{y}}=dx\Rightarrow\frac{3}{2}y^{\frac{2}{3}}=x+C$, 且 $y=0$.

② $\frac{dy}{dx}=y$. $\Rightarrow\frac{dy}{y}=dx\Rightarrow\ln y=x+C$. 且 $y=0$.

③. $\frac{dy}{dx}=y^2 \Rightarrow \frac{dy}{y^2}=dx \Rightarrow -\frac{1}{y}=x+C$、且 $y=0$.

习题 2-3. 一阶线性方程.

1、(2). $\frac{dy}{dx}+\tan x y=\sin 2x$.

由积分因子法.

左右同乘 $e^{\int \tan x dx}$. 则 $e^{\int \tan x dx}\frac{dy}{dx}+\tan x e^{\int \tan x} y \frac{dy}{dx}=e^{\int \tan x}\sin 2x$.

$$\Rightarrow d(e^{\int \tan x dx}y)=.d(\int e^{\int \tan x}\sin 2x\,dx).$$

$$\Rightarrow. e^{\int \tan x dx}y=.\int e^{\int \tan x}\sin 2x dx+C$$

$$y=e^{-\int \tan x dx}(\int e^{\int \tan x dx}\sin 2x dx+C)$$

$$=e^{+\ln|\cos x|}(\int e^{-\ln|\cos x|}\sin 2x dx+C)$$

$$=\cos x(\int 2\sin x dx+C).$$

$$=\cos x(-2\cos x+C).$$

(3) $x\frac{dy}{dx}+2y=\sin x$. $y(\pi)=\frac{1}{\pi}$

则 $\frac{dy}{dx}+\frac{2}{x}y=\frac{\sin x}{x}$.

积分因子: $e^{\int \frac{2}{x}dx}$.

$$e^{\int \frac{2}{x}dx}dy+\frac{2}{x}e^{\int \frac{2}{x}dx}dx=e^{\int \frac{2}{x}dx}\frac{\sin x}{x}dx.$$

$$d(e^{\int \frac{2}{x}dx}y)=d(\int e^{\int \frac{2}{x}dx}\frac{\sin x}{x}dx)$$

$$e^{\int \frac{2}{x}dx}y=\int e^{\int \frac{2}{x}dx}\frac{\sin x}{x}dx+C.$$

有 $y=e^{-\int \frac{2}{x}dx}(\int e^{\int \frac{2}{x}dx}\frac{\sin x}{x}dx+C)$.

$$=\frac{1}{x^2}(\int x\sin x dx+C)$$

$$=\frac{1}{x^2}(-x\cos x+\sin x+C).$$ 代 $y(\pi)=\frac{1}{\pi}$

2、(3). $3xy^2\frac{dy}{dx}+y^3+x^3=0$.

有伯努利方程. $\frac{dy}{dx}+\frac{1}{3x}y=-\frac{x^2}{3}y^{-2}$

左右同乘 $(1-n)y^{-n}$ $n=-2$.

有 $3y^2\frac{dy}{dx}+\frac{y^3}{x}=-x^2$.

令 $Z=y^3$. $\frac{d(z)}{dx}+\frac{1}{x}\cdot z=-x^2$.

由积分因子法: $y=e^{-\int\frac{1}{x}dx}(\int -x^2e^{\int\frac{1}{x}dx}dx$.

$$dy\, e^{\int_0^x a(s)ds}+a(x)ye^{\int_0^x a(s)ds}dx\le 0$$

$$d(ye^{\int_0^x a(s)ds}\le 0$$

2.4. $\frac{dy}{dx}=\frac{x+2y+1}{2x+4y+1}$. 令 $\lambda=x+2y$

则 $\frac{d\lambda}{dx}=1+2\frac{dy}{dx}=1+2\frac{\lambda+1}{2\lambda-1}=\frac{2\lambda+2+2\lambda-1}{2\lambda-1}=\frac{4\lambda+1}{2\lambda-1}$

$\frac{2\lambda+1}{4\lambda+3}d\lambda=dx$

(2). $(3uv+v^2)du+(u^2+uv)dv=0$.

$\frac{\partial P}{\partial v}=3u+2v$ $\frac{\partial Q}{\partial u}=2u+v$.

$\frac{1}{Q}(\frac{\partial P}{\partial v}-\frac{\partial Q}{\partial u})=(u+v)\times\frac{1}{u^2+uv}=\frac{1}{u}$.

则积分因子 $e^{\int\frac{1}{u}du}=u$.

$3u^2v+uv^2du+(u^3+u^2v)dv=0$.

$\Rightarrow u^3v+\frac{1}{2}u^2v^2=C$.

