第十三章 多元函数的极限和连续

1. 证明$\mathbb{R}^n$中的三角不等式：$\forall x,y,z\in\mathbb{R}^n$，成立$|x-z|\le|x-y|+|y-z|$

证明：设 $x=(x_1,\cdots,x_n)$, $y=(y_1,\cdots,y_n)$, $z=(z_1,\cdots,z_n)$. $x_i,y_i,z_i\in\mathbb{R}$ $(i=1,\cdots,n)$，则 $|x-z|=\sqrt{\sum_{i=1}^n (x_i-z_i)^2}$

$=\sqrt{\sum_{i=1}^n (x_i-y_i)^2+\sum_{i=1}^n(y_i-z_i)^2+2\sum_{i=1}^n(x_i-y_i)(y_i-z_i)} \le \left(\sum_{i=1}^n(x_i-y_i)^2+\sum_{i=1}^n(y_i-z_i)^2+2\sqrt{\sum_{i=1}^n(x_i-y_i)^2\sum_{i=1}^n(y_i-z_i)^2}\right)^{1/2}=|x-y|+|y-z|$ #

2. 设点列$\{x_k=(x_1^k,\cdots,x_n^k)\}$趋于$\infty$，问下面命题是否正确？(1) $\forall 1\le i\le n$，$\{x_i^k\}$趋于∞ (2) $\exists 1\le i_0\le n$，$\{x_{i_0}^k\}$趋于∞

解：(1) 不对，反例如 $x_k=(k,0,\cdots,0)$ (2) 不对，反例如：让x_k的第$k \bmod n$个分量为k，其余分量为0

3. 求下面集合的聚点集：(1) $E_1=\{(\frac{q}{p},\frac{q}{p},1): p,q\in\mathbb{Z}_+$互素, $q<p\}$ (2) $E_2=\{(\ln(1+\frac{1}{k})^k,\sin\frac{k\pi}{2}): k=1,2,\cdots\}$

(3) $E_3=\{(r\cos(\tan\frac{\pi}{2}r), r\sin(\tan\frac{\pi}{2}r)): 0\le r<1\}$

解：(1) 若$x=(x_1,x_2,x_3)\in\mathbb{R}^3$是$E_1$聚点，则显然$x_3=1$, $x_1,x_2\in[0,1]$，而这样的x都是聚点，故 $E_1'=\{(t,t,1)|0\le t\le1\}$

(2) 若$x=(x_1,x_2)\in\mathbb{R}^2$是$E_2$聚点，则显然$x_2=0,\pm1$. 而$\{\ln(1+\frac{1}{k})^k: k=1,2,\cdots\}$的任何子列趋于$1\Rightarrow x_1=1$. 故$E_2'=\{(1,0),(1,1),(1,-$

(3) 若$x=(x_1,x_2)\in\mathbb{R}^2$是$E_3$聚点，则 $x_1^2+x_2^2\le1$. 若 $|x|<1$，则由$r\cos(\tan\frac{\pi}{2}r)$与$r\sin(\tan\frac{\pi}{2}r)$的连续性

知$x\in E_3$且此时x是E_3聚点. 若$|x|=1$，显然x是E_3聚点，故$E_3'=E_3\cup\{x\in\mathbb{R}^2: |x|=1\}$

4. 求下面集合的内部、外部、边界、闭包：(1) $E_1=\{(x,y,z)\in\mathbb{R}^3: x>0,y>0,z=1\}$ (2) $E_2=\{(x,y)\in\mathbb{R}^2: x>0, x^2+y^2-2x>1\}$

解：(1) $E_1^\circ=\phi$. $(E_1^c)^\circ=\{(x,y,z)\in\mathbb{R}^3: x<0,y<0,z\ne1\}$. $\partial E_1=\{(x,y,z)\in\mathbb{R}^3: x\ge0,y\ge0,z=1\}$. $\overline{E_1}=\{(x,y,z)\in\mathbb{R}^3: x\ge0, y\ge0, z=1\}$

(2) $E_2^\circ=E_2$. $(E_2^c)^\circ=\mathbb{R}^2\setminus\{(x,y): x\ge0, x^2+y^2-2x\ge1\}$. $\partial E_2=\{(x,y): x=0$且$|y|\ge1$, 或$x>0$且$x^2+y^2-2x=1\}$

$\overline{E}=\{(x,y)\in\mathbb{R}^2: x\ge0, x^2+y^2-2x\ge1\}$

5. 设点列$\{(x_k,y_k)\}\subset\mathbb{R}^2$，判断命题真假：$\{(x_k,y_k)\}$在$\mathbb{R}^2$中有聚点的充要条件是$\{x_ky_k\}$在$\mathbb{R}$中有聚点.

解：假. 且两方面都不对. 不充分：$x_k=k$, $y_k=\frac{1}{k^2}$. $\{x_ky_k\}=\{\frac{1}{k}: k\in\mathbb{Z}_+\}$有聚点0，但$\{(x_k,y_k)\}$无聚点.

不必要：$x_k=\frac{1}{k}$, $y_k=0$. $\{(x_k,y_k)\}$有聚点$(0,0)$，但$\{x_ky_k\}=\{0\}$无聚点

6. 设$E\subset\mathbb{R}^n$. 证明：(1) $\overline{E}=E^\circ\cup\partial E$ (2) $\overline{E}'=E'$ ...

证明：(1) $\overline{E}=E\cup E'$. 一方面，我们证明$\overline{E}\subset E^\circ\cup\partial E$. 显然$E\subset E^\circ\cup\partial E$. 若$E$有聚点$x$，假设$x\in(E^c)^\circ\Rightarrow\exists\delta>0$,

$U(x,\delta)\cap E=\phi\Rightarrow U(x,\delta)$中无$E$的点，矛盾. 从而$x\in E^\circ\cup\partial E\Rightarrow E'\subset E^\circ\cup\partial E\Rightarrow\overline{E}\subset E^\circ\cup\partial E$. 另一方面，我们

证明$E^\circ\cup\partial E\subset\overline{E}$. 显然$E^\circ\subset\overline{E}$. 设$x\in\partial E$. 若$x\in E$，自然有$x\in\overline{E}$. 若$x\notin E$，下面证明$x\in E'$. 若不然，$\exists\delta>0$,

$U(x,\delta)$中没有E中的点（注意$x\notin E$）$\Rightarrow x\notin\partial E$，矛盾. 故$E^\circ\cup\partial E\subset\overline{E}$. 由这两方面，得$\overline{E}=E^\circ\cup\partial E$

(2) 显然$E'\subset\overline{E}'$. 设$x\in\overline{E}'$是$\overline{E}$的聚点. $\forall\delta>0$, $\exists y$, $0<|x-y|<\frac{\delta}{2}$，使得$y\in\overline{E}=E\cup E'$. 若$y\notin E$，则$y\in E'\Rightarrow\exists z$,

$0<|y-z|<|x-y|$，使得$z\in E$. $0=|x-y|-|x-y|<|x-y|-|y-z|\le|x-z|\le|x-y|+|y-z|<2|x-y|<\delta$. 若$y\in E$，则

$y\in U(x,\frac{\delta}{2})\subset U(x,\delta)$，且$y\ne x$. 总之，在$x$的任何邻域中总有$E$中不是$x$的元素$\Rightarrow x\in E'\Rightarrow\overline{E}'\subset E'$

故$E'=\overline{E}'$ #

7. 设$\{A_\lambda\}_{\lambda\in\Lambda}$为$\mathbb{R}^n$中集合族. 证明：(1) 当$\Lambda$有限时，$\overline{\bigcup_{\lambda\in\Lambda}A_\lambda}\subseteq\bigcup_{\lambda\in\Lambda}\overline{A_\lambda}$，$\bigcap_{\lambda\in\Lambda}A_\lambda^\circ\subseteq(\bigcap_{\lambda\in\Lambda}A_\lambda)^\circ$

(2) $\bigcup_{\lambda\in\Lambda}A_\lambda^\circ\subseteq(\bigcup_{\lambda\in\Lambda}A_\lambda)^\circ$，$\overline{\bigcap_{\lambda\in\Lambda}A_\lambda}\subseteq\bigcap_{\lambda\in\Lambda}\overline{A_\lambda}$

证明：(1) $\overline{A_\lambda}$是闭集，Λ有限$\Rightarrow\bigcup_{\lambda\in\Lambda}\overline{A_\lambda}$是闭集$\Rightarrow\bigcup_{\lambda\in\Lambda}\overline{A_\lambda}=\overline{\bigcup_{\lambda\in\Lambda}\overline{A_\lambda}}$. 由$\bigcup_{\lambda\in\Lambda}A_\lambda\subseteq\bigcup_{\lambda\in\Lambda}\overline{A_\lambda}$，有$\overline{\bigcup_{\lambda\in\Lambda}A_\lambda}\subseteq\overline{\bigcup_{\lambda\in\Lambda}\overline{A_\lambda}}=\bigcup_{\lambda\in\Lambda}\overline{A_\lambda}$

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(2) 和(1)类似，只需注意$\bigcup_{\lambda\in\Lambda}A_\lambda^\circ$是开集，$\bigcap_{\lambda\in\Lambda}\overline{A_\lambda}$是闭集

(1)(2)中用到事实：$A\subseteq B\Rightarrow\overline{A}\subseteq\overline{B}$, $A^\circ\subseteq B^\circ$. 前者因为A的聚点必是B的聚点，后者因为A的内点必是B的内点 #

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证明：(1) $\overline{E}=E\cup E'$. 一方面，我们证明 $\overline{E}\subset E^\circ\cup\partial E$. 显然 $E\subset E^\circ\cup\partial E$. 若 E 有聚点 x，假设 $x\in(E^c)^\circ\Rightarrow\exists\delta>0$，$U(x,\delta)\cap E=\varnothing\Rightarrow U(x,\delta)$ 中无 E 的点，矛盾. 从而 $x\in E^\circ\cup\partial E\Rightarrow E'\subset E^\circ\cup\partial E\Rightarrow\overline{E}\subset E^\circ\cup\partial E$. 另一方面，我们证明 $E^\circ\cup\partial E\subset\overline{E}$. 显然 $E^\circ\subset\overline{E}$. 设 $x\in\partial E$. 若 $x\in E$，自然有 $x\in\overline{E}$. 若 $x\notin E$，下面证明 $x\in E'$. 若不然，$\exists\delta>0$，$U(x,\delta)$ 中没有 E 中的点（注意 $x\notin E$）$\Rightarrow x\notin\partial E$，矛盾. 故 $E^\circ\cup\partial E\subset\overline{E}$. 由这两方面，得 $\overline{E}=E^\circ\cup\partial E$

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$0<|y-z|<|x-y|$，使得 $z\in E$. $0=|x-y|-|x-y|<|x-y|-|y-z|\le|x-z|\le|x-y|+|y-z|<2|x-y|<\delta$. 若 $y\in E$，则 $y\in U(x,\frac{\delta}{2})\subset U(x,\delta)$，且 $y\ne x$. 总之，在 x 的任何邻域中总有 E 中不是 x 的元素 $\Rightarrow x\in E'\Rightarrow\overline{E}'\subset E'$

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(2) 和(1)类似，只需注意 $\bigcup_{\lambda\in\Lambda}A_\lambda^\circ$ 是开集，$\bigcap_{\lambda\in\Lambda}\overline{A_\lambda}$ 是闭集

(1)(2)中用到事实：$A\subseteq B\Rightarrow\overline{A}\subseteq\overline{B}$，$A^\circ\subseteq B^\circ$. 前者因为 A 的聚点必是 B 的聚点，后者因为 A 的内点必是 B 的内点 #

8. 设 $E\subset\mathbb{R}^n$. 证明: (1) E'是闭集, (2) ∂E是闭集.

证明: (1) 设 $E'\neq\phi$. $\forall x\in(E')'$, $\forall\delta>0$, $\exists y$, $0<|x-y|<\frac{\delta}{2}$, 使得 $y\in E'$. $\Rightarrow\exists z$, $0<|y-z|<|x-y|$, 使得 $z\in E\Rightarrow 0=|x-y|-|x-y|<|x-y|-|y-z|\le|x-z|\le|x-y|+|y-z|<2|x-y|<\delta\Rightarrow x\in E'\Rightarrow(E')'\subseteq E'$

故 E'是闭集

(2) $\partial E=\partial E^c$, $\partial E\subset\overline{E}$, $\partial E^c\subset\overline{E^c}\Rightarrow\partial E\subset\overline{E}\cap\overline{E^c}$. $\forall x\in\overline{E}\cap\overline{E^c}$, 有 $x\in\overline{E}=E^\circ\cup\partial E$. 若 $x\notin\partial E$, 则 $x\in E^\circ$, 与 $x\in\overline{E^c}$ 矛盾. 从而 $x\in\partial E\Rightarrow\partial E=\overline{E}\cap\overline{E^c}$. 由 $\overline{E},\overline{E^c}$ 都是闭集, 知 ∂E是闭集 #

9. 设 $E\subset\mathbb{R}^2$, $E_1=\{x\in\mathbb{R}:\exists(x,y)\in E\}$, $E_2=\{y\in\mathbb{R}:\exists(x,y)\in E\}$, 判断真假: (1) E为开(闭)集时, E_1,E_2均为开(闭)集 (2) E_1,E_2均为开(闭)集时, E为开(闭)集

解: (1) 对开集结论为真. $\forall x\in E_1$, $\exists y$, $(x,y)\in E\Rightarrow\exists\delta>0$, $U((x,y),\delta)\subset E\Rightarrow U(x,\delta)\subset E_1\Rightarrow E_1$是开集, E_2同理

对闭集结论为假. 如 $E=\{(k,\frac{1}{k}):k\in\mathbb{Z}_+\}$ 是闭集, 但 $E_2=\{\frac{1}{k}:k\in\mathbb{Z}_+\}$ 是开集

(2) 无论开闭皆不对. 开: $E_1=E_2=(-1,1)$, $E=\{(x,y):\frac{1}{2}\le x^2+y^2<1\}$ 非开非闭. 闭: $E_1=E_2=[-1,1]$, $E=\{(x,y):\frac{1}{2}<x^2+y^2\le1\}$ 非开非闭

10. 构造 $\mathbb{R}^2$ 中单位圆盘 $\Delta=\{(x,y):x^2+y^2<1\}$ 内的一个点列 $\{(x_k,y_k)\}$, 使得该集合的聚点集为 $\partial\Delta$

解: 令 $a_k=1-\frac{1}{k}$, $x_k=a_k\cos(\tan\frac{\pi}{2}a_k)$, $y_k=a_k\sin(\tan\frac{\pi}{2}a_k)$, 则 $\{(x_k,y_k)\}\subseteq E=\{(r\cos(\tan\frac{\pi}{2}r), r\sin(\tan\frac{\pi}{2}r)):0\le r<1\}\Rightarrow\{(x_k,y_k)\}$的聚点集含于 $E\cup\partial\Delta$ (第3题(3)). 显然 E中任一点不是$\{(x_k,y_k)\}$的聚点. 故$\{(x_k,y_k)\}$的聚点集为 $\partial\Delta$

11. 设 $E_1,E_2\subset\mathbb{R}^n$ 非空, 定义 E_1,E_2 间的距离 $d(E_1,E_2)=\inf_{x\in E_1,y\in E_2}|x-y|$. 举例说明存在开(闭)集 E_1,E_2, 使得 $E_1\cap E_2=\phi$, 但 $d(E_1,E_2)=0$, 并证明: 若紧集 E_1,E_2 满足 $d(E_1,E_2)=0$, 则 $E_1\cap E_2\neq\phi$

解: 开集: $E_1=\{x\in\mathbb{R}^n:|x|<1\}$, $E_2=\{x\in\mathbb{R}^n:|x|>1\}$. 闭集: $E_1=\{(x,0,\cdots,0):x\in\mathbb{R}\}$, $E_2=\{(x,e^{-x},0,\cdots,0):x\in\mathbb{R}_{\ge0}\}$ ($n\ge2$). $n=1$时 (也可用于$n\ge1$), $E_1=\{k+\frac{1}{k}:k\in\mathbb{Z}_+\}$, $E_2=\{k+\frac{2}{k}:k\in\mathbb{Z}_+\}$. 结论的证明: 由 $d(E_1,E_2)=0$, E_1中有一列$\{x_k\}$, E_2中有一列$\{y_k\}$, 使得 $\lim_{k\to\infty}|x_k-y_k|=0$. 因 E_1,E_2是紧集, $\{x_k\}$有收敛子列 (设收敛至 $x\in E_1$), $\{y_k\}$对应收敛子列 (设收敛至 $y\in E_2$), 则有 $x=y$. 故 $E_1\cap E_2\neq\phi$ #

12. 设 $F\subset\mathbb{R}^n$是紧集, $E\subset\mathbb{R}^n$是开集, $F\subset E$. 证明: 存在开集 $O\subset\mathbb{R}^n$, 使得 $F\subset O\subset\overline{O}\subset E$

证: 由E是开集, $\forall x\in E$, $\exists\delta_x>0$, 使得 $U(x,\delta_x)\subset E$. 由 $\bigcup_{x\in E}U(x,\frac{\delta_x}{2})$ 是紧集F的开覆盖, 必存在有限覆盖 $\bigcup_{j=1}^{J}U(x_j,\frac{\delta_{x_j}}{2})$, 现取 $O=\bigcup_{j=1}^{J}U(x_j,\frac{\delta_{x_j}}{2})$, 则 $F\subset O$, $\overline{O}=\overline{\bigcup_{j=1}^{J}U(x_j,\frac{\delta_{x_j}}{2})}\subset\bigcup_{j=1}^{J}\overline{U(x_j,\frac{\delta_{x_j}}{2})}\subset\bigcup_{j=1}^{J}U(x_j,\delta_{x_j})\subset E$. 故有 $F\subset O\subset\overline{O}\subset E$, $O=\bigcup_{j=1}^{J}U(x_j,\frac{\delta_{x_j}}{2})$是开集 #

13. 求定义域: (1) $f(x,y,z)=\ln(y-x^2-z^2)$. (2) $f(x,y,z)=\sqrt{x^2+y^2-z^2}$. (3) $f(x,y,z)=\frac{\ln(x^2+y^2-z)}{\sqrt{z}}$

解: (1) $\{(x,y,z)\in\mathbb{R}^3:x^2+z^2<y\}$ (2) $\{(x,y,z)\in\mathbb{R}^3:x^2+y^2\ge z^2\}$ (3) $\{(x,y,z)\in\mathbb{R}^3:z>0, x^2+y^2>z\}$

14. 确定极限是否存在, 若存在则求之: (1) $\lim_{E\ni(x,y)\to(0,0)}\frac{\sin(x^3+y^3)}{x^2+y^2}$, $E=\{(x,y):y>x^2\}$ (2) $\lim_{(x,y)\to(0,0)}x\ln(x^2+y^2)$

(3) $\lim_{|(x,y)|\to+\infty}(x^2+y^2)e^{-(|x|+|y|)}$ (4) $\lim_{|(x,y)|\to+\infty}\left(1+\frac{1}{|x|+|y|}\right)^{x^2/(|x|+|y|)}$ (5) $\lim_{(x,y,z)\to(0,0,0)}\left(\frac{xyz}{x^2+y^2+z^2}\right)^{x+y}$

(6) $\lim_{E\ni(x,y,z)\to(0,0,0)}x^{yz}$, $E=\{(x,y,z):x,y,z>0\}$. (7) $\lim_{(x,y,z)\to(0,1,0)}\frac{\sin(xyz)}{x^2+z^2}$ (8) $\lim_{(x,y,z)\to(0,0,0)}\frac{\sin(xyz)}{\sqrt{x^2+y^2+z^2}}$

(9) $\lim_{x=(x_1,x_2,\cdots,x_n)\to0}\frac{(\sum_{i=1}^{n}x_i)^2}{|x|^2}$

解: (1) $\left|\frac{\sin(x^3+y^3)}{x^2+y^2}\right|\le\frac{|x|^3+|y|^3}{x^2+|y|}\le(|x|+|y|)^2\to0\Rightarrow$原式$=0$

(2) 当$|x|,|y|$充分小时, $x^2+y^2<1\Rightarrow|x\ln(x^2+y^2)|\le|x\ln x^2|\to0\Rightarrow$原式$=0$

(3) $0\le(x^2+y^2)e^{-(|x|+|y|)}\le(|x|+|y|)^2e^{-(|x|+|y|)}\to0\Rightarrow$原式$=0$

(4) 取 $y_k=0$, $x_k=k$, $\lim_{k\to\infty}\left(1+\frac{1}{|x_k|+|y_k|}\right)^{\frac{x_k^2}{|x_k|+|y_k|}}=\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k=e$. 取 $x_k'=y_k'=k$, $\lim_{k\to\infty}\left(1+\frac{1}{|x_k'|+|y_k'|}\right)^{\frac{x_k'^2}{|x_k'|+|y_k'|}}=\lim_{k\to\infty}\left(1+\frac{1}{2k}\right)^{\frac{k}{2}}=e^{1/4}$. 故原极限不存在

8. 设 $E\subset\mathbb{R}^n$. 证明: (1) E' 是闭集. (2) ∂E 是闭集.