(3). $(x^2+y^2+3)\frac{dy}{dx}=2x(2y-\frac{x^2}{y})$

3. (2) $x^2y'=x^2y^2+xy+1$.

$y'=y^2+\frac{1}{x}y+\frac{1}{x^2}$

特解: $y=-\frac{1}{x}$

令 $y=u-\frac{1}{x}$ 代入: $\frac{du}{dx}+\frac{1}{x^2}=u^2-\frac{2u}{x}+\frac{1}{x^2}+\frac{u}{x}-\frac{1}{x^2}+\frac{1}{x^2}$

$$\frac{du}{dx}=u^2+\frac{u}{x}$$

则 $\frac{du}{dx}-\frac{1}{x}u=u^2$.

则左右同乘 $(1-n)u^{-n}$ $n=2$.

$$-u^{-2}\frac{du}{dx}+\frac{1}{x}u^{-1}=-1$$

$z=\frac{1}{u}$ $\frac{dz}{dx}+\frac{1}{x}z=-1$. $z=e^{-\int\frac{1}{x}dx}\left(\int -e^{\int\frac{1}{x}dx}dx+C\right)$.

定 $y=y_0e^{\int_{x_0}^{x}p(t)dt}+\int_{x_0}^{x}q(s)e^{-\int_s^x p(t)dt}ds$.

$$y=y_0e^{-\int_{x_0}^{x}p(t)dt}+\int_{x_0}^{x}q(s)e^{-\int_s^x p(t)dt}ds$$

(1) $(3x^2y+2xy+y^3)dx+(x^2+y^2)dy=0$.

$\frac{\partial P}{\partial y}=3x^2+2x+3y^2$. $\frac{\partial Q}{\partial x}=2x$

$\frac{1}{Q(x,y)}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=3$. 积分因子：$e^{\int 3dx}=e^{3x}$.

$e^{3x}(3x^2y+2xy+y^3)dx+e^{3x}(x^2+y^2)dy=0$.

$e^{3x}(3x^2+2x+3y^2)$ $e^{3x}(3x^2+3y^2+2x)$.

$e^{3x}(x^2y+\frac{1}{3}y^3)+\phi(x)$.

$e^{3x}(3x^2y+y^3+2xy)+\phi'(x)$. $\phi'(x)=0$ $\phi(x)=C$

(4) $e^x dx+(e^x\cot y+2y\cos y)dy=0$.

则有 $\frac{\partial P}{\partial y}=0$. $\frac{\partial Q}{\partial x}=e^x\cot y$.

则 $\frac{1}{P}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=\cot y$.

则有：$e^{\int\cot y dy}=e^{\ln|\sin y|}=\sin y$.

$e^x\sin y\,dx+(e^x\cos y+2y\sin y\cos y)dy=0$

$$\Phi=e^x\sin y+\phi(y).$$

$e^x\cos y+\phi'(y)=e^x\cos y+y\sin 2y.$

$\phi(y)=y\sin y.$

$x^2+C^2y^2=1.$ $C^2=\frac{1-x^2}{y^2}.$

$2xdx+C^2 2y\,dy=0.$

$2xdx+\frac{2(1-x^2)}{y}dy=0.$ $\frac{dy}{dx}=\frac{xy}{(x^2-1)}$ 则有：$\left(\frac{dy}{dx}\right)_1=\frac{1-x^2}{xy}$

$2xy\,dx+2(1-x^2)dy=0.$

$(x^2-1)dx+xy\,dy=0$

$\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=\frac{1}{xy}(0-y)=-\frac{1}{x}.$

则 $e^{-\int\frac{1}{x}dx}=\frac{1}{x}.$

则 $(x-\frac{1}{x})dx+ydy=0.$ $(x-\frac{1}{x})dx=-ydy.$

$\frac{1}{2}x^2-\ln x=-\frac{1}{2}y^2+C.$

$y^2=4ax.$

$2ydy=4adx.$

$\frac{dy}{dx}=\frac{4a}{2y}.$ 则 $H(x,y)=\frac{4a}{2y}$ $\frac{dy}{dx}=\frac{\frac{4a}{2y}+1}{1-\frac{4a}{2y}}=\frac{4a+2y}{2y-4a}$

$x-2y=C.$

$dx-2dy=0.$ 则 $\frac{dy}{dx}=\frac{1}{2}.$ $\frac{\frac{1}{2}+1}{1-\frac{1}{2}\times1}=3.$ $y=3x+M.$

4-1.

(2). $y = px\ln x + (xp)^2$.

则 $p = \frac{dy}{dx} = y'_x + y'_p \frac{dp}{dx}$.

$p = p\ln x + 2p^2x + \frac{dp}{dx}(x\ln x + 2x^2 p)$.

$p\ln x + 2p^2x + \frac{dp}{dx}(x\ln x) + \frac{dp}{dx} 2x^2p = 0$.

$p(\ln x + 2px) + \frac{dp}{dx} x(\ln x + 2px) = 0$

$(p + \frac{dp}{dx}x)(\ln x + 2px) = 0$

则 $\frac{dp}{dx} = -\frac{p}{x}$

$-\frac{dp}{p} = \frac{1}{x}dx$

$-\ln p = \ln x + C$.

$\ln px = C'$.

$px = e^{C'}$

$\frac{dy}{dx} x = e^{C_1}$

$dy = \frac{1}{x} dx\, e^{C_1}$

$y = e^{C_1}\ln x + C_2$.

$\begin{cases} \ln x + 2px = 0. \\ y = px\ln x + (xp)^2. \end{cases}$

消 p

2. (2). $x^2 - 3(\frac{dy}{dx})^2 = 1$.

$x = \frac{1}{\sin u}$ $\frac{dy}{dx} = -\frac{1}{\sqrt{3}} \cdot \frac{\cos u}{\sin u}$.

令 $\tan\frac{u}{2} = t$. $x = \frac{1}{\sin u} = \frac{1+t^2}{2t}$.

$\tan u = \frac{\tan\frac{u}{2} + \tan\frac{u}{2}}{1 - \tan^2\frac{u}{2}} = \frac{2t}{1-t^2}$.

则 x 表示出来，则 $dy = -\frac{1}{\sqrt{3}} \cdot \frac{\cos u}{\sin u} \cdot -\frac{\cos u}{\sin^2 u} = \frac{1}{\sqrt{3}} \frac{\cos^2 u}{\sin^3 u}$

y 用 t 表示

(4). $x^3+\left(\frac{dy}{dx}\right)^3=4x\frac{dy}{dx}$.

令 $\frac{dy}{dx}=xt$. $\quad x^3+x^3t^3=4x^2t$

解出 $x=\frac{4t}{1+t^3}$

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}=xt.$$

4.2.

1. $y=x\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2$.

$y=xp+p^2$.

$x+2p=0$. $\Rightarrow y=-\frac{x^2}{4}$.

$F'_y(x,-\frac{x^2}{4},-\frac{x}{2})=1$. $\quad F'_{pp}(x,-\frac{x^2}{4},-\frac{x}{2})=2\neq0$. $\quad F'_p(x,-\frac{x^2}{4},-\frac{x^2}{2})=0$.

(2). $y=2x\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2$.

$$\begin{cases} y=2xp+p^2. \\ 2x+2p=0. \end{cases} \Rightarrow y=-x^2. \quad p=-2x.$$

(3) $(y-1)^2\left(\frac{dy}{dx}\right)^2=\frac{4}{9}y$.

$$\begin{cases} (y-1)^2p^2=\frac{4}{9}y \\ 2p(y-1)^2=0. \end{cases} \qquad y=0.\text{ 显然.}$$

4.3.

1. $y=xp+f(p)$. $\quad (p=\frac{dy}{dx})$.

$p=p+x\frac{dp}{dx}+f'(p)\frac{dp}{dx}$.

$(x+f'(p))\frac{dp}{dx}=0$. $\quad \frac{dp}{dx}=0\Rightarrow p=C$

则 $y=cx+f(c)$. $\qquad V(x,y,C)=0$.

$$\begin{cases} y=cx+f(c). \\ x+f'(c)=0. \end{cases} \qquad V_C(x,y,C)=0.$$

$$\frac{dy}{dt} = A(t)\vec{y}.$$

(1). $\frac{dy_1}{dt} = y_1 + y_2.$

$\frac{dy_2}{dt} = y_2.$ $\Rightarrow y_2 = ke^t$

$$y_0 e^{-\int_{x_0}^{x} p(s)ds} + \int q(s) e^{\int_s^x p(t)dt} ds.$$

取 y_2

$$\vec{y} = \phi(x)\phi^{-1}(x_0)y_0 + \phi(x)\int_{x_0}^{x} \phi^{-1}(s)f(s)\,ds$$

4. $|A-\lambda E| = \begin{vmatrix} -5-\lambda & -10 & -20 \\ 5 & 5-\lambda & 10 \\ 2 & 4 & 9-\lambda \end{vmatrix}$

$= \begin{vmatrix} -5-\lambda & 2\lambda & -20 \\ 5 & -\lambda-5 & 10 \\ 2 & 0 & 9-\lambda \end{vmatrix}$

$= 2\begin{vmatrix} 2\lambda & -20 \\ -\lambda-5 & 10 \end{vmatrix} + (9-\lambda)\begin{vmatrix} -5-\lambda & 2\lambda \\ 5 & -\lambda-5 \end{vmatrix}.$

$= 2(20\lambda - 20\lambda - 100) + (9-\lambda)((5+\lambda)^2 - 10\lambda).$

$= -200 + (9-\lambda)(25+\lambda^2)$

$= -\lambda^3 - 25\lambda + 9\lambda^2 + 25$

$= -(\lambda^3 - 9\lambda^2 + 25\lambda - 25).$

$= -(\lambda-5)(\lambda^2 - 4\lambda + 5)$

4. 5. $2+i$ $2-i$.