证明: (1) 设 $E'\neq\phi$. $\forall x\in(E')'$, $\forall\delta>0$, $\exists y$, $0<|x-y|<\frac{\delta}{2}$, 使得 $y\in E'$. $\Rightarrow\exists z$, $0<|y-z|<|x-y|$, 使得 $z\in E\Rightarrow 0=|x-y|-|x-y|<|x-y|-|y-z|\le|x-z|\le|x-y|+|y-z|<2|x-y|<\delta\Rightarrow x\in E'\Rightarrow(E')'\subseteq E'$

故 E' 是闭集

(2) $\partial E=\partial E^c$, $\partial E\subset\overline{E}$, $\partial E^c\subset\overline{E^c}\Rightarrow\partial E\subset\overline{E}\cap\overline{E^c}$. $\forall x\in\overline{E}\cap\overline{E^c}$, 有 $x\in\overline{E}=E^\circ\cup\partial E$. 若 $x\notin\partial E$, 则 $x\in E^\circ$, 与 $x\in\overline{E^c}$ 矛盾. 从而 $x\in\partial E\Rightarrow\partial E=\overline{E}\cap\overline{E^c}$. 由 $\overline{E}$, $\overline{E^c}$ 都是闭集, 知 ∂E 是闭集 #

9. 设 $E\subset\mathbb{R}^2$, $E_1=\{x\in\mathbb{R}:\exists(x,y)\in E\}$, $E_2=\{y\in\mathbb{R}:\exists(x,y)\in E\}$, 判断真假: (1) E 为开(闭)集时, E_1、E_2 均为开(闭)集 (2) E_1、E_2 均为开(闭)集时, E 为开(闭)集

解: (1) 对开集结论为真. $\forall x\in E_1$, $\exists y$, $(x,y)\in E\Rightarrow\exists\delta>0$, $U((x,y),\delta)\subset E\Rightarrow U(x,\delta)\subset E_1\Rightarrow E_1$ 是开集, E_2 同理

对闭集结论为假. 如 $E=\{(k,\frac{1}{k}):k\in\mathbb{Z}_+\}$ 是闭集, 但 $E_2=\{\frac{1}{k}:k\in\mathbb{Z}_+\}$ 是开集

(2) 无论开闭皆不对. 开: $E_1=E_2=(-1,1)$, $E=\{(x,y):\frac{1}{2}\le x^2+y^2<1\}$ 非开非闭. 闭: $E_1=E_2=[-1,1]$, $E=\{(x,y):\frac{1}{2}<x^2+y^2\le1\}$ 非开非闭

10. 构造 $\mathbb{R}^2$ 中单位圆盘 $\Delta=\{(x,y):x^2+y^2<1\}$ 内的一个点列 $\{(x_k,y_k)\}$, 使得该集合的聚点集为 $\partial\Delta$

解: 令 $a_k=1-\frac{1}{k}$, $x_k=a_k\cos(\tan\frac{\pi}{2}a_k)$, $y_k=a_k\sin(\tan\frac{\pi}{2}a_k)$, 则 $\{(x_k,y_k)\}\subseteq E=\{(r\cos(\tan\frac{\pi}{2}r), r\sin(\tan\frac{\pi}{2}r)):0\le r<1\}\Rightarrow\{(x_k,y_k)\}$ 的聚点集含于 $E\cup\partial\Delta$ (第3题(3)). 显然 E 中任一点不是 $\{(x_k,y_k)\}$ 的聚点. 故 $\{(x_k,y_k)\}$ 的聚点集为 $\partial\Delta$

11. 设 $E_1,E_2\subset\mathbb{R}^n$ 非空, 定义 E_1,E_2 间的距离 $d(E_1,E_2)=\inf\limits_{x\in E_1,y\in E_2}|x-y|$. 举例说明存在开(闭)集 E_1,E_2, 使得 $E_1\cap E_2=\phi$, 但 $d(E_1,E_2)=0$, 并证明: 若紧集 E_1,E_2 满足 $d(E_1,E_2)=0$, 则 $E_1\cap E_2\neq\phi$

解: 开集: $E_1=\{x\in\mathbb{R}^n:|x|<1\}$, $E_2=\{x\in\mathbb{R}^n:|x|>1\}$. 闭集: $E_1=\{(x,0,\cdots,0):x\in\mathbb{R}\}$, $E_2=\{(x,e^{-x},0,\cdots,0):x\in\mathbb{R}_{\ge0}\}$ $(n\ge2)$. $n=1$ 时, (也可用于 $n\ge1$) $E_1=\{k+\frac{1}{k}:k\in\mathbb{Z}_+\}$, $E_2=\{k+\frac{2}{k}:k\in\mathbb{Z}_+\}$. 结论的证明: 由 $d(E_1,E_2)=0$, E_1 中有一列 $\{x_k\}$, E_2 中有一列 $\{y_k\}$, 使得 $\lim\limits_{k\to\infty}|x_k-y_k|=0$. 因 E_1,E_2 是紧集, $\{x_k\}$ 有收敛子列(设收敛至 $x\in E_1$), $\{y_k\}$ 有收敛子列(设收敛至 $y\in E_2$), 则有 $x=y$. 故 $E_1\cap E_2\neq\phi$ #

12. 设 $F\subset\mathbb{R}^n$ 是紧集, $E\subset\mathbb{R}^n$ 是开集, $F\subset E$. 证明: 存在开集 $O\subset\mathbb{R}^n$, 使得 $F\subset O\subset\overline{O}\subset E$

证: 由 E 是开集, $\forall x\in E$, $\exists\delta_x>0$, 使得 $U(x,\delta_x)\subset E$. 由 $\bigcup\limits_{x\in E}U(x,\frac{\delta_x}{2})$ 是紧集 F 的开覆盖, 必存在有限覆盖 $\bigcup\limits_{j=1}^{J}U(x_j,\frac{\delta_{x_j}}{2})$. 现取 $O=\bigcup\limits_{j=1}^{J}U(x_j,\frac{\delta_{x_j}}{2})$, 则 $F\subset O$, $\overline{O}=\overline{\bigcup\limits_{j=1}^{J}U(x_j,\frac{\delta_{x_j}}{2})}\subset\bigcup\limits_{j=1}^{J}\overline{U(x_j,\frac{\delta_{x_j}}{2})}\subset\bigcup\limits_{j=1}^{J}U(x_j,\delta_{x_j})\subset E$. 故有 $F\subset O\subset\overline{O}\subset E$, $O=\bigcup\limits_{j=1}^{J}U(x_j,\frac{\delta_{x_j}}{2})$ 是开集 #

13. 求定义域: (1) $f(x,y,z)=\ln(y-x^2-z^2)$. (2) $f(x,y,z)=\sqrt{x^2+y^2-z^2}$. (3) $f(x,y,z)=\frac{\ln(x^2+y^2-z)}{\sqrt{z}}$

解: (1) $\{(x,y,z)\in\mathbb{R}^3:x^2+z^2<y\}$ (2) $\{(x,y,z)\in\mathbb{R}^3:x^2+y^2\ge z^2\}$ (3) $\{(x,y,z)\in\mathbb{R}^3:z>0, x^2+y^2>z\}$

14. 确定极限是否存在, 若存在则求之: (1) $\lim\limits_{E\ni(x,y)\to(0,0)}\frac{\sin(x^3+y^3)}{x^2+y^2}$, $E=\{(x,y):y>x^2\}$ (2) $\lim\limits_{(x,y)\to(0,0)}x\ln(x^2+y^2)$

(3) $\lim\limits_{|(x,y)|\to+\infty}(x^2+y^2)e^{-(|x|+|y|)}$ (4) $\lim\limits_{|(x,y)|\to+\infty}\left(1+\frac{1}{|x|+|y|}\right)^{x^2/(|x|+|y|)}$ (5) $\lim\limits_{(x,y,z)\to(0,0,0)}\left(\frac{xyz}{x^2+y^2+z^2}\right)^{x+y}$

(6) $\lim\limits_{E\ni(x,y,z)\to(0,0,0)}x^{yz}$, $E=\{(x,y,z):x,y,z>0\}$. (7) $\lim\limits_{(x,y,z)\to(0,1,0)}\frac{\sin(xyz)}{x^2+z^2}$ (8) $\lim\limits_{(x,y,z)\to(0,0,0)}\frac{\sin(xyz)}{\sqrt{x^2+y^2+z^2}}$

(9) $\lim\limits_{x=(x_1,x_2,\cdots,x_n)\to0}\frac{(\sum_{i=1}^{n}x_i)^2}{|x|^2}$

解: (1) $\left|\frac{\sin(x^3+y^3)}{x^2+y^2}\right|\le\frac{|x|^3+|y|^3}{x^2+|y|^2}\le(|x|+|y|)^2\to0\Rightarrow$ 原式 $=0$

(2) 当 $|x|,|y|$ 充分小时, $x^2+y^2<1\Rightarrow|x\ln(x^2+y^2)|\le|x\ln x^2|\to0\Rightarrow$ 原式 $=0$

(3) $0\le(x^2+y^2)e^{-(|x|+|y|)}\le(|x|+|y|)^2e^{-(|x|+|y|)}\to0\Rightarrow$ 原式 $=0$

(4) 取 $y_k=0$, $x_k=k$, $\lim\limits_{k\to\infty}\left(1+\frac{1}{|x_k|+|y_k|}\right)^{\frac{x_k^2}{|x_k|+|y_k|}}=\lim\limits_{k\to\infty}(1+\frac{1}{k})^k=e$. 取 $x_k'=y_k'=k$, $\lim\limits_{k\to\infty}\left(1+\frac{1}{|x_k'|+|y_k'|}\right)^{\frac{x_k'^2}{|x_k'|+|y_k'|}}=\lim\limits_{k\to\infty}(1+\frac{1}{2k})^{\frac{k}{2}}=e^{1/4}$. 故原极限不存在

(5) 取$x=y=z$，$\left(\frac{xyz}{x^2+y^2+z^2}\right)^{x+y}=\left(\frac{x}{3}\right)^{2x}\to 1$. 取$x_k=y_k=\frac{1}{k}$，$z_k\equiv 0$，$\left(\frac{x_k y_k z_k}{x_k^2+y_k^2+z_k^2}\right)^{x_k+y_k}=0^{\frac{2}{k}}\to 0$

故原极限不存在

(6) 取$x=y=z$，则$x^{yz}=x^{z^2}=e^{z^2\ln x}\to e^0=1$. 取$y=z=\frac{1}{\sqrt{-\ln x}}$ $(0<x<1)$，则$x^{yz}=x^{-\frac{1}{\ln x}}=e^{-1}$

故原极限不存在

(7) 取$x=z$，$y=1$，则$\frac{\sin(xyz)}{x^2+z^2}=\frac{\sin x^2}{2x^2}\to\frac{1}{2}$. 取$z=x^2$，$y=1$，则$\frac{\sin(xyz)}{x^2+z^2}=\frac{\sin x^3}{x^2+x^4}\sim\frac{x^3}{x^2+x^4}\to 0$

故原极限不存在

(8) $\left|\frac{\sin(xyz)}{\sqrt{x^2+y^2+z^2}}\right|\le\frac{|xyz|}{\max\{|x|,|y|,|z|\}}\le\max\{|xy|,|yz|,|zx|\}\to 0$. 故原式$=0$

(9) 若$n=1$，则极限为1. 若$n\ge 2$，取$x=(t,0,\cdots,0)$，$\frac{(\sum_{i=1}^n x_i)^2}{|x|^2}=1$. 取$x=(t,t,0,\cdots,0)$，$\frac{(\sum_{i=1}^n x_i)^2}{|x|^2}=2$，极限不存在

15. 给出三元函数$f(x,y,z)$累次极限$\lim\limits_{x\to x_0}\lim\limits_{y\to y_0}\lim\limits_{z\to z_0}f(x,y,z)$的定义，并构造一个$f(x,y,z)$，满足$\lim\limits_{(x,y,z)\to(0,0,0)}f(x,y,z)$存在，但$\lim\limits_{x\to 0}\lim\limits_{y\to 0}\lim\limits_{z\to 0}f(x,y,z)$不存在

解：设$f(x,y,z)$在$E\subset\mathbb{R}^3$有定义，$N_0((x_0,y_0,z_0),\delta_0)\subset E$. 若在$N_0((x_0,y_0,z_0),\delta_0)$内，对每个$(x,y)\ne(x_0,y_0)$，都有$\lim\limits_{z\to z_0}f(x,y,z)=\varphi(x,y)$存在，对每个$x\ne x_0$，都有$\lim\limits_{y\to y_0}\varphi(x,y)=\psi(x)$存在，且$\lim\limits_{x\to x_0}\psi(x)=A$，则称$A$为$f(x,y,z)$在$(x,y,z)$趋于$(x_0,y_0,z_0)$时先$z$后$y$后$x$的累次极限.

例子：$f(x,y,z)=\begin{cases}(x+y+z)\sin\frac{1}{x}\sin\frac{1}{y}\sin\frac{1}{z}, & xyz\ne 0\\ 0, & xyz=0\end{cases}$，则$|f(x,y,z)|\le|x|+|y|+|z|\to 0$. $\lim\limits_{(x,y,z)\to(0,0,0)}f(x,y,z)=0$.

对$x,y\ne 0$，$\lim\limits_{z\to 0}f(x,y,z)$不存在，因此$\lim\limits_{x\to 0}\lim\limits_{y\to 0}\lim\limits_{z\to 0}f(x,y,z)$不存在

16. 设$y=f(x)$在$U_0(0,\delta_0)\subset\mathbb{R}$中有定义，满足$\forall x\in U_0(0,\delta_0)$，$f(x)\ne 0$，且$\lim\limits_{x\to 0}f(x)=0$. 记$E=\{(x,y):xy\ne 0\}$.

证明：(1) $\lim\limits_{E\ni(x,y)\to(0,0)}\frac{f(x)f(y)}{f^2(x)+f^2(y)}$不存在 (2) $\lim\limits_{E\ni(x,y)\to(0,0)}\frac{yf^2(x)}{f^4(x)+y^2}$不存在

证明：(1) 任取$a_1\in U_0(0,\delta_0)$. 由$f(x)\to 0$ $(x\to 0)$，$\exists a_2\in U_0(0,\delta_0)$. 使得$|a_2|\le\frac{1}{2}|a_1|$，且$|f(a_2)|\le\frac{1}{2}|f(a_1)|$. 类似归纳构造$\{a_k\}$，满足$|a_{k+1}|\le\frac{1}{2}|a_k|$，$|f(a_{k+1})|\le\frac{1}{2^k}|f(a_k)|$，则$\lim\limits_{k\to\infty}a_k=0$，$\lim\limits_{k\to\infty}\frac{f(a_{k+1})}{f(a_k)}=0$，于是

$\frac{f(a_k)f(a_k)}{f^2(a_k)+f^2(a_k)}=\frac{1}{2}$，$\frac{f(a_{k+1})f(a_k)}{f^2(a_{k+1})+f^2(a_k)}=\frac{\frac{f(a_{k+1})}{f(a_k)}}{1+\left(\frac{f(a_{k+1})}{f(a_k)}\right)^2}\to 0$. 故$\lim\limits_{E\ni(x,y)\to(0,0)}\frac{f(x)f(y)}{f^2(x)+f^2(y)}$不存在

(2) 取$y=f(x)$，则$\frac{yf^2(x)}{f^4(x)+y^2}=\frac{f(x)}{1+f^2(x)}\to 0$. 取$y=f^2(x)$，则$\frac{yf^2(x)}{f^4(x)+y^2}=\frac{1}{2}$. 故$\lim\limits_{E\ni(x,y)\to(0,0)}\frac{yf^2(x)}{f^4(x)+y^2}$不存在 #

17. 构造二元函数$f(x,y)$ $((x,y)\in\mathbb{R}^2)$，使得对$1\le k\le K$，$\lim\limits_{x\to 0}f(x,x^k)=0$，但$\lim\limits_{(x,y)\to(0,0)}f(x,y)$不存在.

解：$f(x,y)=\frac{x^{K+1}}{x^{K+1}+y}$. $\forall 1\le k\le K$，$f(x,x^k)=\frac{x^{K+1}}{x^{K+1}+x^k}=\frac{1}{1+\frac{1}{x^{K+1-k}}}\to 0$. 而$f(x,0)=1$，$f(0,y)=0$，$\lim\limits_{(x,y)\to(0,0)}f(x,y)$不存在

18. 设$f(x,y)$在$\mathbb{R}^2$内除直线$x=a$与$y=b$外处处有定义，且：(a) $\lim\limits_{y\to b}f(x,y)=g(x)$存在 (b) $\lim\limits_{x\to a}f(x,y)=h(y)$

一致存在，即$\forall\varepsilon>0$，$\exists\delta>0$，对$\forall(x,y)\in\{(x,y):0<|x-a|<\delta\}$，有$|f(x,y)-h(y)|<\varepsilon$. 证明：$\exists C\in\mathbb{R}$，使得

(1) $\lim\limits_{x\to a}\lim\limits_{y\to b}f(x,y)=\lim\limits_{x\to a}g(x)=C$ (2) $\lim\limits_{y\to b}\lim\limits_{x\to a}f(x,y)=\lim\limits_{y\to b}h(y)=C$ (3) $\lim\limits_{E\ni(x,y)\to(a,b)}f(x,y)=C$，$E=\mathbb{R}^2\setminus\{(x,y):x=a\text{或}y=b\}$

证明：$\forall\varepsilon>0$，$\exists\delta_1>0$，当$0<|x-a|<\delta_1$时，$|f(x,y)-h(y)|<\frac{\varepsilon}{3}$对$\forall y\ne b$成立$^{(*)}$. 取定$x$，并任取$y',y''\ne b$. $\exists\delta>0$，

当$y',y''\in U_0(b,\delta)$时，$|f(x,y')-f(x,y'')|<\frac{\varepsilon}{3}$. 此时，$|h(y')-h(y'')|\le|h(y')-f(x,y')|+|f(x,y')-f(x,y'')|+|f(x,y'')-h(y'')|$

$<\varepsilon\Rightarrow\lim\limits_{y\to b}h(y)$存在，设为$C$. 下面证明$C$符合(1)(3). 在(*)中令$y\to b$，则$|g(x)-C|\le\frac{\varepsilon}{3}<\varepsilon$. 故$\lim\limits_{x\to a}g(x)=C$

对上述ε，$\exists\delta_2>0$，当$0<|y-b|<\delta_2$时，$|h(y)-C|<\frac{\varepsilon}{3}$. 取$\delta_0=\min\{\delta_1,\delta_2\}$，当$(x,y)\in E$且$\sqrt{(x-a)^2+(y-b)^2}<\delta$时，有

$0<|x-a|<\delta_1$，$0<|y-b|<\delta_2$，于是$|f(x,y)-C|\le|f(x,y)-h(y)|+|h(y)-C|<\frac{2\varepsilon}{3}<\varepsilon$. 故$\lim\limits_{E\ni(x,y)\to(a,b)}f(x,y)=C$ #

(5) 取 $x=y=z$，$\left(\frac{xyz}{x^2+y^2+z^2}\right)^{x+y}=\left(\frac{x}{3}\right)^{2x}\to 1$. 取 $x_k=y_k=\frac{1}{k}$，$z_k\equiv 0$，$\left(\frac{x_ky_kz_k}{x_k^2+y_k^2+z_k^2}\right)^{x_k+y_k}=0^{\frac{2}{k}}\to 0$
故原极限不存在

(6) 取 $x=y=z$，则 $x^{yz}=x^{z^2}=e^{z^2\ln x}\to e^0=1$. 取 $y=z=\frac{1}{\sqrt{-\ln x}}$ $(0<x<1)$，则 $x^{yz}=x^{-\frac{1}{\ln x}}=e^{-1}$
故原极限不存在

(7) 取 $x=z$，$y=1$，则 $\frac{\sin(xyz)}{x^2+z^2}=\frac{\sin x^2}{2x^2}\to\frac{1}{2}$. 取 $z=x^2$，$y=1$，则 $\frac{\sin(xyz)}{x^2+z^2}=\frac{\sin x^3}{x^2+x^4}\sim\frac{x^3}{x^2+x^4}\to 0$
故原极限不存在

(8) $\left|\frac{\sin(xyz)}{\sqrt{x^2+y^2+z^2}}\right|\le\frac{|xyz|}{\max\{|x|,|y|,|z|\}}\le\max\{|xy|,|yz|,|zx|\}\to 0$. 故原式 $=0$

(9) 若 $n=1$，则极限为1. 若 $n\ge 2$，取 $x=(t,0,\cdots,0)$，$\frac{(\sum_{i=1}^n x_i)^2}{|x|^2}=1$. 取 $x=(t,t,0,\cdots,0)$，$\frac{(\sum_{i=1}^n x_i)^2}{|x|^2}=2$，极限不存在

15. 给出三元函数 $f(x,y,z)$ 累次极限 $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0}\lim\limits_{z\to z_0}f(x,y,z)$ 的定义，并构造一个 $f(x,y,z)$，满足 $\lim\limits_{(x,y,z)\to(0,0,0)}f(x,y,z)$ 存在，但 $\lim\limits_{x\to 0}\lim\limits_{y\to 0}\lim\limits_{z\to 0}f(x,y,z)$ 不存在

解：设 $f(x,y,z)$ 在 $E\subset\mathbb{R}^3$ 有定义，$N_0((x_0,y_0,z_0),\delta_0)\subset E$. 若在 $N_0((x_0,y_0,z_0),\delta_0)$ 内，对每个 $(x,y)\ne(x_0,y_0)$，都有 $\lim\limits_{z\to z_0}f(x,y,z)=\varphi(x,y)$ 存在，对每个 $x\ne x_0$，都有 $\lim\limits_{y\to y_0}\varphi(x,y)=\psi(x)$ 存在，且 $\lim\limits_{x\to x_0}\psi(x)=A$，则称 A 为 $f(x,y,z)$ 在 (x,y,z) 趋于 (x_0,y_0,z_0) 时先 z 再 y 后 x 的累次极限.