根5. 根.

$$\begin{bmatrix} -10 & -10 & -20 \\ 5 & 0 & 10 \\ 2 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$x_1 = -2x_3.$

$x_2 = 0x_3$

$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{5x}.$

$$\begin{bmatrix} -7-i & -10 & -20 \\ 5 & 3-i & 10 \\ 2 & 4 & 7-i \end{bmatrix} = \begin{bmatrix} 7+i & 10 & 20 \\ 1 & -i-5 & 2i-4 \\ 2 & 4 & 7-i \end{bmatrix} = \begin{bmatrix} 1 & -i-5 & 2i-4 \\ 0 & 14+2i & 15-5i \\ 7+i & 10 & 20 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -i-5 & 2i-4 \\ 0 & 14+2i & 15-5i \\ & & \end{bmatrix}$$

$y'' + 4y' + 4y = \cos 2x.$

$r^2 + 4r + 4 = 0 \quad (r+2)^2 = 0. \quad r = -2$ 二重.

设 $y^* = a\sin 2x + b\cos 2x.$

$(\lambda+1)^3 = 0. \qquad e^{\alpha x}\cos\beta x + e^{\alpha x}\sin\beta x.$

$y'' - 2y' + 2y = 4e^x\cos x.$

$r^2 - 2r + 2 = 0. \quad r_1 r_2 = 2 \quad r_1 + r_2 = 2. \quad 1+i \quad 1-i$

$y^* = e^x[a\cos x + b\sin x] \qquad C_1 e^x\cos x + C_2 e^x \sin x.$

$y' = e^x[a\cos x + b\sin x - a\sin x + b\cos x]$

例：$\frac{dy}{dx} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\vec{y}$.

特 $\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2+1 = \lambda^2-2\lambda+2$. $\lambda_1 = 1+i$. $\lambda_2 = 1-i$.

$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} i \\ -1 \end{bmatrix}$. $\begin{bmatrix} 1 \\ i \end{bmatrix}$ $\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} \ \\ \ \end{bmatrix} = 0$ $\begin{bmatrix} i \\ 1 \end{bmatrix}$

$\phi[x] = \begin{bmatrix} e^{(1+i)x} \\ e^{(1+i)x} i \end{bmatrix}$ $\begin{matrix} e^x(\cos x + i\sin x) \\ e^x(i\cos x - \sin x) \end{matrix}$ $\begin{bmatrix} e^x\cos x & e^x\sin x \\ e^x(-\sin x) & e^x\cos x \end{bmatrix}$

1. $\begin{cases} \frac{dx}{dt} = 2x+y \\ \frac{dx}{dt} = 3x+4y \end{cases}$ $\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8-3=5$. 初.

$p = -\operatorname{tr}A = -(2+4) = -6$. $q = 5$.

$q<0$.

$q>0$, 且 $p^2-4q>0$. 二向. 且. $p<0$. 不稳.

$q>0$. $p=0$. 中

$\begin{cases} x = C \\ y = \sin C \end{cases} \Rightarrow \begin{cases} \Delta(x-C) + (y - \sin C) = 0 \\ \Delta'(x-C) - \Delta - \cos C = 0 \end{cases}$

$x = \frac{1}{p}\tan y + \frac{1}{2}p^2\cos^2 y$

$\frac{1}{p} = \frac{\frac{1}{\cos^2 y}p - \tan y\frac{dp}{dy}}{p^2} + p\cos^2 y\frac{dp}{dy} - p^2\cos y\sin y$.

$0 = \frac{-1}{p^2}\tan y\frac{1}{p}\frac{dp}{dx}$
$+\tan^2 y\cdot\frac{1}{p} +$
$\cos^2 y\frac{dp}{dx} - p^2\cos y\sin y$

$$p = \frac{1}{\cos^2 y} p - \tan y \frac{dp}{dy} + p^3 \cos^2 y \frac{dp}{dy} - p^4 \cos^2 y \sin y.$$

$$0 = \tan^2 y \, p - \tan y \frac{dp}{dy} + p^3 \cos^2 y \frac{dp}{dy} - p^4 \cos^3 y \sin y.$$