例子：$f(x,y,z)=\begin{cases}(x+y+z)\sin\frac{1}{x}\sin\frac{1}{y}\sin\frac{1}{z}, & xyz\ne 0\\ 0, & xyz=0\end{cases}$，则 $|f(x,y,z)|\le|x|+|y|+|z|\to 0$. $\lim\limits_{(x,y,z)\to(0,0,0)}f(x,y,z)=0$.

对 $x,y\ne 0$，$\lim\limits_{z\to 0}f(x,y,z)$ 不存在，因此 $\lim\limits_{x\to 0}\lim\limits_{y\to 0}\lim\limits_{z\to 0}f(x,y,z)$ 不存在

16. 设 $y=f(x)$ 在 $U_0(0,\delta_0)\subset\mathbb{R}$ 中有定义，满足 $\forall x\in U_0(0,\delta_0)$，$f(x)\ne 0$，且 $\lim\limits_{x\to 0}f(x)=0$. 记 $E=\{(x,y):xy\ne 0\}$.
证明：(1) $\lim\limits_{E\ni(x,y)\to(0,0)}\frac{f(x)f(y)}{f^2(x)+f^2(y)}$ 不存在 (2) $\lim\limits_{E\ni(x,y)\to(0,0)}\frac{yf^2(x)}{f^4(x)+y^2}$ 不存在

证明：(1) 任取 $a_1\in U_0(0,\delta_0)$. 由 $f(x)\to 0$ $(x\to 0)$，$\exists a_2\in U_0(0,\delta_0)$，使得 $|a_2|\le\frac{1}{2}|a_1|$，且 $|f(a_2)|\le\frac{1}{2}|f(a_1)|$. 类似归纳构造 $\{a_k\}$，满足 $|a_{k+1}|\le\frac{1}{2}|a_k|$，$|f(a_{k+1})|\le\frac{1}{2^k}|f(a_k)|$，则 $\lim\limits_{k\to\infty}a_k=0$，$\lim\limits_{k\to\infty}\frac{f(a_{k+1})}{f(a_k)}=0$，于是
$\frac{f(a_k)f(a_k)}{f^2(a_k)+f^2(a_k)}=\frac{1}{2}$，$\frac{f(a_{k+1})f(a_k)}{f^2(a_{k+1})+f^2(a_k)}=\frac{\frac{f(a_{k+1})}{f(a_k)}}{1+\left(\frac{f(a_{k+1})}{f(a_k)}\right)^2}\to 0$. 故 $\lim\limits_{E\ni(x,y)\to(0,0)}\frac{f(x)f(y)}{f^2(x)+f^2(y)}$ 不存在

(2) 取 $y=f(x)$，则 $\frac{yf^2(x)}{f^4(x)+y^2}=\frac{f(x)}{1+f^2(x)}\to 0$. 取 $y=f^2(x)$，则 $\frac{yf^2(x)}{f^4(x)+y^2}=\frac{1}{2}$. 故 $\lim\limits_{E\ni(x,y)\to(0,0)}\frac{yf^2(x)}{f^4(x)+y^2}$ 不存在 #

17. 构造二元函数 $f(x,y)$ $((x,y)\in\mathbb{R}^2)$，使得对 $1\le k\le K$，$\lim\limits_{x\to 0}f(x,x^k)=0$，但 $\lim\limits_{(x,y)\to(0,0)}f(x,y)$ 不存在.
解：$f(x,y)=\frac{x^{K+1}}{x^{K+1}+y}$. $\forall 1\le k\le K$，$f(x,x^k)=\frac{x^{K+1}}{x^{K+1}+x^k}=\frac{1}{1+\frac{1}{x^{K+1-k}}}\to 0$. 而 $f(x,0)=1$，$f(0,y)=0$，$\lim\limits_{(x,y)\to(0,0)}f(x,y)$ 不存在

18. 设 $f(x,y)$ 在 $\mathbb{R}^2$ 内除直线 $x=a$ 与 $y=b$ 外处处有定义，且：(a) $\lim\limits_{y\to b}f(x,y)=g(x)$ 存在 (b) $\lim\limits_{x\to a}f(x,y)=h(y)$ 一致存在，即 $\forall\varepsilon>0$，$\exists\delta>0$，对 $\forall(x,y)\in\{(x,y):0<|x-a|<\delta\}$，有 $|f(x,y)-h(y)|<\varepsilon$. 证明：$\exists C\in\mathbb{R}$，使得
(1) $\lim\limits_{x\to a}\lim\limits_{y\to b}f(x,y)=\lim\limits_{x\to a}g(x)=C$ (2) $\lim\limits_{y\to b}\lim\limits_{x\to a}f(x,y)=\lim\limits_{y\to b}h(y)=C$ (3) $\lim\limits_{E\ni(x,y)\to(a,b)}f(x,y)=C$，$E=\mathbb{R}^2\setminus\{(x,y):x=a\text{ 或 }y=b\}$

证明：$\forall\varepsilon>0$，$\exists\delta_1>0$，当 $0<|x-a|<\delta_1$ 时，$|f(x,y)-h(y)|<\frac{\varepsilon}{3}$ 对 $\forall y\ne b$ 成立 $(*)$. 取定 x，并任取 $y',y''\ne b$. $\exists\delta>0$，当 $y',y''\in U_0(b,\delta)$ 时，$|f(x,y')-f(x,y'')|<\frac{\varepsilon}{3}$. 此时，$|h(y')-h(y'')|\le|h(y')-f(x,y')|+|f(x,y')-f(x,y'')|+|f(x,y'')-h(y'')|<\varepsilon\Rightarrow\lim\limits_{y\to b}h(y)$ 存在，设为 C. 下面证明 C 符合(1)(3). 在 $(*)$ 中令 $y\to b$，则 $|g(x)-C|\le\frac{\varepsilon}{3}<\varepsilon$. 故 $\lim\limits_{x\to a}g(x)=C$
对上述 ε，$\exists\delta_2>0$，当 $0<|y-b|<\delta_2$ 时，$|h(y)-C|<\frac{\varepsilon}{3}$. 取 $\delta_0=\min\{\delta_1,\delta_2\}$，当 $(x,y)\in E$ 且 $\sqrt{(x-a)^2+(y-b)^2}<\delta$ 时，有 $0<|x-a|<\delta_1$，$0<|y-b|<\delta_2$，于是 $|f(x,y)-C|\le|f(x,y)-h(y)|+|h(y)-C|<\frac{2\varepsilon}{3}<\varepsilon$. 故 $\lim\limits_{E\ni(x,y)\to(a,b)}f(x,y)=C$ #

17. 设 $f(x)$ 在 $[0,1]$ 上连续, $g(y)$ 在 $[0,1]$ 上除 $y_0=\frac{1}{2}$ 外连续, y_0 是 $g(y)$ 的第一类间断点. 求 $F(x,y)=f(x)g(y)$ 在 $[0,1]^2$ 上的全体间断点

解: $F(x,y)$ 在 $\{(x,y)\in[0,1]^2: y\neq\frac{1}{2}\}$ 上连续, 只需考虑 $(x,\frac{1}{2})$ $(0\le x\le 1)$ 的点. 若 $f(x_0)\neq 0$, 则 $\lim_{y\to\frac{1}{2}}F(x_0,y)=f(x_0)\lim_{y\to\frac{1}{2}}g(y)$ 或者不存在, 或者存在但不等于 $f(x_0)g(\frac{1}{2})=F(x_0,\frac{1}{2})$. 于是 $F(x,y)$ 在 $(x_0,\frac{1}{2})$ 不连续.

若 $f(x_0)=0$, $\forall\varepsilon>0$, $\exists\delta>0$, 当 $|x-x_0|<\delta$ 且 $0\le x\le 1$ 时, $|f(x)|<\varepsilon$. 因 $g(y)$ 在 $[0,1]\setminus\{\frac{1}{2}\}$ 上连续, $\frac{1}{2}$ 是第一类间断点, $g(y)$ 在 $[0,1]$ 上有界: $|g(y)|\le M$ $(0\le y\le 1)$. 当 $0<\sqrt{(x-x_0)^2+(y-\frac{1}{2})^2}<\delta$ 且 $(x,y)\in[0,1]^2$ 时, $|F(x,y)|=|f(x)||g(y)|\le M\varepsilon$

$\Longrightarrow \lim_{(x,y)\to(x_0,\frac{1}{2})}F(x,y)=0=F(x_0,\frac{1}{2})$, 即 $F(x,y)$ 在 $(x_0,\frac{1}{2})$ 连续. 故 $F(x,y)$ 在 $[0,1]^2$ 上的全体间断点为 $\{(x,\frac{1}{2}): 0\le x\le 1, f(x)\neq 0\}$

20. 设 $f(x,y)$ 在 $D=[0,1]^2$ 上有定义, 对固定的 x, $f(x,y)$ 是 y 的连续函数, 对固定的 y, $f(x,y)$ 是 x 的连续函数. 证明: 若 $f(x,y)$ 满足下面二者之一: (1) 对固定的 x, $f(x,y)$ 关于 y 单调增 (2) $\forall\varepsilon>0$, $\exists\delta>0$, 当 $y_1,y_2\in[0,1]$ 且 $|y_1-y_2|<\delta$ 时, $|f(x,y_1)-f(x,y_2)|<\varepsilon$ 对 $\forall x\in[0,1]$ 成立. 则 $f(x,y)$ 在 D 内连续

证明: (1) 任取 $(x_0,y_0)\subset(0,1)^2$ (若在边界可类似考虑). $\forall\varepsilon>0$, $\exists\delta_1>0$, 当 $y\in[y_0-\delta_1,y_0+\delta_1]\subset(0,1)$ 时, $|f(x_0,y)-f(x_0,y_0)|<\frac{\varepsilon}{6}\Longrightarrow 0\le f(x_0,y_0+\delta_1)-f(x_0,y_0-\delta_1)<\frac{\varepsilon}{3}$. $\exists\delta_2>0$, 当 $|x-x_0|<\delta_2$ 时, $|f(x,y_0+\delta_1)-f(x_0,y_0+\delta_1)|<\frac{\varepsilon}{6}$, $|f(x,y_0-\delta_1)-f(x_0,y_0-\delta_1)|<\frac{\varepsilon}{6}$. 取 $\delta=\min\{\delta_1,\delta_2\}$, 当 $|x-x_0|,|y-y_0|<\delta$ 时, $|f(x,y)-f(x_0,y_0)|\le|f(x,y)-f(x,y_0+\delta_1)|+|f(x,y_0+\delta_1)-f(x_0,y_0+\delta_1)|+|f(x_0,y_0+\delta_1)-f(x_0,y_0)|\le f(x,y_0+\delta_1)-f(x,y_0-\delta_1)+\frac{\varepsilon}{6}+\frac{\varepsilon}{6}\le|f(x,y_0+\delta_1)-f(x_0,y_0+\delta_1)|+|f(x_0,y_0+\delta_1)-f(x_0,y_0-\delta_1)|+|f(x_0,y_0-\delta_1)-f(x,y_0-\delta_1)|+\frac{\varepsilon}{3}<\varepsilon$. 故 $f(x,y)$ 在 D 内连续

(2) 任取 $(x_0,y_0)\subset(0,1)^2$. $\forall\varepsilon>0$, $\exists\delta_1>0$, 当 $|y-y_0|<\delta_1$ 时, $|f(x,y)-f(x,y_0)|<\frac{\varepsilon}{2}$ 对 $\forall x\in[0,1]$ 成立. $\exists\delta_2>0$, 当 $|x-x_0|<\delta_2$ 时, $|f(x,y_0)-f(x_0,y_0)|<\frac{\varepsilon}{2}$. 取 $\delta=\min\{\delta_1,\delta_2\}$, 当 $|x-x_0|,|y-y_0|<\delta$ 时, $|f(x,y)-f(x_0,y_0)|\le|f(x,y)-f(x,y_0)|+|f(x,y_0)-f(x_0,y_0)|<\varepsilon$. 故 $f(x,y)$ 在 D 内连续 #

21. 设 $E\subset\mathbb{R}^n$. 证明: 向量函数 $f(x): E\to\mathbb{R}^m$ 在 $x_0\in E$ 处连续的充要条件是对任何在 $U(f(x_0),\delta)$ 内连续的函数 $h(y)$, $h(f(x))$ 在 x_0 处连续

证明: $(\Longrightarrow)$ 显然, 下证明 $(\Longleftarrow)$. 反设 $f(x)=(f_1(x),f_2(x),\cdots,f_m(x))$ 在 x_0 处不连续, 则 $\exists 1\le j_0\le m$, 使得 $f_{j_0}(x)$ 在 x_0 处不连续. 取 $U(f(x_0),\delta)$ 内的连续函数 $h(y)=h(y_1,y_2,\cdots,y_m)=y_{j_0}$. 则 $h(f(x))=f_{j_0}(x)$ 在 x_0 处不连续, 矛盾 #

22. 设 $\emptyset\neq U\subset\mathbb{R}^n$ 是开集. 证明: $f: U\to\mathbb{R}^m$ 在 U 内连续的充要条件是 对 $\mathbb{R}^m$ 中的任意开集 E, $f^{-1}(E)$ 是 $\mathbb{R}^n$ 中的开集

证明: $(\Longrightarrow)$ 不妨设 $E\subset f(U)$ (开集). $\forall x_0\in f^{-1}(E)$, 设 $y_0=f(x_0)\in E$, 由 E 是开集, $\exists\delta_1>0$, 使得 $U(y_0,\delta_1)\subset E$. 由连续性, $\exists\delta>0$, 当 $|x-x_0|<\delta$ 时, $|f(x)-f(x_0)|<\delta_1\Longrightarrow f(x)\in U(y_0,\delta_1)\subset E\Longrightarrow x\in f^{-1}(E)\Longrightarrow U(x_0,\delta)\subset f^{-1}(E)\Longrightarrow f^{-1}(E)$ 是开集

$(\Longleftarrow)$ $\forall x_0\in U$, $\forall\varepsilon>0$, 对 $\mathbb{R}^m$ 中开集 $U(f(x_0),\varepsilon)$, 有 $f^{-1}(U(f(x_0),\varepsilon))$ 是开集, 而 $x_0\in f^{-1}(U(f(x_0),\varepsilon))\Longrightarrow\exists\delta>0$, $U(x_0,\delta)\subset U$ 且 $U(x_0,\delta)\subset f^{-1}(U(f(x_0),\varepsilon))\Longrightarrow$ 当 $|x-x_0|<\delta$ 时, $|f(x)-f(x_0)|<\varepsilon\Longrightarrow f(x)$ 在 x_0 处连续. 由 x_0 的任意性, $f(x)$ 在 U 内连续 #

23. 设 $D\subset\mathbb{R}^2$ 是有界区域, $z=f(x,y)$ 是 $\bar{D}$ 上的连续函数, $\forall(x,y)\in D$, $f(x,y)>0$, 设 $z=g(x,y)$ 在 $\bar{D}$ 上有定义, $\exists(x_0,y_0)\in D$, $g(x_0,y_0)>0$, $\forall(x,y)\in\bar{D}\setminus\{(x_0,y_0)\}$, $f(x,y)=g(x,y)$. 问: (1) 当 $g(x_0,y_0)$ 满足什么条件时, $\{(x,y,z): (x,y)\in D, 0<z<g(x,y)\}$ 是 $\mathbb{R}^3$ 中的开集? (2) 当 $g(x_0,y_0)$ 满足什么条件时, $\{(x,y,z): (x,y)\in\bar{D}, 0\le z\le g(x,y)\}$ 是 $\mathbb{R}^3$ 中的闭集?

解: (1) 当 $g(x_0,y_0)>f(x_0,y_0)$ 时, 显然 $(x_0,y_0,\frac{f(x_0,y_0)+g(x_0,y_0)}{2})\in\{(x,y,z): (x,y)\in D, 0<z<g(x,y)\}$ 不是其内点, 不成立. 当 $g(x_0,y_0)\le f(x_0,y_0)$ 时, $\forall(\bar{x},\bar{y},\bar{z})\in\{(x,y,z): (x,y)\in D, 0<z<g(x,y)\}$, $\exists\delta_1>0$, 使得 $U((\bar{x},\bar{y}),\delta_1)\subset D$ 且 $\forall(x,y)\in U((\bar{x},\bar{y}),\delta_1)$, 有 $g(x,y)>\frac{g(\bar{x},\bar{y})+\bar{z}}{2}$. 取 $\delta=\min\{\delta_1,\frac{g(\bar{x},\bar{y})-\bar{z}}{2},\bar{z}\}$, 则 $U((\bar{x},\bar{y},\bar{z}),\delta)\subset\{(x,y,z): (x,y)\in D, 0<z<g(x,y)\}$

$\Longrightarrow\{(x,y,z): (x,y)\in D, 0<z<g(x,y)\}$ 是开集. 故 $g(x_0,y_0)\le f(x_0,y_0)$

(2) 显然 $(x_0,y_0,f(x_0,y_0))$ 是 $\{(x,y,z): (x,y)\in\bar{D}, 0\le z\le g(x,y)\}$ 的聚点, 从而必有 $g(x_0,y_0)\ge f(x_0,y_0)$. 此时, 显然 $\{(x,y,z): (x,y)\in\bar{D}, 0\le z\le f(x,y)\}$ 和 $\{(x_0,y_0,z): f(x_0,y_0)\le z\le g(x_0,y_0)\}$ 是闭集, 于是二者之并, 即 $\{(x,y,z): (x,y)\in\bar{D}, 0\le z\le g(x,y)\}$ 是闭集. 故 $g(x_0,y_0)\ge f(x_0,y_0)$

24. 设 $E=\{(x,y): x\in\mathbb{Q}, y\in\mathbb{Q}\}$. 证明: (1) E 是可数集 (2) $\mathbb{R}^2\setminus E$ 是连通集

证明: (1) E 可以写成可数个可数集之并: $E=\bigcup_{x\in\mathbb{Q}}\{(x,y): y\in\mathbb{Q}\}$, 从而是可数集

(2) 将对任何可数集 $A\subset\mathbb{R}^2$ 证明 $\mathbb{R}^2\setminus A$ 连通. $\forall(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2\setminus A$, 考虑以其连线的线段为弦的所有圆弧, 这是不可数个 ([illegible]) 从而必有一条圆弧上无 A 的点, 这是 $\mathbb{R}^2\setminus A$ 中连接两点的一条道路 #

19. 设$f(x)$在$[0,1]$上连续，$g(y)$在$[0,1]$上除$y_0=\frac{1}{2}$外连续，y_0是$g(y)$的第一类间断点. 求 $F(x,y)=f(x)g(y)$ 在 $[0,1]^2$ 上的全体间断点

解：$F(x,y)$在 $\{(x,y)\in[0,1]^2: y\neq\frac{1}{2}\}$ 上连续，只需考虑 $(x,\frac{1}{2})$ $(0\le x\le 1)$ 的点. 若 $f(x)\neq 0$，则 $\lim\limits_{y\to\frac{1}{2}}F(x_0,y) = f(x_0)\lim\limits_{y\to\frac{1}{2}}g(y)$ 或者不存在，或者存在但不等于 $f(x_0)g(\frac{1}{2})=F(x_0,\frac{1}{2})$. 于是 $F(x,y)$ 在 $(x_0,\frac{1}{2})$ 不连续.
若 $f(x_0)=0$，$\forall\varepsilon>0$，$\exists\delta>0$，当 $|x-x_0|<\delta$ 且 $0\le x\le 1$ 时，$|f(x)|<\varepsilon$. 因 $g(y)$在 $[0,1]\setminus\{\frac{1}{2}\}$ 上连续，$\frac{1}{2}$是第一类间断点，$g(y)$在$[0,1]$上有界：$|g(y)|\le M$ $(0\le y\le 1)$. 当 $0<\sqrt{(x-x_0)^2+(y-\frac{1}{2})^2}<\delta$ 且 $(x,y)\in[0,1]^2$ 时，$|F(x,y)|=|f(x)||g(y)|\le M\varepsilon$
$\Longrightarrow \lim\limits_{(x,y)\to(x_0,\frac{1}{2})}F(x,y)=0=F(x_0,\frac{1}{2})$，即$F(x,y)$在$(x_0,\frac{1}{2})$连续. 故$F(x,y)$在$[0,1]^2$上的全体间断点为 $\{(x,\frac{1}{2}):0\le x\le 1, f(x)\neq 0\}$

20. 设$f(x,y)$在$D=[0,1]^2$上有定义，对固定的x，$f(x,y)$是y的连续函数，对固定的y，$f(x,y)$是x的连续函数. 证明：若$f(x,y)$满足下面二者之一：(1) 对固定的x，$f(x,y)$关于y单调增 (2) $\forall\varepsilon>0$，$\exists\delta>0$，当 $y_1,y_2\in[0,1]$ 且 $|y_1-y_2|<\delta$ 时，$|f(x,y_1)-f(x,y_2)|<\varepsilon$ 对 $\forall x\in[0,1]$ 成立. 则 $f(x,y)$ 在D内连续

证明：(1) 任取 $(x_0,y_0)\subset(0,1)^2$（若在边界可类似考虑）. $\forall\varepsilon>0$，$\exists\delta_1>0$，当 $y\in[y_0-\delta_1,y_0+\delta_1]\subset(0,1)$ 时，$|f(x_0,y)-f(x_0,y_0)|<\frac{\varepsilon}{6}\Longrightarrow 0\le f(x_0,y_0+\delta_1)-f(x_0,y_0-\delta_1)<\frac{\varepsilon}{3}$. $\exists\delta_2>0$，当 $|x-x_0|<\delta_2$ 时，$|f(x,y_0+\delta_1)-f(x_0,y_0+\delta_1)|<\frac{\varepsilon}{6}$，$|f(x,y_0-\delta_1)-f(x_0,y_0-\delta_1)|<\frac{\varepsilon}{6}$. 取 $\delta=\min\{\delta_1,\delta_2\}$，当 $|x-x_0|,|y-y_0|<\delta$ 时，$|f(x,y)-f(x_0,y_0)|\le |f(x,y)-f(x,y_0+\delta_1)|+|f(x,y_0+\delta_1)-f(x_0,y_0+\delta_1)|+|f(x_0,y_0+\delta_1)-f(x_0,y_0)|\le f(x,y_0+\delta_1)-f(x,y_0-\delta_1)+\frac{\varepsilon}{6}+\frac{\varepsilon}{6}\le|f(x,y_0+\delta_1)-f(x_0,y_0+\delta_1)|+|f(x_0,y_0+\delta_1)-f(x_0,y_0-\delta_1)|+|f(x_0,y_0-\delta_1)-f(x,y_0-\delta_1)|+\frac{\varepsilon}{3}<\varepsilon$. 故$f(x,y)$在$D$内连续
(2) 任取 $(x_0,y_0)\subset(0,1)^2$. $\forall\varepsilon>0$，$\exists\delta_1>0$，当 $|y-y_0|<\delta_1$ 时，$|f(x,y)-f(x,y_0)|<\frac{\varepsilon}{2}$ 对 $\forall x\in[0,1]$ 成立. $\exists\delta_2>0$，当 $|x-x_0|<\delta_2$ 时，$|f(x,y_0)-f(x_0,y_0)|<\frac{\varepsilon}{2}$. 取 $\delta=\min\{\delta_1,\delta_2\}$，当 $|x-x_0|,|y-y_0|<\delta$ 时，$|f(x,y)-f(x_0,y_0)|\le|f(x,y)-f(x,y_0)|+|f(x,y_0)-f(x_0,y_0)|<\varepsilon$. 故$f(x,y)$在$D$内连续 #

21. 设$E\subset\mathbb{R}^n$. 证明：向量函数 $f(x):E\to\mathbb{R}^m$ 在 $x_0\in E$ 处连续的充要条件是对任何在 $U(f(x_0),\delta)$ 内连续的函数 $h(y)$，$h(f(x))$ 在 x_0 处连续

证明：($\Rightarrow$) 显然，下证明($\Leftarrow$). 反设 $f(x)=(f_1(x),f_2(x),\cdots,f_m(x))$ 在 x_0 处不连续，则 $\exists\, 1\le j_0\le m$，使得 $f_{j_0}(x)$ 在 x_0 处不连续. 取 $U(f(x_0),\delta)$ 内的连续函数 $h(y)=h(y_1,y_2,\cdots,y_m)=y_{j_0}$. 则 $h(f(x))=f_{j_0}(x)$ 在 x_0 处不连续，矛盾 #

22. 设 $\emptyset\neq U\subset\mathbb{R}^n$ 是开集. 证明：$f:U\to\mathbb{R}^m$ 在U内连续的充要条件是对$\mathbb{R}^m$中的任意开集E，$f^{-1}(E)$是$\mathbb{R}^n$中的开集
证明：($\Rightarrow$) 不妨设 $E\subset f(U)$. (开集) $\forall x_0\in f^{-1}(E)$，设 $y_0=f(x_0)\in E$，由E是开集，$\exists\delta_1>0$，使得 $U(y_0,\delta_1)\subset E$. 由连续性，$\exists\delta>0$，当 $|x-x_0|<\delta$ 时，$|f(x)-f(x_0)|<\delta_1\Longrightarrow f(x)\in U(y_0,\delta_1)\subset E\Longrightarrow x\in f^{-1}(E)\Longrightarrow U(x_0,\delta)\subset f^{-1}(E)\Longrightarrow f^{-1}(E)$是开集
($\Leftarrow$) $\forall x_0\in U$，$\forall\varepsilon>0$，对$\mathbb{R}^m$中开集 $U(f(x_0),\varepsilon)$，有 $f^{-1}(U(f(x_0),\varepsilon))$ 是开集，而 $x_0\in f^{-1}(U(f(x_0),\varepsilon))\Longrightarrow\exists\delta>0$，$U(x_0,\delta)\subset U$ 且 $U(x_0,\delta)\subset f^{-1}(U(f(x_0),\varepsilon))\Longrightarrow$ 当 $|x-x_0|<\delta$ 时，$|f(x)-f(x_0)|<\varepsilon\Longrightarrow f(x)$在$x_0$处连续. 由$x_0$的任意性，$f(x)$在$U$内连续 #

23. 设 $D\subset\mathbb{R}^2$ 是有界区域，$z=f(x,y)$ 是 $\bar{D}$ 上的连续函数，$\forall(x,y)\in D$，$f(x,y)>0$，设 $z=g(x,y)$ 在 $\bar{D}$ 上有定义，$\exists(x_0,y_0)\in D$，$g(x_0,y_0)>0$，$\forall(x,y)\in\bar{D}\setminus\{(x_0,y_0)\}$，$f(x,y)=g(x,y)$. 问：(1) 当 $g(x_0,y_0)$ 满足什么条件时，$\{(x,y,z):(x,y)\in D, 0<z<g(x,y)\}$ 是 $\mathbb{R}^3$ 中的开集？(2) 当 $g(x_0,y_0)$ 满足什么条件时，$\{(x,y,z):(x,y)\in\bar{D}, 0\le z\le g(x,y)\}$ 是 $\mathbb{R}^3$ 中的闭集？
解：(1) 当 $g(x_0,y_0)>f(x_0,y_0)$ 时，显然 $(x_0,y_0,\frac{f(x_0,y_0)+g(x_0,y_0)}{2})\in\{(x,y,z):(x,y)\in D,0<z<g(x,y)\}$ 不是其内点，不成立. 当 $g(x_0,y_0)\le f(x_0,y_0)$ 时，$\forall(\bar{x},\bar{y},\bar{z})\in\{(x,y,z):(x,y)\in D,0<z<g(x,y)\}$，$\exists\delta_1>0$，使得 $U((\bar{x},\bar{y}),\delta_1)\subset D$ 且 $\forall(x,y)\in U((\bar{x},\bar{y}),\delta_1)$，有 $g(x,y)>\frac{g(\bar{x},\bar{y})+\bar{z}}{2}$. 取 $\delta=\min\{\delta_1,\frac{g(\bar{x},\bar{y})-\bar{z}}{2},\bar{z}\}$，则 $U((\bar{x},\bar{y},\bar{z}),\delta)\subset\{(x,y,z):(x,y)\in D,0<z<g(x,y)\}$
$\Longrightarrow\{(x,y,z):(x,y)\in D,0<z<g(x,y)\}$是开集. 故 $g(x_0,y_0)\le f(x_0,y_0)$
(2) 显然 $(x_0,y_0,f(x_0,y_0))$ 是 $\{(x,y,z):(x,y)\in\bar{D},0\le z\le g(x,y)\}$ 的聚点，从而必有 $g(x_0,y_0)\ge f(x_0,y_0)$. 此时，显然 $\{(x,y,z):(x,y)\in\bar{D},0\le z\le f(x,y)\}$ 和 $\{(x_0,y_0,z): f(x_0,y_0)\le z\le g(x_0,y_0)\}$ 是闭集，于是二者之并，即 $\{(x,y,z):(x,y)\in\bar{D},0\le z\le g(x,y)\}$ 是闭集. 故 $g(x_0,y_0)\ge f(x_0,y_0)$

24. 设 $E=\{(x,y):x\in\mathbb{Q},y\in\mathbb{Q}\}$. 证明：(1) E是可数集 (2) $\mathbb{R}^2\setminus E$ 是连通集
证明：(1) E可以写成可数个可数集之并：$E=\bigcup\limits_{x\in\mathbb{Q}}\{(x,y):y\in\mathbb{Q}\}$，从而是可数集
(2) 将对任何可数集 $A\subset\mathbb{R}^2$ 证明 $\mathbb{R}^2\setminus A$ 连通. $\forall(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2\setminus A$，考虑以其连线的线段为弦的所有圆弧，这是不可数个（互不交），从而必有一条圆弧上无A的点，这是 $\mathbb{R}^2\setminus A$ 中连接两点的一条道路 #

25. 设$f(x,y)$在$D=[0,1]^2$上连续，最大值为M，最小值为m. 证明：$\forall C\in(m,M)$（假设$M>m$），存在点$(\xi,\eta)\in D$，使得$f(\xi,\eta)=C$

证明：设$(x_1,y_1),(x_2,y_2)\in D$使得$f(x_1,y_1)=m$，$f(x_2,y_2)=M$. 显然D中连接这两点的道路有无穷多条，任取一条：$h(t)=(x(t),y(t))$ $(t\in[\alpha,\beta])$，$h(\alpha)=(x_1,y_1)$，$h(\beta)=(x_2,y_2)$，则$f(h(t))$在$[\alpha,\beta]$上连续. 又$f(h(\alpha))=m$，$f(h(\beta))=M \Longrightarrow \exists t_0\in(\alpha,\beta)$，使得$f(h(t_0))=C$. 如此得到无穷多个$(\xi,\eta)\in D$，使得$f(\xi,\eta)=C$ #

26. 设A是$n\times n$ $(n\geq 2)$非退化矩阵，证明：$\exists\lambda>0$，$\forall x\in\mathbb{R}^n$，有$|Ax|\geq\lambda|x|$（x为列向量）

证明：连续函数$f(x')=|Ax'|$在紧集$\{x'\in\mathbb{R}^n: |x'|=1\}$上有最小值$\lambda>0$. 若$x=0$，则$|Ax|=0=\lambda|x|$. 若$x\neq 0$，则$f(\frac{x}{|x|})=\frac{|Ax|}{|x|}\geq\lambda \Longrightarrow |Ax|\geq\lambda|x|$ #

27. 设$E\subset\mathbb{R}^n$，证明：$f(x)=\inf\limits_{y\in E}|x-y|$在$\mathbb{R}^n$内一致连续.

证明：$\forall\varepsilon>0$，$\exists\delta=\varepsilon$，当$x',x''\in\mathbb{R}^n$且$|x'-x''|<\delta$时，$-|x'-x''|=\inf\limits_{y\in E}|x'-y|-\inf\limits_{y\in E}(|x'-y|+|x'-x''|)\leq\inf\limits_{y\in E}|x'-y|-\inf\limits_{y\in E}|x''-y|$ $=f(x')-f(x'')\leq\inf\limits_{y\in E}(|x'-x''|+|x''-y|)-\inf\limits_{y\in E}|x''-y|=|x'-x''| \Longrightarrow |f(x')-f(x'')|\leq|x'-x''|<\delta=\varepsilon \Longrightarrow f(x)$在$\mathbb{R}^n$内一致连续 #

28. 证明：$f(x,y)=\sqrt{xy}$在$D=\{(x,y): x\geq 0, y\geq 0\}$上不一致连续

证明：在D中取两列点：$x'_k=(0,k)$，$x''_k=(\frac{1}{k},k)$，则$|x'_k-x''_k|\to 0$，但$|f(x'_k)-f(x''_k)|=1$. 故$f(x,y)$在D上不一致连续

29. 用两种方法证明$\mathbb{R}^n$中紧集上的连续函数必一致连续：(1) 有限覆盖定理 (2) 聚点原理.

证明：设（向量）函数$f(x)=(f_1(x),f_2(x),\cdots,f_m(x))$在紧集$E\subset\mathbb{R}^n$上连续.

(1) $\forall\varepsilon>0$，$\forall x\in E$，$\exists\delta_x>0$，当$x',x''\in U(x,2\delta_x)$时，$|f(x')-f(x'')|<\varepsilon$. $\exists x_1,x_2,\cdots,x_K\in E$，使得$E\subset\bigcup\limits_{k=1}^{K}U(x_k,\delta_{x_k})$.
令$\delta=\min\{\delta_{x_1},\delta_{x_2},\cdots,\delta_{x_K}\}$，当$x',x''\in E$且$|x'-x''|<\delta$时，设$x'\in U(x_j,\delta_{x_j})$ $(1\leq j\leq K)$，则$|x''-x_j|\leq|x''-x'|+|x'-x_j|$ $<2\delta_{x_j} \Longrightarrow x',x''$落在同一个$U(x_j,2\delta_{x_j})$内 $\Longrightarrow |f(x')-f(x'')|<\varepsilon$. 故$f(x)$在$E$上一致连续

(2) 反设$f(x)$在E上不一致连续. $\exists\varepsilon_0>0$，$\forall k$，$\exists x'_k,x''_k\in E$，$|x'_k-x''_k|<\frac{1}{k}$，使得$|f(x'_k)-f(x''_k)|\geq\varepsilon_0$
由$\{x'_k\}\subset E$，$\{x'_k\}$有收敛子列$\{x'_{k_l}\}$. 设$x'_{k_l}\to x_0$ $(l\to\infty)$，则$x''_{k_l}\to x_0$ $(l\to\infty)$. $\exists L$，当$l>L$时，$|f(x'_{k_l})-f(x_0)|<\frac{\varepsilon_0}{2}$，$|f(x''_{k_l})-f(x_0)|<\frac{\varepsilon_0}{2} \Longrightarrow \varepsilon_0\leq|f(x'_{k_l})-f(x''_{k_l})|\leq|f(x'_{k_l})-f(x_0)|+|f(x_0)-f(x''_{k_l})|<\varepsilon_0$. 矛盾 #

30. 证明：$f(x)$在$U(0,1)\subset\mathbb{R}^n$内一致连续的充要条件是存在$\overline{U(0,1)}$上的连续函数$g(x)$，使得在$U(0,1)$内处处有$g(x)=f(x)$

证明：($\Longleftarrow$) $g(x)$在紧集$\overline{U(0,1)}$上连续，从而一致连续 $\Longrightarrow f(x)$在$U(0,1)$上一致连续.

($\Longrightarrow$) $\forall x_0\in\partial U(0,1)$，由一致连续性和Cauchy准则，$U(0,1)$内任何收敛到$x_0$的点列，其对应的函数值收敛. 设$\{x_k\}$是其中之一，令$g(x_0)=\lim\limits_{k\to\infty}f(x_k)$. 对其他趋于$x_0$的$\{x'_k\}$，我们证明$\lim\limits_{k\to\infty}f(x'_k)=g(x_0)$. $\forall\varepsilon>0$，$\exists\delta>0$，当$x',x''\in U(0,1)$，$|x'-x''|<\delta$时，$|f(x')-f(x'')|<\varepsilon$. $\exists K$，当$k>K$时，$|x_k-x'_k|<\delta \Longrightarrow |f(x_k)-f(x'_k)|<\varepsilon \Longrightarrow \lim\limits_{k\to\infty}f(x'_k)$ $=\lim\limits_{k\to\infty}f(x_k)=g(x_0) \Longrightarrow \lim\limits_{U(0,1)\ni x\to x_0}f(x)=g(x_0)$. $\forall x\in U(0,1)$，取$g(x)=f(x)$，则$g(x)$在x_0处连续. 于是$g(x)$在$\overline{U(0,1)}$连续 #

31. 设$E\subset\mathbb{R}^n$是开集，$D\subset E$称为E的一个分支，若D是区域，且对任何区域$D'\subset E$，只要$D\cap D'\neq\emptyset$，就有$D'\subset D$. 证明：$\mathbb{R}^n$中任何开集都是可数个分支的并

证明：在E上定义关系：$x\sim y \Longleftrightarrow$ 存在E中从x到y的道路. 显然，$\sim$是等价关系，从而诱导出$E=\bigcup\limits_{x\in E}[x]$，$[x]$是$x$对应的等价类. $\forall x\in E$，$[x]$是E的分支，因为：$\forall x'\in[x]$，$\exists\delta>0$，使得$U(x',\delta)\subset E$，显然$U(x',\delta)\subset[x] \Longrightarrow [x]$是开集. 由定义$[x]$道路连通 $\Longrightarrow [x]$是区域. 若区域D'与$[x]$交非空，设$x_0\in D'$，$x_0\in[x]$，则$\forall y\in D'$，有$y\sim x_0$，又$x_0\sim x \Longrightarrow y\sim x \Longrightarrow y\in[x] \Longrightarrow D'\subset[x]$
由此E可表示为若干个分支之并：$E=\bigcup\limits_{x\in E}[x]$. 因为每个$[x]$中必含有理点（即各分量均为有理数的点），而$\mathbb{Q}^n$可数，所以分支必是可数个. 故$\mathbb{R}^n$中任何开集都是可数个分支的并 #

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$<2\delta_{x_j}\Rightarrow x',x''$ 属在同一个 $U(x_j,\delta_{x_j})$ 内 $\Rightarrow|f(x')-f(x'')|<\varepsilon$. 故 $f(x)$ 在 E 上一致连续

(2) 反设 $f(x)$ 在 E 上不一致连续. $\exists\varepsilon_0>0$，$\forall k$，$\exists x_k',x_k''\in E$，$|x_k'-x_k''|<\frac{1}{k}$，使得 $|f(x_k')-f(x_k'')|\geq\varepsilon_0$
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解：$(u,v)=\left(x/(1-\sqrt{x^2+y^2}),\ y/(1-\sqrt{x^2+y^2})\right)$，$(x,y)\in\Delta$

第十四章 多元微分学

1. 设 $u=f(x)$ 在 $U(x_0,\delta)\subset R^n$ 内有各个偏导数，且所有偏导数在该邻域内有界，证明 $f(x)$ 在 x_0 处连续。举例说明存在 $u=g(x)$ 在 x_0 的某个邻域内存在无界的各个偏导数，但在 x_0 处连续

证明：设 $|f_{x_i}'(x)|\le M$ ($\forall x\in U(x_0,\delta)$, $\forall 1\le i\le n$)，记 $x_0=(x_1^0,x_2^0,\dots,x_n^0)$，$x=(x_1,x_2,\dots,x_n)\in U(x_0,\delta)$，由一元函数的微分中值定理，$|f(x_1,x_2,\dots,x_n)-f(x_1^0,x_2^0,\dots,x_n^0)|\le|f(x_1,x_2,\dots,x_n)-f(x_1^0,x_2,\dots,x_n)|+|f(x_1^0,x_2,\dots,x_n)-f(x_1^0,x_2^0,\dots,x_n)|+\dots+|f(x_1^0,x_2^0,\dots,x_{n-1}^0,x_n)-f(x_1^0,x_2^0,\dots,x_n^0)|\le M\sum_{i=1}^{n}|x_i-x_i^0|\to0$，故 $f(x)$ 在 x_0 处连续。#

例子：$g(x_1,x_2,\dots,x_n)=\begin{cases}(x_1^2+x_2^2+\dots+x_n^2)\sin\dfrac{1}{x_1^2+x_2^2+\dots+x_n^2}, & x_1^2+x_2^2+\dots+x_n^2\ne0\\ 0, & x_1^2+x_2^2+\dots+x_n^2=0\end{cases}$，$|g(x_1,\dots,x_n)|\le(x_1^2+\dots+x_n^2)\to0$

$g_{x_i}'(x_1,\dots,x_n)=2x_i\sin\frac{1}{x_1^2+\dots+x_n^2}-\frac{2x_i}{x_1^2+\dots+x_n^2}\cos\frac{1}{x_1^2+\dots+x_n^2}$ ($x\ne0$) 无界

2. 举例说明在 R^2 内存在 $z=f(x,y)$，使得 $f(x,y)$ 在 R^2 内处处不连续，但在原点处有两个偏导数

解：$f(x,y)=\begin{cases}1, & x或y\in Q\\ 0, & 其他\end{cases}$，则 $f(x,y)$ 在 R^2 处处不连续，但 $f_x'(0,0)=\lim\limits_{t\to0}\frac{f(t,0)-f(0,0)}{t}=0=f_y'(0,0)$

3. (1) $f(x,y)=xy\ln(x^2+\sin(xy^2)+\sin(xy))$，求 $f_x'(1,-1)$，$f_y'(1,-1)$ (2) $f(x,y,z)=(x^2+y^2)\cos\frac{xz}{x+y}$，求 $f_x'(1,0,\pi)$

解 (1) $f_x'(1,-1)=\lim\limits_{x\to1}\frac{f(x,-1)-f(1,-1)}{x-1}=\lim\limits_{x\to1}\frac{-2x\ln x}{x-1}=-2\lim\limits_{x\to1}\frac{x}{1}=-2$

$f_y'(1,-1)=\lim\limits_{y\to-1}\frac{f(1,y)-f(1,-1)}{y+1}=\lim\limits_{y\to-1}\frac{y\ln(1+\sin y^2+\sin y)}{y+1}=-\lim\limits_{y\to-1}\frac{\sin y^2+\sin y}{y+1}=-\lim\limits_{y\to-1}(2y\cos y^2+\cos y)=\cos1$

(2) $f_x'(x,y,z)=2x\cos\frac{xz}{x+y}-\frac{(x^2+y^2)yz}{(x+y)^2}\sin\frac{xz}{x+y}$，易得 $f_x'(1,0,\pi)=-2$

4. 求各个偏导数：(1) $z=\frac{x}{2x^2+y^2+xy}$ (2) $z=x\sqrt{x^2-y^2}$ (3) $z=\tan(x^2+2y^3)$ (4) $u=(1+x+y+z)e^{xyz}$ (5) $u=\sin(ye^{zx})$ (6) $u=\ln(xy+x^2+z^2)$ (7) $u=\sqrt[3]{1-z\sin^2(x+y)}$ (8) $u=\frac{\sin xz}{\cos(x^2+y)}$ (9) $u=\ln(\sec\sqrt{x+y-z})$ (10) $u=e^{-xz}\tan y$ (11) $u=e^z(x^2+y^2+z^2)$ (12) $u=(\frac{x}{y})^z$ (13) $u=\ln(1+\sqrt{\sum_{i=1}^{n}x_i^2})$ (14) $u=x_1x_2\cdots x_n+(x_1+x_2+\dots+x_n)^n$

解：(1) $\frac{\partial z}{\partial x}=\frac{-2x^2+y^2}{(2x^2+y^2+xy)^2}$，$\frac{\partial z}{\partial y}=\frac{-x(2y+x)}{(2x^2+y^2+xy)^2}$ (2) $\frac{\partial z}{\partial x}=\frac{2x^2-y^2}{\sqrt{x^2-y^2}}$，$\frac{\partial z}{\partial y}=-\frac{xy}{\sqrt{x^2-y^2}}$ (3) $\frac{\partial z}{\partial x}=\frac{2x}{\cos^2(x^2+2y^3)}$，$\frac{\partial z}{\partial y}=\frac{6y^2}{\cos^2(x^2+2y^3)}$

(4) $\frac{\partial u}{\partial x}=(1+yz(x+y+z+1))e^{xyz}$，$\frac{\partial u}{\partial y}=(1+xz(x+y+z+1))e^{xyz}$，$\frac{\partial u}{\partial z}=(1+xy(x+y+z+1))e^{xyz}$ (5) $\frac{\partial u}{\partial x}=\cos(ye^{zx})\cdot yz e^{zx}$

$\frac{\partial u}{\partial y}=\cos(ye^{zx})\cdot e^{zx}$，$\frac{\partial u}{\partial z}=\cos(ye^{zx})\cdot xy\cdot e^{zx}$ (6) $\frac{\partial u}{\partial x}=\frac{y+2x}{xy+x^2+z^2}$，$\frac{\partial u}{\partial y}=\frac{x}{xy+x^2+z^2}$，$\frac{\partial u}{\partial z}=\frac{2z}{xy+x^2+z^2}$

(7) $\frac{\partial u}{\partial x}=-\frac{1}{3}z(1-z\sin^2(x+y))^{-2/3}\sin2(x+y)$，$\frac{\partial u}{\partial y}=-\frac{1}{3}z(1-z\sin^2(x+y))^{-2/3}\sin2(x+y)$，$\frac{\partial u}{\partial z}=-\frac{1}{3}(1-z\sin^2(x+y))^{-2/3}\sin^2(x+y)$

(8) $\frac{\partial u}{\partial x}=\frac{z\cos xz\cos(x^2+y)+2x\sin(x^2+y)\sin xz}{(\cos(x^2+y))^2}$，$\frac{\partial u}{\partial y}=-\frac{\sin xz}{(\cos(x^2+y))^2}$，$\frac{\partial u}{\partial z}=\frac{x\cos xz}{\cos(x^2+y)}$

(9) $\frac{\partial u}{\partial x}=\frac{\tan\sqrt{x+y-z}}{2\sqrt{x+y-z}}$，$\frac{\partial u}{\partial y}=\frac{\tan\sqrt{x+y-z}}{2\sqrt{x+y-z}}$，$\frac{\partial u}{\partial z}=\frac{-\tan\sqrt{x+y-z}}{2\sqrt{x+y-z}}$ (10) $\frac{\partial u}{\partial x}=-ze^{-xz}\tan y$，$\frac{\partial u}{\partial y}=e^{-xz}\sec^2y$，$\frac{\partial u}{\partial z}=-xe^{-xz}\tan y$

(11) $\frac{\partial u}{\partial x}=2xe^z$，$\frac{\partial u}{\partial y}=2ye^z$，$\frac{\partial u}{\partial z}=(x^2+y^2+z^2+2z)e^z$ (12) $\frac{\partial u}{\partial x}=\frac{z}{x}(\frac{x}{y})^z$，$\frac{\partial u}{\partial y}=-\frac{z}{y}(\frac{x}{y})^z$，$\frac{\partial u}{\partial z}=\ln\frac{x}{y}(\frac{x}{y})^z$

(13) $\frac{\partial u}{\partial x_i}=\frac{x_i}{\sum_{j=1}^{n}x_j^2+(\sum_{j=1}^{n}x_j^2)^{1/2}}$ ($1\le i\le n$) (14) $\frac{\partial u}{\partial x_i}=\prod_{j\ne i}x_j+n(\sum_{j=1}^{n}x_j)^{n-1}$ ($1\le i\le n$)

5. 对下面的 $u(x,y)$，$v(x,y)$ 证明 $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$，$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$，(1) $u(x,y)=e^x\cos y$，$v(x,y)=e^x\sin y$

(2) $u(x,y)=\cos x\cosh y+\sin x\sinh y$，$v(x,y)=\cos x\cosh y-\sin x\sinh y$

证明：(1) $\frac{\partial u}{\partial x}=e^x\cos y=\frac{\partial v}{\partial y}$，$\frac{\partial u}{\partial y}=-e^x\sin y=-\frac{\partial v}{\partial x}$

(2) $\frac{\partial u}{\partial x}=-\sin x\cosh y+\cos x\sinh y=\frac{\partial v}{\partial y}$，$\frac{\partial u}{\partial y}=\cos x\sinh y+\sin x\cosh y=-\frac{\partial v}{\partial x}$ #

6. 求 $f(x,y)=x^2\sin y$ 在 $(1,0)$ 处分别沿 $\vec{i}$，$-\vec{j}$，$\frac{1}{\sqrt2}(\vec{i}+\vec{j})$ 的方向导数

解 $\frac{\partial f(1,0)}{\partial\vec{i}}=\lim\limits_{t\to0^+}\frac{f(1+t,0)-f(1,0)}{t}=0$，$\frac{\partial f(1,0)}{\partial(-\vec{j})}=\lim\limits_{t\to0^+}\frac{f(1,-t)-f(1,0)}{t}=-1$，$\frac{\partial f(1,0)}{\partial(\frac{1}{\sqrt2}(\vec{i}+\vec{j}))}=\lim\limits_{t\to0^+}\frac{f(1+\frac{t}{\sqrt2},\frac{t}{\sqrt2})-f(1,0)}{t}=\frac{\sqrt2}{2}$

7. 设 $f(x,y,z)=x^2-xy+y^2+z^2$，求其在 $(1,1,1)$ 处沿各个方向的方向导数，并求方向导数的最大值、最小值和方向导数为零的所有方向。

解：设 $\vec{v}=(p,q,r)$ 是任一方向单位向量，$\mathrm{grad}f(1,1,1)=(\frac{\partial f(1,1,1)}{\partial x},\frac{\partial f(1,1,1)}{\partial y},\frac{\partial f(1,1,1)}{\partial z})=(1,1,2)$，于是

$\frac{\partial f(1,1,1)}{\partial\vec{v}}=(1,1,2)\cdot\vec{v}=p+q+2r$，方向导数最大值、最小值分别为 $\pm|\mathrm{grad}f(1,1,1)|=\pm\sqrt6$，若 $\frac{\partial f(1,1,1)}{\partial\vec{v}}=0$，则

$\begin{cases}p+q+2r=0\\ p^2+q^2+r^2=1\end{cases}\Rightarrow-\frac{\sqrt3}{3}\le r\le\frac{\sqrt3}{3}$，$p=-r+\frac{1}{2}\sqrt{2-6r^2}$，$q=-r-\frac{1}{2}\sqrt{2-6r^2}$，使方向导数为零的所有方向为

$(-r+\frac{1}{2}\sqrt{2-6r^2},-r-\frac{1}{2}\sqrt{2-6r^2},r)$，$-\frac{\sqrt3}{3}\le r\le\frac{\sqrt3}{3}$

第十四章 多元微分学

1. 设 $u=f(x)$ 在 $U(x_0,\delta_0)\subset\mathbb{R}^n$ 内有各个偏导数，且所有偏导数在该邻域内有界，证明 $f(x)$ 在 x_0 处连续。举例说明存在 $u=g(x)$，在 x_0 的某个邻域内存在无界的各个偏导数，但在 x_0 处连续

证明：设 $|f'_{x_i}(x)|\le M$ ($\forall x\in U(x_0,\delta_0)$，$\forall 1\le i\le n$)，记 $x_0=(x_1^0,x_2^0,\dots,x_n^0)$，$x=(x_1,x_2,\dots,x_n)\in U(x_0,\delta_0)$，由一元函数的微分中值定理，$|f(x_1,x_2,\dots,x_n)-f(x_1^0,x_2^0,\dots,x_n^0)|\le|f(x_1,x_2,\dots,x_n)-f(x_1^0,x_2,\dots,x_n)|+|f(x_1^0,x_2,\dots,x_n)-f(x_1^0,x_2^0,\dots,x_n)|+\dots+|f(x_1^0,x_2^0,\dots,x_{n-1}^0,x_n)-f(x_1^0,x_2^0,\dots,x_n^0)|\le M\sum_{i=1}^{n}|x_i-x_i^0|\to0$，故 $f(x)$ 在 x_0 处连续。#

例子：$g(x_1,x_2,\dots,x_n)=\begin{cases}(x_1^2+x_2^2+\dots+x_n^2)\sin\frac{1}{x_1^2+x_2^2+\dots+x_n^2}, & x_1^2+x_2^2+\dots+x_n^2\ne0\\ 0, & x_1^2+x_2^2+\dots+x_n^2=0\end{cases}$ $|g(x_1,\dots,x_n)|\le(x_1^2+\dots+x_n^2)\to0$

$g'_{x_i}(x_1,\dots,x_n)=2x_i\sin\frac{1}{x_1^2+\dots+x_n^2}-\frac{2x_i}{x_1^2+\dots+x_n^2}\cos\frac{1}{x_1^2+\dots+x_n^2}$ (当 $x\ne0$) 无界

2. 举例说明在 $\mathbb{R}^2$ 内存在 $z=f(x,y)$，使得 $f(x,y)$ 在 $\mathbb{R}^2$ 内处处不连续，但在原点处有两个偏导数

解：$f(x,y)=\begin{cases}1, & x或y为有理数\\ 0, & 其它\end{cases}$，则 $f(x,y)$ 在 $\mathbb{R}^2$ 处处不连续，但 $f'_x(0,0)=\lim\limits_{t\to0}\frac{f(t,0)-f(0,0)}{t}=0=f'_y(0,0)$

3. (1) $f(x,y)=xy\ln(x^2+\sin(xy^2)+\sin(xy))$，求 $f'_x(1,-1)$，$f'_y(1,-1)$ (2) $f(x,y,z)=(x^2+y^2)\cos\frac{xz}{x+y}$，求 $f'_x(1,0,\pi)$

解 (1) $f'_x(1,-1)=\lim\limits_{x\to1}\frac{f(x,-1)-f(1,-1)}{x-1}=\lim\limits_{x\to1}\frac{-2x\ln x}{x-1}=-2\lim\limits_{x\to1}\frac{x}{1}=-2$

$f'_y(1,-1)=\lim\limits_{y\to-1}\frac{f(1,y)-f(1,-1)}{y+1}=\lim\limits_{y\to-1}\frac{y\ln(1+\sin y^2+\sin y)}{y+1}=-\lim\limits_{y\to-1}\frac{\sin y^2+\sin y}{y+1}=-\lim\limits_{y\to-1}(2y\cos y^2+\cos y)=\cos1$

(2) $f'_x(x,y,z)=2x\cos\frac{xz}{x+y}-\frac{(x^2+y^2)yz}{(x+y)^2}\sin\frac{xz}{x+y}$，于是得 $f'_x(1,0,\pi)=-2$

4. 求各个偏导数 (1) $z=\frac{x}{2x^2+y^2+xy}$ (2) $z=x\sqrt{x^2-y^2}$ (3) $z=\tan(x^2+2y^3)$ (4) $u=(x+y+z)e^{xyz}$ (5) $u=\sin(ye^{xz})$
(6) $u=\ln(xy+x^2+z^2)$ (7) $u=\sqrt[3]{1-z\sin^2(xy)}$ (8) $u=\frac{\sin xz}{\cos(x^2+y)}$ (9) $u=\ln(\sec\sqrt{x+y-z})$ (10) $u=e^{-xz}\tan y$
(11) $u=e^{z}(x^2+y^2+z^2)$ (12) $u=(\frac{x}{y})^z$ (13) $u=\ln(1+\sqrt{\sum_{i=1}^{n}x_i^2})$ (14) $u=x_1x_2\cdots x_n+(x_1+x_2+\dots+x_n)^n$

解 (1) $\frac{\partial z}{\partial x}=\frac{-2x^2+y^2}{(2x^2+y^2+xy)^2}$，$\frac{\partial z}{\partial y}=\frac{-x(2y+x)}{(2x^2+y^2+xy)^2}$ (2) $\frac{\partial z}{\partial x}=\frac{2x^2-y^2}{\sqrt{x^2-y^2}}$，$\frac{\partial z}{\partial y}=-\frac{xy}{\sqrt{x^2-y^2}}$ (3) $\frac{\partial z}{\partial x}=\frac{2x}{\cos^2(x^2+2y^3)}$，$\frac{\partial z}{\partial y}=\frac{6y^2}{\cos^2(x^2+2y^3)}$

(4) $\frac{\partial u}{\partial x}=(1+yz(x+y+z))e^{xyz}$，$\frac{\partial u}{\partial y}=(1+xz(x+y+z))e^{xyz}$，$\frac{\partial u}{\partial z}=(1+xy(x+y+z))e^{xyz}$ (5) $\frac{\partial u}{\partial x}=\cos(ye^{xz})\cdot yz e^{xz}$

$\frac{\partial u}{\partial y}=\cos(ye^{xz})e^{xz}$，$\frac{\partial u}{\partial z}=\cos(ye^{xz})\cdot xy e^{xz}$ (6) $\frac{\partial u}{\partial x}=\frac{y+2x}{xy+x^2+z^2}$，$\frac{\partial u}{\partial y}=\frac{x}{xy+x^2+z^2}$，$\frac{\partial u}{\partial z}=\frac{2z}{xy+x^2+z^2}$

(7) $\frac{\partial u}{\partial x}=-\frac{1}{3}yz(1-z\sin^2(xy))^{-2/3}\sin2(xy)$，$\frac{\partial u}{\partial y}=-\frac{1}{3}xz(1-z\sin^2(xy))^{-2/3}\sin2(xy)$，$\frac{\partial u}{\partial z}=-\frac{1}{3}(1-z\sin^2(xy))^{-2/3}\sin^2(xy)$

(8) $\frac{\partial u}{\partial x}=\frac{z\cos xz\cos(x^2+y)+2x\sin(x^2+y)\sin xz}{(\cos(x^2+y))^2}$，$\frac{\partial u}{\partial y}=-\frac{\sin xz}{(\cos(x^2+y))^2}$，$\frac{\partial u}{\partial z}=\frac{x\cos xz}{\cos(x^2+y)}$

(9) $\frac{\partial u}{\partial x}=\frac{\tan\sqrt{x+y-z}}{2\sqrt{x+y-z}}$，$\frac{\partial u}{\partial y}=\frac{\tan\sqrt{x+y-z}}{2\sqrt{x+y-z}}$，$\frac{\partial u}{\partial z}=\frac{-\tan\sqrt{x+y-z}}{2\sqrt{x+y-z}}$ (10) $\frac{\partial u}{\partial x}=-ze^{-xz}\tan y$，$\frac{\partial u}{\partial y}=e^{-xz}\sec^2y$，$\frac{\partial u}{\partial z}=-xe^{-xz}\tan y$

(11) $\frac{\partial u}{\partial x}=2xe^{z}$，$\frac{\partial u}{\partial y}=2ye^{z}$，$\frac{\partial u}{\partial z}=(x^2+y^2+z^2+2z)e^{z}$ (12) $\frac{\partial u}{\partial x}=\frac{z}{x}(\frac{x}{y})^z$，$\frac{\partial u}{\partial y}=-\frac{z}{y}(\frac{x}{y})^z$，$\frac{\partial u}{\partial z}=\ln\frac{x}{y}(\frac{x}{y})^z$

(13) $\frac{\partial u}{\partial x_i}=\frac{x_i}{\sum_{j=1}^{n}x_j^2+(\sum_{j=1}^{n}x_j^2)^{1/2}}$ $(1\le i\le n)$ (14) $\frac{\partial u}{\partial x_i}=\prod_{j\ne i}x_j+n(\sum_{j=1}^{n}x_j)^{n-1}$ $(1\le i\le n)$

5. 对下面的 $u(x,y)$，$v(x,y)$ 验证 $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$，$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$：(1) $u(x,y)=e^x\cos y$，$v(x,y)=e^x\sin y$
(2) $u(x,y)=\cos x\cosh y+\sin x\sinh y$，$v(x,y)=\cos x\sinh y-\sin x\cosh y$

证明：(1) $\frac{\partial u}{\partial x}=e^x\cos y=\frac{\partial v}{\partial y}$，$\frac{\partial u}{\partial y}=-e^x\sin y=-\frac{\partial v}{\partial x}$

(2) $\frac{\partial u}{\partial x}=-\sin x\cosh y+\cos x\sinh y=\frac{\partial v}{\partial y}$，$\frac{\partial u}{\partial y}=\cos x\sinh y+\sin x\cosh y=-\frac{\partial v}{\partial x}$ #

6. 求 $f(x,y)=x^2\sin y$ 在 $(1,0)$ 处分别沿 $\vec{i}$，$-\vec{j}$，$\frac{1}{\sqrt{2}}(\vec{i}+\vec{j})$ 的方向导数

解 $\frac{\partial f(1,0)}{\partial\vec{i}}=\lim\limits_{t\to0^+}\frac{f(1+t,0)-f(1,0)}{t}=0$，$\frac{\partial f(1,0)}{\partial(-\vec{j})}=\lim\limits_{t\to0^+}\frac{f(1,-t)-f(1,0)}{t}=-1$，$\frac{\partial f(1,0)}{\partial(\frac{1}{\sqrt{2}}(\vec{i}+\vec{j}))}=\lim\limits_{t\to0^+}\frac{f(1+\frac{t}{\sqrt{2}},\frac{t}{\sqrt{2}})-f(1,0)}{t}=\frac{\sqrt{2}}{2}$

7. 设 $f(x,y,z)=x^2-xy+y^2+z^2$，求其在 $(1,1,1)$ 处沿各个方向的方向导数，并求方向导数的最大值、最小值和方向导数为零的所有方向。

解：设 $\vec{v}=(p,q,r)$ 是任一方向的单位向量，$\text{grad} f(1,1,1)=(\frac{\partial f(1,1,1)}{\partial x},\frac{\partial f(1,1,1)}{\partial y},\frac{\partial f(1,1,1)}{\partial z})=(1,1,2)$，于是 $\frac{\partial f(1,1,1)}{\partial\vec{v}}=(1,1,2)\cdot\vec{v}=p+q+2r$，方向导数最大值、最小值分别为 $\pm|\text{grad} f(1,1,1)|=\pm\sqrt{6}$，若 $\frac{\partial f(1,1,1)}{\partial\vec{v}}=0$，则 $\begin{cases}p+q+2r=0\\ p^2+q^2+r^2=1\end{cases}\Longrightarrow-\frac{\sqrt{3}}{3}\le r\le\frac{\sqrt{3}}{3}$，$p=-r+\frac{1}{2}\sqrt{2-6r^2}$，$q=-r-\frac{1}{2}\sqrt{2-6r^2}$，故方向导数为零的所有方向为

$(-r+\frac{1}{2}\sqrt{2-6r^2},-r-\frac{1}{2}\sqrt{2-6r^2},r)$，$-\frac{\sqrt{3}}{3}\le r\le\frac{\sqrt{3}}{3}$

8. 设 $z=u(x,y)$ 在 $\mathbb{R}^2\setminus\{(0,0)\}$ 内可微. 令 $x=r\cos\theta$, $y=r\sin\theta$ 在 Oxy 平面上作单位向量 e_r, e_θ. e_r 表示 θ 固定时沿 r 增加的方向, e_θ 表示 r 固定时沿 θ 增加的方向. 证明: $\frac{\partial u}{\partial e_r}=\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial e_\theta}=\frac{1}{r}\cdot\frac{\partial u}{\partial \theta}$

证明: $z=u(x,y)=u(r\cos\theta, r\sin\theta)$. $\vec{e_r}=(\cos\theta,\sin\theta)$. $\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta$. $\frac{\partial u}{\partial e_r}=\operatorname{grad}u(x,y)\cdot\vec{e_r}$ $=(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y})\cdot\vec{e_r}=\frac{\partial u}{\partial r}$. $\vec{e_\theta}=(-\sin\theta,\cos\theta)$, $\frac{\partial u}{\partial \theta}=-\frac{\partial u}{\partial x}r\sin\theta+\frac{\partial u}{\partial y}r\cos\theta$, $\frac{\partial u}{\partial e_\theta}=\operatorname{grad}u(x,y)\cdot\vec{e_\theta}=$ $(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y})\cdot(-\sin\theta,\cos\theta)=\frac{1}{r}\cdot\frac{\partial u}{\partial \theta}$ #

9. 举出一个 $f(x)$ $(x\in\mathbb{R}^n)$ 满足以下三条: (1) $f(x)$ 在 $x=0$ 处各方向导数存在 (2) $f(x)$ 在 $x=0$ 处各偏导数存在 (3) $f(x)$ 在 $x=0$ 处连续但不可微

解: $f(x_1,x_2,\cdots,x_n)=\begin{cases}\frac{x_1^2x_2^2\cdots x_n^2}{(x_1^2+x_2^2+\cdots+x_n^2)^{(2n-1)/2}}, & x_1^2+x_2^2+\cdots+x_n^2\neq 0\\ 0, & 其他\end{cases}$. $|f(x)|\le\frac{1}{n^{(2n-1)/2}}|x_1x_2\cdots x_n|^{1/n}\to 0\Rightarrow$ $f(x)$ 在 $x=0$ 处连续. 设 $\vec{v}=(v_1,v_2,\cdots,v_n)$ 是任一方向, 则 $\frac{\partial f(0)}{\partial \vec{v}}=\lim\limits_{t\to 0}\frac{f(tv_1,tv_2,\cdots,tv_n)-f(0)}{t}=v_1^2v_2^2\cdots v_n^2$ $\frac{\partial f(0)}{\partial x_i}=0$. 若 $f(x)$ 在 $x=0$ 处可微, 则 $f(x)=o(|x|)$ $(|x|\to 0)$, 即 $\lim\limits_{|x|\to 0}\frac{x_1^2x_2^2\cdots x_n^2}{(x_1^2+x_2^2+\cdots+x_n^2)^n}=0$. 但取 $x_1=x_2=\cdots=x_n$, 式子等于 $\frac{1}{n^n}$, 矛盾. 故 $f(x)$ 在 $x=0$ 处不可微

10. 设 $\mathbb{R}^n$ 上函数 $f(x)=\begin{cases}|x|^2\sin\frac{1}{|x|^2}, & |x|\neq 0\\ 0, & |x|=0\end{cases}$, 证明: $\frac{\partial f(x)}{\partial x_i}$ 在 $x=0$ 处不连续, 但 $f(x)$ 在 $\mathbb{R}^n$ 上处处可微

证明: $\frac{\partial f(x)}{\partial x_i}=\begin{cases}2x_i\sin\frac{1}{|x|^2}-\frac{2x_i}{|x|^2}\cos\frac{1}{|x|^2}, & |x|\neq 0\\ 0, & |x|=0\end{cases}$ 在 $x=0$ 的邻域内无界, 所以不连续. 显然 $f(x)$ 在 $\mathbb{R}^n\setminus\{0\}$ 可微.
在 $x=0$ 附近, $f(\Delta x)=|\Delta x|^2\sin\frac{1}{|\Delta x|^2}=o(|\Delta x|)=\sum\limits_{i=1}^{n}\frac{\partial f(0)}{\partial x_i}\Delta x_i+o(|\Delta x|)$ $(|\Delta x|\to 0)$, 从而 $f(x)$ 在 $x=0$ 可微 #

11. 求函数在某点处的微分: (1) $f(x,y)=3x^2-xy^2+y^2$, $(1,2)$ (2) $f(x,y)=xe^y+x^y$, $(1,0)$

解: (1) $\frac{\partial f(1,2)}{\partial x}=(6x-y^2)|_{(1,2)}=2$, $\frac{\partial f(1,2)}{\partial y}=(-2xy+2y)|_{(1,2)}=0$. $df(1,2)=2dx$

(2) $\frac{\partial f(1,0)}{\partial x}=(e^y+yx^{y-1})|_{(1,0)}=1$, $\frac{\partial f(1,0)}{\partial y}=(xe^y+x^y\ln x)|_{(1,0)}=1$. $df(1,0)=dx+dy$

12. 求微分: (1) $f(x,y)=y^2\sin x+2x^2y$ (2) $f(x,y)=xe^{-2y}+3y^4$ (3) $f(x,y,z)=y^2\ln[(x^2+2)\cdot(z^2+1)]$
(4) $f(x)=|x|$, $x\in\mathbb{R}^n\setminus\{0\}$ (5) $f(x)=\ln|x|$, $x\in\mathbb{R}^n\setminus\{0\}$

解: (1) $df=(y^2\cos x+4xy)dx+(2y\sin x+2x^2)dy$ (2) $df=e^{-2y}dx+(-2xe^{-2y}+12y^3)dy$
(3) $df=\frac{2xy^2}{x^2+2}dx+2y\ln[(x^2+2)(z^2+1)]dy+\frac{2y^2z}{z^2+1}dz$ (4) $df=\sum\limits_{i=1}^{n}\frac{x_i}{|x|}dx_i$ (5) $df=\sum\limits_{i=1}^{n}\frac{x_i}{|x|^2}dx_i$

13. $f(x,y)=x^2y-3y$, 求 $f(x,y)$ 的微分和 $f(5.12,6.85)$ 的近似值

解: $df=2xydx+(x^2-3)dy$. $f(5.12,6.85)\approx f(5,7)+2\times5\times7\times0.12+(5^2-3)\times(-0.15)=159.1$

14. 求近似值: (1) $\sqrt{1.02^3+2.03^2+3.02^2}$ (2) $3.01^{0.99}$

解: (1) 令 $f(x,y,z)=\sqrt{x^3+y^2+z^2}$, $df=\frac{3x^2}{2\sqrt{x^3+y^2+z^2}}dx+\frac{y}{\sqrt{x^3+y^2+z^2}}dy+\frac{z}{\sqrt{x^3+y^2+z^2}}dz$. $\sqrt{1.02^3+2.03^2+3.02^2}=f(1.02,2.03,3.02)$ $\approx f(1,2,3)+\frac{3}{2\sqrt{14}}\times0.02+\frac{2}{\sqrt{14}}\times0.03+\frac{3}{\sqrt{14}}\times0.02\approx3.78$

(2) 令 $f(x,y)=x^y$. $df=yx^{y-1}dx+x^y\ln x\,dy$. $3.01^{0.99}=f(3.01,0.99)\approx f(3,1)+0.01-3\ln3\cdot0.01\approx2.98$

15. 设 $u=f(x)$ 在 $x_0\in\mathbb{R}^n$ 的邻域 $U(x_0,\delta_0)$ 内有各个偏导数, 且有 $n-1$ 个偏导数在该邻域内连续. 证明: $f(x)$ 在 x_0 处可微

证明: $n=2$ 时, 设 $f'_x(x,y)$ 连续. $f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0+\Delta y)+f(x_0,y_0+\Delta y)-f(x_0,y_0)=$ $(f'_x(x_0+\theta\Delta x,y_0+\Delta y)-f'_x(x_0,y_0))\Delta x+f'_x(x_0,y_0)\Delta x+f'_y(x_0,y_0)\Delta y+o(\Delta y)=f'_x(x_0,y_0)\Delta x+f'_y(x_0,y_0)\Delta y+o(\Delta x)+o(\Delta y)$ $=f'_x(x_0,y_0)\Delta x+f'_y(x_0,y_0)\Delta y+o(\sqrt{(\Delta x)^2+(\Delta y)^2})$ $((\Delta x,\Delta y)\to(0,0))$ $\Rightarrow f(x,y)$ 在 (x_0,y_0) 处可微. 假设该结论对 $n-1$ 成立, 考虑 n 的情形. 设 $x_0=(x_1^0,x_2^0,\cdots,x_n^0)$, $\Delta x=(\Delta x_1,\Delta x_2,\cdots,\Delta x_n)$, $f'_{x_1},\cdots,f'_{x_{n-1}}$ 连续, 则

8. 设 $z=u(x,y)$ 在 $\mathbb{R}^2\setminus\{(0,0)\}$ 内可微. 令 $x=r\cos\theta$, $y=r\sin\theta$. 在 Oxy 平面上作单位向量 e_r, e_θ. e_r 表示 θ 固定时沿 r 增加的方向, $\vec{e_\theta}$ 表示 r 固定时沿 θ 增加的方向. 证明: $\frac{\partial u}{\partial e_r}=\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial e_\theta}=\frac{1}{r}\cdot\frac{\partial u}{\partial\theta}$

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解: (1) 令 $f(x,y,z)=\sqrt{x^3+y^2+z^2}$. $df=\frac{3x^2}{2\sqrt{x^3+y^2+z^2}}dx+\frac{y}{\sqrt{x^3+y^2+z^2}}dy+\frac{z}{\sqrt{x^3+y^2+z^2}}dz$. $\sqrt{1.02^3+2.03^2+3.02^2}=f(1.02,2.03,3.02)$ $\approx f(1,2,3)+\frac{3}{2\sqrt{14}}\times0.02+\frac{2}{\sqrt{14}}\times0.03+\frac{3}{\sqrt{14}}\times0.02\approx3.78$

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证明: $n=2$ 时, 设 $f'_x(x,y)$ 连续. $f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0+\Delta y)+f(x_0,y_0+\Delta y)-f(x_0,y_0)=$ $(f'_x(x_0+\theta\Delta x,y_0+\Delta y)-f'_x(x_0,y_0))\Delta x+f'_x(x_0,y_0)\Delta x+f'_y(x_0,y_0)\Delta y+o(\Delta y)=f'_x(x_0,y_0)\Delta x+f'_y(x_0,y_0)\Delta y+o(\Delta x)+o(\Delta y)$ $=f'_x(x_0,y_0)\Delta x+f'_y(x_0,y_0)\Delta y+o(\sqrt{(\Delta x)^2+(\Delta y)^2})$ $((\Delta x,\Delta y)\to(0,0))$ $\Rightarrow f(x,y)$ 在 (x_0,y_0) 处可微. 假设该结论对 $n-1$ 成立, 考虑 n 的情形. 设 $x_0=(x_1^0,x_2^0,\cdots,x_n^0)$, $\Delta x=(\Delta x_1,\Delta x_2,\cdots,\Delta x_n)$, $f'_{x_1},\cdots,f'_{x_{n-1}}$ 连续, 则

$f(x_1^0+\Delta x_1,\cdots,x_n^0+\Delta x_n)-f(x_1^0,\cdots,x_n^0)=f(x_1^0+\Delta x_1,\cdots,x_n^0+\Delta x_n)-f(x_1^0,\cdots,x_{n-1}^0,x_n^0+\Delta x_n)+f(x_1^0,\cdots,x_{n-1}^0,x_n^0+\Delta x_n)$
$-f(x_1^0,\cdots,x_n^0)=\sum_{i=1}^{n-1}f'_{x_i}(x_1^0,\cdots,x_{n-1}^0,x_n^0+\Delta x_n)\Delta x_i+o(\sqrt{\sum_{i=1}^{n-1}(\Delta x_i)^2})+f'_{x_n}(x_1^0,\cdots,x_n^0)\Delta x_n+o(\Delta x_n)$
$=\sum_{i=1}^{n}f'_{x_i}(x_1^0,\cdots,x_n^0)\Delta x_i+o(\sqrt{\sum_{i=1}^{n}(\Delta x_i)^2})$ $(\sqrt{\sum_{i=1}^{n}(\Delta x_i)^2}\to 0)$. 即 $f(x_1,\cdots,x_n)$ 在 x_0 处可微 #

16. 求梯度：(1) $f(x,y,z)=x^2\sin yz+y^2e^{xz}+z^2$ (2) $f(x)=|x|e^{-|x|}$, $x\in\mathbb{R}^n\setminus\{0\}$ $(n\geq 2)$

解：(1) $\mathrm{grad}f=(2x\sin yz+y^2ze^{xz},\ x^2z\cos yz+2ye^{xz},\ x^2y\cos yz+xy^2e^{xz}+2z)$

(2) $\mathrm{grad}f=((\frac{1}{|x|}-1)x_1e^{-|x|},\cdots,(\frac{1}{|x|}-1)x_ne^{-|x|})$

17. 求 $f(x,y,z)=x^3+y^3+z^3-3xyz$ 在 $\mathbb{R}^3$ 中各点处的梯度. 并求 (x,y,z)，使得该点处的梯度分别垂直于z轴、平行于z轴、为0

解：$\mathrm{grad}f=(3x^2-3yz,3y^2-3xz,3z^2-3xy)$. 垂直于z轴：$3z^2-3xy=0$，曲面 $z^2=xy$. 平行于z轴：$3x^2-3yz=0$ 且 $3y^2-3xz=0$，曲线 $\begin{cases}x^2=yz\\ y^2=xz\end{cases}$. 为0：$3x^2-3yz=3y^2-3xz=3z^2-3xy=0$，直线 $x=y=z$

18. 设 $f(x,y)$ 在 (x_0,y_0) 处可微，沿 $(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$ 的方向导数为 $\frac{3\sqrt{2}}{2}$，沿 $(\frac{\sqrt{3}}{2},\frac{1}{2})$ 的方向导数为 $1+\frac{3\sqrt{2}}{2}$，求 $f(x,y)$ 在 (x_0,y_0) 处沿 $\vec{i},\vec{j}$ 的方向导数、梯度

解：设 $\mathrm{grad}f(x_0,y_0)=(a,b)$，则 $(a,b)\cdot(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})=\frac{3\sqrt{2}}{2}=\frac{\sqrt{2}}{2}(a+b)$，$(a,b)\cdot(\frac{\sqrt{3}}{2},\frac{1}{2})=1+\frac{3\sqrt{2}}{2}=\frac{\sqrt{3}}{2}a+\frac{1}{2}b\Rightarrow\begin{cases}a=\frac{3\sqrt{6}-\sqrt{3}+3\sqrt{2}-1}{2}\\ b=\frac{-3\sqrt{6}+\sqrt{3}-3\sqrt{2}+7}{2}\end{cases}$

$\mathrm{grad}f(x_0,y_0)=(\frac{3\sqrt{6}-\sqrt{3}+3\sqrt{2}-1}{2},\frac{-3\sqrt{6}+\sqrt{3}-3\sqrt{2}+7}{2})$. $\frac{\partial f(x_0,y_0)}{\partial\vec{i}}=\frac{3\sqrt{6}-\sqrt{3}+3\sqrt{2}-1}{2}$, $\frac{\partial f(x_0,y_0)}{\partial\vec{j}}=\frac{-3\sqrt{6}+\sqrt{3}-3\sqrt{2}+7}{2}$

19. 求 $f(x,y,z)=2x^3y-3y^2z$ 在 $(1,2,-1)$ 处所有方向导数的集合

解：设 $\vec{v}=(u,v,w)$ 是任一方向 $\frac{\partial f(1,2,-1)}{\partial\vec{v}}=\mathrm{grad}f(1,2,-1)\cdot\vec{v}=12u+14v-12w$. $|12u+14v-12w|\leq(12^2+14^2+12^2)^{1/2}\cdot(u^2+v^2+w^2)^{1/2}=22$. 且显然能取等号. 故所求集合为 $[-22,22]$

20. 设 $x\in\mathbb{R}^n$ $(n\geq 2)$，求向量函数的导数：(1) $f(x)=x|x|$ (2) $f(x)=\frac{x}{|x|}$ $(|x|\neq 0)$ (3) 设A为 $n\times n$ 矩阵，$f(x)=|Ax|^2$，x是列向量

解：(1) $f'(x)=\begin{pmatrix}|x|+\frac{x_1^2}{|x|} & \frac{x_1x_2}{|x|} & \cdots & \frac{x_1x_n}{|x|}\\ \frac{x_1x_2}{|x|} & |x|+\frac{x_2^2}{|x|} & \cdots & \frac{x_2x_n}{|x|}\\ \vdots & \vdots & & \vdots\\ \frac{x_1x_n}{|x|} & \frac{x_2x_n}{|x|} & \cdots & |x|+\frac{x_n^2}{|x|}\end{pmatrix}$ (2) $f'(x)=\begin{pmatrix}\frac{1}{|x|}-\frac{x_1^2}{|x|^3} & -\frac{x_1x_2}{|x|^3} & \cdots & -\frac{x_1x_n}{|x|^3}\\ -\frac{x_1x_2}{|x|^3} & \frac{1}{|x|}-\frac{x_2^2}{|x|^3} & \cdots & -\frac{x_2x_n}{|x|^3}\\ \vdots & \vdots & & \vdots\\ -\frac{x_1x_n}{|x|^3} & -\frac{x_2x_n}{|x|^3} & \cdots & \frac{1}{|x|}-\frac{x_n^2}{|x|^3}\end{pmatrix}$

(3) $f(x)=\sum_{j=1}^{n}(\sum_{k=1}^{n}a_{jk}x_k)^2$, $f'(x)=(2\sum_{j=1}^{n}a_{j1}\sum_{k=1}^{n}a_{jk}x_k,\ 2\sum_{j=1}^{n}a_{j2}\sum_{k=1}^{n}a_{jk}x_k,\cdots,\ 2\sum_{j=1}^{n}a_{jn}\sum_{k=1}^{n}a_{jk}x_k)$

21. 设 $f(u)=f(u_1,u_2,\cdots,u_m)$ 在区域 $\Omega\subset\mathbb{R}^m$ 内有定义，在 $u_0=(u_1^0,u_2^0,\cdots,u_m^0)\in\Omega$ 处可微. 设 $u=u(x)=(u_1(x),u_2(x),\cdots,u_m(x))$ 在区域 $D\subset\mathbb{R}^n$ 内有定义，在 $x_0=(x_1^0,x_2^0,\cdots,x_n^0)\in D$ 处可偏导，$u_0=u(x_0)$.

证明：$\forall 1\leq i\leq n$，$f(u(x))$ 在 x_0 处关于 x_i 可偏导，且 $\frac{\partial f(u(x_0))}{\partial x_i}=\sum_{j=1}^{m}\frac{\partial f(u_0)}{\partial u_j}\cdot\frac{\partial u_j(x_0)}{\partial x_i}$

证明：由 $f(u)$ 在 u_0 处可微，$f(u_0+\Delta u)-f(u_0)=\sum_{j=1}^{m}\frac{\partial f(u_0)}{\partial u_j}\Delta u_j+o(|\Delta u|)$ $(|\Delta u|\to 0)$. 由 $u(x)$ 在 x_0 处可偏导，$u(x_1^0,\cdots,x_i^0+\Delta x_i,\cdots,x_n^0)-u(x_1^0,\cdots,x_n^0)=(\frac{\partial u_1(x_0)}{\partial x_i}\Delta x_i+o(\Delta x_i),\cdots,\frac{\partial u_m(x_0)}{\partial x_i}\Delta x_i+o(\Delta x_i))$ $(\Delta x_i\to 0)$. 于是

$f(u(x_1^0,\cdots,x_i^0+\Delta x_i,\cdots,x_n^0))-f(u(x_1^0,\cdots,x_n^0))=\sum_{j=1}^{m}\frac{\partial f(u_0)}{\partial u_j}(\frac{\partial u_j(x_0)}{\partial x_i}\Delta x_i+o(\Delta x_i))+o(|\Delta u|)$ <u>$\frac{|\Delta u|}{\Delta x_i}$ 在 x_0 附近有界</u>

$\sum_{j=1}^{m}\frac{\partial f(u_0)}{\partial u_j}\cdot\frac{\partial u_j(x_0)}{\partial x_i}\Delta x_i+o(\Delta x_i)\Rightarrow\frac{\partial f(u(x_0))}{\partial x_i}=\sum_{j=1}^{m}\frac{\partial f(u_0)}{\partial u_j}\cdot\frac{\partial u_j(x_0)}{\partial x_i}$ #

22. 设 f 可微，求偏导数：(1) $z=f(xe^y,xe^{-y})$ (2) $u=f(\sum_{i=1}^{n}x_i^2,\prod_{i=1}^{n}x_i^2,x_3,\cdots,x_n)$

解：(1) $\frac{\partial z}{\partial x}=e^yf_1'+e^{-y}f_2'$, $\frac{\partial z}{\partial y}=xe^yf_1'-xe^{-y}f_2'$ (2) $\frac{\partial f}{\partial x_1}=2x_1f_1'+2x_1x_2^2\cdots x_n^2f_2'$, $\frac{\partial f}{\partial x_2}=2x_2f_1'+2x_1^2x_2x_3^2\cdots x_n^2f_2'$

对 $3\leq k\leq n$，$\frac{\partial f}{\partial x_k}=2x_kf_1'+2x_k\prod_{i\neq k}x_i^2f_2'+f_k'$

23. 设 $u=f(x)$ 在区域 $D\subset\mathbb{R}^n$ 内存在n个连续偏导数，且各偏导数有界. (1) 证明D是凸区域时，$f(x)$ 在D内一致连续 (2) 说明D不是凸区域时，$f(x)$ 在D内可能不一致连续

证明：(1) 设 $f(x)$ 的任一偏导数绝对值 $\leq M$. $\forall\varepsilon>0$，取 $\delta=\frac{\varepsilon}{nM}$，当 $x'=(x_1',\cdots,x_n')$, $x''=(x_1'',\cdots,x_n'')\in D$，且 $|x_i'-x_i''|<\delta$ $(i=1,2,\cdots,n)$ 时，连接 x',x'' 的线段在D内. 取N充分大，把该线段N等分：$x'=t_0,t_1,\cdots,t_N=x''$，存在包含 t_{k-1},t_k 的方形区域含于D $(k=1,\cdots,N)$ 设 $t_{k-1}=(t_1^{k-1},\cdots,t_n^{k-1})$, $t_k=(t_1^k,\cdots,t_n^k)$. $|f(t_{k-1})-f(t_k)|$
$\leq|f(t_1^{k-1},\cdots,t_n^{k-1})-f(t_1^{k-1},\cdots,t_{n-1}^{k-1},t_n^k)|+|f(t_1^{k-1},\cdots,t_{n-1}^{k-1},t_n^k)-f(t_1^{k-1},\cdots,t_{n-2}^{k-1},t_{n-1}^k,t_n^k)|+\cdots+|f(t_1^{k-1},t_2^k,\cdots,t_n^k)-f(t_1^k,\cdots,t_n^k)|$
$\leq M\sum_{i=1}^{n}|t_i^{k-1}-t_i^k|\Rightarrow|f(x')-f(x'')|\leq M\sum_{i=1}^{n}|x_i'-x_i''|<\varepsilon$. 故 $f(x)$ 在D内一致连续

(2) 取 $D=N((0,0),1)\setminus\{(x,y):x=0,0\leq y<1\}$，$f(x,y)=\begin{cases}0, & x>0,y>0,(x,y)\in D\\ y^2, & (x,y)\in D\text{ 且 }x\leq 0\text{ 或 }y\leq 0\end{cases}$. 则 $f'_x(x,y)=0$，$f'_y(x,y)=\begin{cases}0, & x>0,y>0,(x,y)\in D\\ 2y, & (x,y)\in D\text{ 且 }x\leq 0\text{ 或 }y\leq 0\end{cases}$

在D连续、有界. 但对 $\varepsilon_0=\frac{1}{4}$，$\forall\delta>0$，取 $(x',y')=(-\frac{\delta}{2},\frac{1}{2})$, $(x'',y'')=(\frac{\delta}{2},\frac{1}{2})$. $f(x',y')-f(x'',y'')=\frac{1}{4}$. $f(x,y)$ 在D不一致连续 #

$f(x_1^0+\Delta x_1,\cdots,x_n^0+\Delta x_n)-f(x_1^0,\cdots,x_n^0)=f(x_1^0+\Delta x_1,\cdots,x_n^0+\Delta x_n)-f(x_1^0,\cdots,x_{n-1}^0,x_n^0+\Delta x_n)+f(x_1^0,\cdots,x_{n-1}^0,x_n^0+\Delta x_n)$
$-f(x_1^0,\cdots,x_n^0)=\sum_{i=1}^{n-1}f'_{x_i}(x_1^0,\cdots,x_{n-1}^0,x_n^0+\Delta x_n)\Delta x_i+o(\sqrt{\sum_{i=1}^{n-1}(\Delta x_i)^2})+f'_{x_n}(x_1^0,\cdots,x_n^0)\Delta x_n+o(\Delta x_n)$
$=\sum_{i=1}^{n}f'_{x_i}(x_1^0,\cdots,x_n^0)\Delta x_i+o(\sqrt{\sum_{i=1}^{n}(\Delta x_i)^2})$ $(\sqrt{\sum_{i=1}^{n}(\Delta x_i)^2}\to 0)$. 即 $f(x_1,\cdots,x_n)$ 在 x_0 处可微 #

16. 求梯度：(1) $f(x,y,z)=x^2\sin yz+y^2e^{xz}+z^2$ (2) $f(x)=|x|e^{-|x|}$, $x\in\mathbb{R}^n\setminus\{0\}$ $(n\geq 2)$

解：(1) $\operatorname{grad}f=(2x\sin yz+y^2ze^{xz},\ x^2z\cos yz+2ye^{xz},\ x^2y\cos yz+xy^2e^{xz}+2z)$

(2) $\operatorname{grad}f=((\frac{1}{|x|}-1)x_1e^{-|x|},\cdots,(\frac{1}{|x|}-1)x_ne^{-|x|})$

17. 求 $f(x,y,z)=x^3+y^3+z^3-3xyz$ 在 $\mathbb{R}^3$ 中各点处的梯度，并求 (x,y,z)，使得该点处的梯度分别垂直于 z 轴、平行于 z 轴、为 0

解：$\operatorname{grad}f=(3x^2-3yz,3y^2-3xz,3z^2-3xy)$. 垂直于 z 轴：$3z^2-3xy=0$，曲面 $z^2=xy$. 平行于 z 轴：$3x^2-3yz=0$ 且 $3y^2-3xz=0$，曲线 $\begin{cases}x^2=yz\\y^2=xz\end{cases}$. 为 0：$3x^2-3yz=3y^2-3xz=3z^2-3xy=0$，直线 $x=y=z$

18. 设 $f(x,y)$ 在 (x_0,y_0) 处可微，沿 $(\frac{\sqrt2}{2},\frac{\sqrt2}{2})$ 的方向导数为 $\frac{3\sqrt2}{2}$，沿 $(\frac{\sqrt3}{2},\frac12)$ 的方向导数为 $1+\frac{3\sqrt2}{2}$，求 $f(x,y)$ 在 (x_0,y_0) 处沿 $\vec{i},\vec{j}$ 的方向导数、梯度

解：设 $\operatorname{grad}f(x_0,y_0)=(a,b)$，则 $(a,b)\cdot(\frac{\sqrt2}{2},\frac{\sqrt2}{2})=\frac{3\sqrt2}{2}=\frac{\sqrt2}{2}(a+b)$，$(a,b)\cdot(\frac{\sqrt3}{2},\frac12)=1+\frac{3\sqrt2}{2}=\frac{\sqrt3}{2}a+\frac12 b\Rightarrow\begin{cases}a=\frac{3\sqrt6-\sqrt3+3\sqrt2-1}{2}\\b=\frac{-3\sqrt6+\sqrt3-3\sqrt2+7}{2}\end{cases}$
$\operatorname{grad}f(x_0,y_0)=(\frac{3\sqrt6-\sqrt3+3\sqrt2-1}{2},\frac{-3\sqrt6+\sqrt3-3\sqrt2+7}{2})$. $\frac{\partial f(x_0,y_0)}{\partial\vec{i}}=\frac{3\sqrt6-\sqrt3+3\sqrt2-1}{2}$, $\frac{\partial f(x_0,y_0)}{\partial\vec{j}}=\frac{-3\sqrt6+\sqrt3-3\sqrt2+7}{2}$

19. 求 $f(x,y,z)=2x^3y-3y^2z$ 在 $(1,2,-1)$ 处所有方向导数的集合

解：设 $\vec{v}=(u,v,w)$ 是任一方向 $\frac{\partial f(1,2,-1)}{\partial\vec{v}}=\operatorname{grad}f(1,2,-1)\cdot\vec{v}=12u+14v-12w$. $|12u+14v-12w|\leq(12^2+14^2+12^2)^{1/2}\cdot(u^2+v^2+w^2)^{1/2}=22$. 且显然能取等号. 故所求集合为 $[-22,22]$

20. 设 $x\in\mathbb{R}^n$ $(n\geq2)$，求向量函数的导数：(1) $f(x)=x|x|$ (2) $f(x)=\frac{x}{|x|}$ $(|x|\neq0)$ (3) 设 A 为 $n\times n$ 矩阵，$f(x)=|Ax|^2$，x 是列向量

解：(1) $f'(x)=\begin{pmatrix}|x|+\frac{x_1^2}{|x|}&\frac{x_1x_2}{|x|}&\cdots&\frac{x_1x_n}{|x|}\\\frac{x_1x_2}{|x|}&|x|+\frac{x_2^2}{|x|}&\cdots&\frac{x_2x_n}{|x|}\\\vdots&\vdots&&\vdots\\\frac{x_1x_n}{|x|}&\frac{x_2x_n}{|x|}&\cdots&|x|+\frac{x_n^2}{|x|}\end{pmatrix}$ (2) $f'(x)=\begin{pmatrix}\frac{1}{|x|}-\frac{x_1^2}{|x|^3}&-\frac{x_1x_2}{|x|^3}&\cdots&-\frac{x_1x_n}{|x|^3}\\-\frac{x_1x_2}{|x|^3}&\frac{1}{|x|}-\frac{x_2^2}{|x|^3}&\cdots&-\frac{x_2x_n}{|x|^3}\\\vdots&\vdots&&\vdots\\-\frac{x_1x_n}{|x|^3}&-\frac{x_2x_n}{|x|^3}&\cdots&\frac{1}{|x|}-\frac{x_n^2}{|x|^3}\end{pmatrix}$

(3) $f(x)=\sum_{j=1}^{n}(\sum_{k=1}^{n}a_{jk}x_k)^2$, $f'(x)=(2\sum_{j=1}^{n}a_{j1}\sum_{k=1}^{n}a_{jk}x_k,\ 2\sum_{j=1}^{n}a_{j2}\sum_{k=1}^{n}a_{jk}x_k,\cdots,\ 2\sum_{j=1}^{n}a_{jn}\sum_{k=1}^{n}a_{jk}x_k)$

21. 设 $f(u)=f(u_1,u_2,\cdots,u_m)$ 在区域 $\Omega\subset\mathbb{R}^m$ 内有定义，在 $u_0=(u_1^0,u_2^0,\cdots,u_m^0)\in\Omega$ 处可微. 设 $u=u(x)=(u_1(x),u_2(x),\cdots,u_m(x))$ 在区域 $D\subset\mathbb{R}^n$ 内有定义，在 $x_0=(x_1^0,x_2^0,\cdots,x_n^0)\in D$ 处可偏导，$u_0=u(x_0)$.

证明：$\forall 1\leq i\leq n$，$f(u(x))$ 在 x_0 处关于 x_i 可偏导，且 $\frac{\partial f(u(x_0))}{\partial x_i}=\sum_{j=1}^{m}\frac{\partial f(u_0)}{\partial u_j}\cdot\frac{\partial u_j(x_0)}{\partial x_i}$

证明：由 $f(u)$ 在 u_0 处可微，$f(u_0+\Delta u)-f(u_0)=\sum_{j=1}^{m}\frac{\partial f(u_0)}{\partial u_i}\Delta u_i+o(|\Delta u|)$ $(|\Delta u|\to0)$. 由 $u(x)$ 在 x_0 处可偏导，
$u(x_1^0,\cdots,x_i^0+\Delta x_i,\cdots,x_n^0)-u(x_1^0,\cdots,x_n^0)=(\frac{\partial u_1(x_0)}{\partial x_i}\Delta x_i+o(\Delta x_i),\cdots,\frac{\partial u_m(x_0)}{\partial x_i}\Delta x_i+o(\Delta x_i))$ $(\Delta x_i\to0)$，于是
$f(u(x_1^0,\cdots,x_i^0+\Delta x_i,\cdots,x_n^0))-f(u(x_1^0,\cdots,x_n^0))=\sum_{j=1}^{m}\frac{\partial f(u_0)}{\partial u_j}(\frac{\partial u_j(x_0)}{\partial x_i}\Delta x_i+o(\Delta x_i))+o(|\Delta u|)$ $\underline{\underline{\frac{|\Delta u|}{\Delta x_i}\text{在}x_0\text{附近有界}}}$
$\sum_{j=1}^{m}\frac{\partial f(u_0)}{\partial u_j}\cdot\frac{\partial u_j(x_0)}{\partial x_i}\Delta x_i+o(\Delta x_i)\Longrightarrow\frac{\partial f(u(x_0))}{\partial x_i}=\sum_{j=1}^{m}\frac{\partial f(u_0)}{\partial u_j}\cdot\frac{\partial u_j(x_0)}{\partial x_i}$ #

22. 设 f 可微，求偏导数：(1) $z=f(xe^y,xe^{-y})$ (2) $u=f(\sum_{i=1}^{n}x_i^2,\prod_{i=1}^{n}x_i^2,x_3,\cdots,x_n)$

解：(1) $\frac{\partial z}{\partial x}=e^yf_1'+e^{-y}f_2'$, $\frac{\partial z}{\partial y}=xe^yf_1'-xe^{-y}f_2'$ (2) $\frac{\partial f}{\partial x_1}=2x_1f_1'+2x_1x_2^2\cdots x_n^2f_2'$, $\frac{\partial f}{\partial x_2}=2x_2f_1'+2x_1^2x_2x_3^2\cdots x_n^2f_2'$
对 $3\leq k\leq n$, $\frac{\partial f}{\partial x_k}=2x_kf_1'+2x_k\prod_{i\neq k}x_i^2f_2'+f_k'$

23. 设 $u=f(x)$ 在区域 $D\subset\mathbb{R}^n$ 内存在 n 个连续偏导数，且各偏导数有界. (1) 证明 D 是凸域时，$f(x)$ 在 D 内一致连续 (2) 说明 D 不是凸域时，$f(x)$ 在 D 内可能不一致连续

证明：(1) 设 $f(x)$ 的任一偏导数绝对值 $\leq M$. $\forall\varepsilon>0$，取 $\delta=\frac{\varepsilon}{nM}$，当 $x'=(x_1',\cdots,x_n')$，$x''=(x_1'',\cdots,x_n'')\in D$，且 $|x_i'-x_i''|<\delta$ $(i=1,2,\cdots,n)$ 时，连接 x',x'' 的线段在 D 内. 取 N 充分大，把该线段 N 等分：$x'=t_0,t_1,\cdots,t_N=x''$，存在包含 t_{k-1},t_k 的方形区域含于 D $(k=1,\cdots,N)$ 设 $t_{k-1}=(t_1^{k-1},\cdots,t_n^{k-1})$，$t_k=(t_1^k,\cdots,t_n^k)$. $|f(t_{k-1})-f(t_k)|$
$\leq|f(t_1^{k-1},\cdots,t_n^{k-1})-f(t_1^{k-1},\cdots,t_{n-1}^{k-1},t_n^k)|+|f(t_1^{k-1},\cdots,t_{n-1}^{k-1},t_n^k)-f(t_1^{k-1},\cdots,t_{n-2}^{k-1},t_{n-1}^k,t_n^k)|+\cdots+|f(t_1^{k-1},t_2^k,\cdots,t_n^k)-f(t_1^k,\cdots,t_n^k)|$
$\leq M\sum_{i=1}^{n}|t_i^{k-1}-t_i^k|\Longrightarrow|f(x')-f(x'')|\leq M\sum_{i=1}^{n}|x_i'-x_i''|<\varepsilon$. 故 $f(x)$ 在 D 内一致连续

(2) 取 $D=N((0,0),1)\setminus\{(x,y):x=0,0\leq y<1\}$，$f(x,y)=\begin{cases}0, & x>0,y>0,(x,y)\in D\\y^2, & (x,y)\in D\text{且}x\leq0\text{或}y\leq0\end{cases}$，则 $f'_x(x,y)=0$，$f'_y(x,y)=\begin{cases}0, & x>0,y>0,(x,y)\in D\\2y, & (x,y)\in D\text{且}x\leq0\text{或}y\leq0\end{cases}$ 在 D 连续、有界，但对 $\varepsilon_0=\frac14$，$\forall\delta>0$，取 $(x',y')=(-\frac{\delta}{2},\frac12)$，$(x'',y'')=(\frac{\delta}{2},\frac12)$. $f(x',y')-f(x'',y'')=\frac14$. $f(x,y)$ 在 D 不一致连续 #

24. 设 $z=f(x,y)$ 在区域 D 内有两个偏导数. (1) 若 D 是凸域且 $\forall (x,y)\in D$, $f'_x(x,y)=0$, 证明存在 $h(y)$, 使得在 D 内, $f(x,y)\equiv h(y)$ (2) 若 $\forall (x,y)\in D$, $f'_x(x,y)=f'_y(x,y)=0$, 证明在 D 内, $f(x,y)\equiv C$ 是常数 (3) 举例说明若 D 不是凸域, (1) 中结论未必成立.

证明: (1) 对固定的 y, 只要 $(x',y),(x'',y)\in D$, 由 D 是凸域, $f(x,y)$ 在 $\{(x,y): x$ 介于 x',x'' 之间$\}$ 有定义, 就有 $f(x',y)-f(x'',y)=f'_x(\xi,y)(x'-x'')=0$. 故存在 $h(y)$, $f(x,y)\equiv h(y)$

(2) 引理: 若 $D=U\cup V$, U,V 是不交的非空开集, 则 D 必不是道路连通的, 从而不是区域

引理的证明: 若 D 道路连通, 则存在道路 $\sigma:[0,1]\to D$ 连接 $p\in U$ 和 $q\in V$, $\sigma(0)=p\neq q=\sigma(1)\Rightarrow$ $[0,1]=\sigma^{-1}(D)=\sigma^{-1}(U\cup V)=\sigma^{-1}(U)\cup\sigma^{-1}(V)$. 由 σ 连续, $\sigma^{-1}(U)$ 和 $\sigma^{-1}(V)$ 是不交的开集, 且 $0\in\sigma^{-1}(U)$, $1\in\sigma^{-1}(V)$. 令 $\xi=\sup\{t\mid t\in\sigma^{-1}(U)\}\in[0,1]$. 显然 $0<\xi<1$. 若 $\xi\in\sigma^{-1}(U)$, 则 $\exists\delta>0$, 使得 $\xi+\delta\in\sigma^{-1}(U)$, 与 ξ 是上确界矛盾 $\Rightarrow\xi\notin\sigma^{-1}(U)$. 若 $\xi\in\sigma^{-1}(V)$, 则 $\exists\delta>0$, 使得 $[\xi-\delta,\xi]\subset\sigma^{-1}(V)$, 亦矛盾 $\Rightarrow\xi\notin\sigma^{-1}(V)\Rightarrow\xi\notin\sigma^{-1}(U)\cup\sigma^{-1}(V)=[0,1]$. 矛盾. 引理获证.

回到原题. $\forall(x,y)\in D$, $\exists\delta>0$, 使得 $N((x,y),\delta)\subset D$. $\forall(x',y')\in N((x,y),\delta)$, $f(x',y')=f(x',y')-f(x',y)+f(x',y)-f(x,y)+f(x,y)=f(x,y)$. 任取 $\vec{x_0}\in D$. 令 $U=\{\vec{x}\in D\mid f(\vec{x})=f(\vec{x_0})\}$, $V=\{\vec{x}\in D\mid f(\vec{x})\neq f(\vec{x_0})\}$. U 是开集. 由第15题 $f(x,y)$ 在 D 可微, 从而连续 $\Rightarrow$ V 是开集. $U\cap V=\phi$ 且 $D=U\cup V$. 由 $\vec{x_0}\in U$ 知 $U\neq\phi$. 从而由引理, 必有 $V=\phi$. 故在 D 内 $f(x,y)$ 为常值

(3) 取 $D=N((0,0),1)\setminus\{(x,y): x=0, 0\le y<1\}$, $f(x,y)=\begin{cases}0, & x>0,y>0,(x,y)\in D\\ y^2, & x\le 0 \text{ 或 } y\le 0,\ (x,y)\in D\end{cases}$ #

25. 若 $D\subset\mathbb{R}^n\ (n\ge 2)$ 上的函数 $f(x)$ 满足 $f(tx)=t^Kf(x)$ 对 $\forall t>0,\forall x\in D$ 成立 $(K\in\mathbb{Z}_+)$, 则称 $f(x)$ 是 K 次齐次函数. 设 K 次齐次函数 $f(x)$ 在 D 内有各个 $k\ (1\le k\le K)$ 阶连续偏导数, 证明 $\left(\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}\right)^k f(x)=K(K-1)\cdots(K-k+1)f(x)$

证明: 在 $f(tx)=t^Kf(x)$ 两边对 t 求 k 阶导数: $\left(\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}\right)^k f(tx)=K(K-1)\cdots(K-k+1)t^{K-k}f(x)$

令 $t=1$: $\left(\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}\right)^k f(x)=K(K-1)\cdots(K-k+1)f(x)$ #

26. 设 $z=e^{xy^2}$, 其中 $x=t\cos t$, $y=t\sin t$, 求 $\left.\frac{dz}{dt}\right|_{t=\frac{\pi}{2}}$

解: $\left.\frac{dz}{dt}\right|_{t=\frac{\pi}{2}}=\left.\left(\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}\right)\right|_{t=\frac{\pi}{2}}=\left.\left(y^2e^{xy^2}(\cos t-t\sin t)+xe^{xy^2}(\sin t+t\cos t)\right)\right|_{t=\frac{\pi}{2}}=-\frac{\pi^3}{8}$

27. 设 $u=z\sin\frac{y}{x}$, 其中 $x=3r^2+2s$, $y=4r-2s^3$, $z=2r^2-3s^2$. 求 $\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial s}$

解: $\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial r}=2z\cos\frac{y}{x}\left(-\frac{3ry}{x^2}+\frac{2}{x}\right)+4r\sin\frac{y}{x}$

$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}=2z\cos\frac{y}{x}\left(-\frac{y}{x^2}-\frac{3s^2}{x}\right)-6s\sin\frac{y}{x}$

28. 设 $x=r\cos\alpha-t\sin\alpha$, $y=r\sin\alpha+t\cos\alpha$, $\alpha\in\mathbb{R}$ 为常数. 证明: 对任何可微的 $f(x,y)$, 有 $\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2=\left(\frac{\partial f}{\partial r}\right)^2+\left(\frac{\partial f}{\partial t}\right)^2$

证明: $\left(\frac{\partial f}{\partial r}\right)^2+\left(\frac{\partial f}{\partial t}\right)^2=\left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}\right)^2+\left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}\right)^2$

$=\left(\frac{\partial f}{\partial x}\right)^2\left(\frac{\partial x}{\partial r}+\frac{\partial y}{\partial r}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\left(\frac{\partial x}{\partial t}+\frac{\partial y}{\partial t}\right)^2=\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2$ #

29. 设 $f(x,y)=\begin{pmatrix}u(x,y)\\v(x,y)\end{pmatrix}=\begin{pmatrix}\frac{x}{x^2+y^2}\\ \frac{-y}{x^2+y^2}\end{pmatrix}$. 证明 $f(x,y)$ 是 $\mathbb{R}^2\setminus\{(0,0)\}$ 到自身的 C^1 同胚映射, 并求 $f(x,y)$ 在 $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$ 处的 Jacobi 行列式

证明: 令 $x=r\cos\theta$, $y=r\sin\theta$, 则 $\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}\cos\theta/r\\-\sin\theta/r\end{pmatrix}$ $(0<r<+\infty, 0\le\theta<2\pi)$

$\Rightarrow$ f 是满的. 且若 $f(x_1,y_1)=f(x_2,y_2)$, 则对应 r,θ 相等. f 是单的. f 连续. $f^{-1}: \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{u}{u^2+v^2}\\-\frac{v}{u^2+v^2}\end{pmatrix}$ 连续. 故 f 是 $\mathbb{R}^2\setminus\{(0,0)\}$ 到自身的 C^1 同胚. $|f'(x,y)|=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\frac{1}{(x^2+y^2)^2}$ #

24. 设 $z=f(x,y)$ 在区域 D 内有两个偏导数. (1) 若 D 是凸域且 $\forall (x,y)\in D$, $f'_x(x,y)=0$, 证明存在 $h(y)$, 使得在 D 内, $f(x,y)\equiv h(y)$ (2) 若 $\forall (x,y)\in D$, $f'_x(x,y)=f'_y(x,y)=0$, 证明在 D 内, $f(x,y)\equiv C$ 是常数 (3) 举例说明若 D 不是凸域, (1) 中结论未必成立.

证明: (1) 对固定的 y, 只要 $(x',y),(x'',y)\in D$, 由 D 是凸域, $f(x,y)$ 在 $\{(x,y): x 介于 x',x'' 之间\}$ 有定义, 就有

~~f(x)-f(x)~~ $f(x',y)-f(x'',y)=f'_x(\xi,y)(x'-x'')=0$. 故存在 $h(y)$, $f(x,y)\equiv h(y)$

(2) 引理: 若 $D=U\cup V$, U,V 是不交的非空开集, 则 D 必不是道路连通的, 从而不是区域

引理的证明: 若 D 道路连通, 则存在道路 $\sigma:[0,1]\to D$ 连接 $p\in U$ 和 $q\in V$, $\sigma(0)=p\neq q=\sigma(1)\Rightarrow$ $[0,1]=\sigma^{-1}(D)=\sigma^{-1}(U\cup V)=\sigma^{-1}(U)\cup\sigma^{-1}(V)$. 由 σ 连续, $\sigma^{-1}(U)$ 和 $\sigma^{-1}(V)$ 是不交的开集, 且 $0\in\sigma^{-1}(U)$, $1\in\sigma^{-1}(V)$. 令 $\xi=\sup\{t\mid t\in\sigma^{-1}(U)\}\in[0,1]$. 显然 $0<\xi<1$. 若 $\xi\in\sigma^{-1}(U)$, 则 $\exists\delta>0$, 使得 $t+\delta\in\sigma^{-1}(U)$. 与 ξ 是上确界矛盾 $\Rightarrow \xi\notin\sigma^{-1}(U)$. 若 $\xi\in\sigma^{-1}(V)$, 则 $\exists\delta>0$, 使得 $[\xi-\delta,\xi]\subset\sigma^{-1}(V)$, 亦矛盾 $\Rightarrow \xi\notin\sigma^{-1}(V)\Rightarrow \xi\notin\sigma^{-1}(U)\cup\sigma^{-1}(V)=[0,1]$. 矛盾. 引理获证.

回到原题. $\forall (x,y)\in D$, $\exists\delta>0$, 使得 $N((x,y),\delta)\subset D$. $\forall (x',y')\in N((x,y),\delta)$, $f(x',y')=f(x',y')-f(x',y)+f(x',y)-f(x,y)+f(x,y)=f(x,y)$. 任取 $x_0\in D$. 令 $U=\{x\in D\mid f(x)=f(x_0)\}$, $V=\{x\in D\mid f(x)\neq f(x_0)\}$. U 是开集. 由第15题 $f(x,y)$ 在 D 可微, 从而连续 $\Rightarrow$ V 是开集. $U\cap V=\phi$ 且 $D=U\cup V$. 由 $x_0\in U$ 知 $U\neq\phi$. 从而由引理, 必有 $V=\phi$. 故在 D 内 $f(x,y)$ 为常值

(3) 取 $D=N((0,0),1)\setminus\{(x,y): x=0, 0\le y<1\}$, $f(x,y)=\begin{cases}0, & x>0,y>0,(x,y)\in D\\ y^2, & x\le 0 或 y\le 0, (x,y)\in D\end{cases}$ #

25. 若 $D\subset\mathbb{R}^n(n\ge 2)$ 上的函数 $f(x)$ 满足 $f(tx)=t^Kf(x)$ 对 $\forall t>0,\forall x\in D$ 成立 $(k\in\mathbb{Z}_+)$. 则称 $f(x)$ 是 K 次齐次函数

设 K 次齐次函数 $f(x)$ 在 D 内有各个 k $(1\le k\le K)$ 阶连续偏导数, 证明 $\left(\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}\right)^k f(x)=K(K-1)\cdots(K-k+1)f(x)$

证明: 在 $f(tx)=t^Kf(x)$ 两边对 t 求 k 阶导数: $\left(\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}\right)^k f(tx)=K(K-1)\cdots(K-k+1)t^{K-k}f(x)$

令 $t=1$: $\left(\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}\right)^k f(x)=K(K-1)\cdots(K-k+1)f(x)$ #

26. 设 $z=e^{xy^2}$, 其中 $x=t\cos t$, $y=t\sin t$, 求 $\frac{dz}{dt}\Big|_{t=\frac{\pi}{2}}$

解: $\frac{dz}{dt}\Big|_{t=\frac{\pi}{2}}=\left(\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}\right)\Big|_{t=\frac{\pi}{2}}=\left(y^2e^{xy^2}(\cos t-t\sin t)+xe^{xy^2}(\sin t+t\cos t)\right)\Big|_{t=\frac{\pi}{2}}=-\frac{\pi^3}{8}$

27. 设 $u=z\sin\frac{y}{x}$, 其中 $x=3r^2+2s$, $y=4r-2s^3$, $z=2r^2-3s^2$. 求 $\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial s}$

解: $\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial r}=2z\cos\frac{y}{x}\left(-\frac{3ry}{x^2}+\frac{2}{x}\right)+4r\sin\frac{y}{x}$

$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}=2z\cos\frac{y}{x}\left(-\frac{y}{x^2}-\frac{3s^2}{x}\right)-6s\sin\frac{y}{x}$

28. 设 $x=r\cos\alpha-t\sin\alpha$, $y=r\sin\alpha+t\cos\alpha$, $\alpha\in\mathbb{R}$ 为常数. 证明: 对任何可微的 $f(x,y)$, 有 $\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2=\left(\frac{\partial f}{\partial r}\right)^2+\left(\frac{\partial f}{\partial t}\right)^2$

证明: $\left(\frac{\partial f}{\partial r}\right)^2+\left(\frac{\partial f}{\partial t}\right)^2=\left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}\right)^2+\left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}\right)^2$

$=\left(\frac{\partial f}{\partial x}\right)^2\left(\frac{\partial x}{\partial r}+\frac{\partial y}{\partial r}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\left(\frac{\partial x}{\partial t}+\frac{\partial y}{\partial t}\right)^2=\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2$ #

29. 设 $f(x,y)=\begin{pmatrix}u(x,y)\\ v(x,y)\end{pmatrix}=\begin{pmatrix}\frac{x}{x^2+y^2}\\ \frac{-y}{x^2+y^2}\end{pmatrix}$. 证明 $f(x,y)$ 是 $\mathbb{R}^2\setminus\{(0,0)\}$ 到自身的 C^1 同胚映射, 并求 $f(x,y)$ 在 $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$ 处的 Jacobi 行列式

证明: 令 $x=r\cos\theta$, $y=r\sin\theta$, 则 $\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}\cos\theta/r\\ -\sin\theta/r\end{pmatrix}$ $(0<r<+\infty, 0\le\theta<2\pi)$

$\Rightarrow$ f 是满的. 且若 $f(x_1,y_1)=f(x_2,y_2)$, 则对应 r,θ 相等. f 是单的. f 连续. $f^{-1}:\begin{pmatrix}u\\ v\end{pmatrix}\mapsto\begin{pmatrix}\frac{u}{u^2+v^2}\\ \text{[illegible]}\end{pmatrix}$ 连续. 故 f 是 $\mathbb{R}^2\setminus\{(0,0)\}$ 到自身的 C^1 同胚. $|f(x,y)|=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\frac{1}{(x^2+y^2)^2}$ #

30. 求高阶偏导数 $\frac{\partial^{\sum_{i=1}^{n} m_i} f(x)}{\partial x_1^{m_1}\partial x_2^{m_2}\cdots\partial x_n^{m_n}}$：(1) $f(x_1,x_2,\cdots,x_n)=e^{\sum_{i=1}^{n} x_i}$ (2) $f(x_1,x_2,\cdots,x_n)=\ln(\sum_{i=1}^{n} a_i x_i)$

解：(1) $\frac{\partial^{\sum_{i=1}^{n} m_i} f(x)}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}}=e^{\sum_{i=1}^{n} x_i}$ (2) $\frac{\partial^{\sum_{i=1}^{n} m_i} f(x)}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}}=(-1)^{\sum_{i=1}^{n} m_i+1}\frac{\prod_{i=1}^{n} a_i^{m_i}(\sum_{i=1}^{n} m_i-1)!}{(\sum_{i=1}^{n} a_i x_i)^{\sum_{i=1}^{n} m_i}}$

31. 求二阶偏导数，其中f有二阶连续导数：(1) $z=f(x^2+y^2,xy)$ (2) $z=f(x_1+x_2+\cdots+x_n)$

解：(1) $\frac{\partial^2 z}{\partial x^2}=2f_1'+4x^2f_{11}''+4xyf_{12}''+y^2f_{22}''$，$\frac{\partial^2 z}{\partial x\partial y}=f_2'+4xyf_{11}''+2(x^2+y^2)f_{12}''+xyf_{22}''$

$\frac{\partial^2 z}{\partial y^2}=2f_1'+4y^2f_{11}''+4xyf_{12}''+x^2f_{22}''$

(2) $\frac{\partial^2 z}{\partial x_i\partial x_j}=f''(x_1+x_2+\cdots+x_n)$

32. 验证函数满足Laplace方程 $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$：(1) $z=\arctan\frac{y}{x}$ (2) $z=\ln\sqrt{x^2+y^2}$

证明：(1) $\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right)=\frac{2xy}{(x^2+y^2)^2}$，$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)=-\frac{2xy}{(x^2+y^2)^2}\Rightarrow\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$

(2) $\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)=\frac{y^2-x^2}{(x^2+y^2)^2}$，$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)=\frac{x^2-y^2}{(x^2+y^2)^2}\Rightarrow\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$ #

33. 验证 $u=(x_1^2+x_2^2+x_3^2+x_4^2)^{-1}$ 满足Laplace方程 $\sum_{i=1}^{4}\frac{\partial^2 u}{\partial x_i^2}=0$

证明：$\frac{\partial^2 u}{\partial x_i^2}=\frac{\partial}{\partial x_i}\left(-\frac{2x_i}{(x_1^2+x_2^2+x_3^2+x_4^2)^2}\right)=2\cdot\frac{x_1^2+x_2^2+x_3^2+x_4^2-4x_i^2}{(x_1^2+x_2^2+x_3^2+x_4^2)^3}\Rightarrow\sum_{i=1}^{4}\frac{\partial^2 u}{\partial x_i^2}=0$ #

34. 设 $f(x)$ 二次可微，证明 $F(x,t)=\frac12(f(x-ct)+f(x+ct))$（c为常数）满足 $\frac{\partial^2 F}{\partial t^2}=c^2\frac{\partial^2 F}{\partial x^2}$

证明：$\frac{\partial^2 F}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac12(-cf'(x-ct)+cf'(x+ct))\right)=\frac12c^2(f''(x-ct)+f''(x+ct))$，$\frac{\partial^2 F}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac12(f'(x-ct)+f'(x+ct))\right)$

$=\frac12(f''(x-ct)+f''(x+ct))$. 故 $\frac{\partial^2 F}{\partial t^2}=c^2\frac{\partial^2 F}{\partial t^2}$ #

35. 证明：在极坐标变换 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$ 下，Laplace方程 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ 的形式为 $\frac{\partial^2 u}{\partial r^2}+\frac1r\cdot\frac{\partial u}{\partial r}+\frac1{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2}=0$

证明：$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta=\frac1r\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$

$\frac{\partial^2 u}{\partial r^2}=\frac1{r^2}\left(r\left(\cos\theta\frac{\partial u}{\partial x}+x\left(\cos\theta\frac{\partial^2 u}{\partial x^2}+\sin\theta\frac{\partial^2 u}{\partial x\partial y}\right)+\sin\theta\frac{\partial u}{\partial y}+y\left(\cos\theta\frac{\partial^2 u}{\partial x\partial y}+\sin\theta\frac{\partial^2 u}{\partial y^2}\right)\right)-x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}\right)$

$=\frac{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}}{r^2}$

$\frac{\partial u}{\partial\theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\theta}=-y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}$.

$\frac{\partial^2 u}{\partial\theta^2}=-x\frac{\partial u}{\partial x}-y\left(-y\frac{\partial^2 u}{\partial x^2}+x\frac{\partial^2 u}{\partial x\partial y}\right)-y\frac{\partial u}{\partial y}+x\left(-y\frac{\partial^2 u}{\partial x\partial y}+x\frac{\partial^2 u}{\partial y^2}\right)=-x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}+y^2\frac{\partial^2 u}{\partial x^2}+x^2\frac{\partial^2 u}{\partial y^2}-2xy\frac{\partial^2 u}{\partial x\partial y}$

故 $\frac{\partial^2 u}{\partial r^2}+\frac1r\frac{\partial u}{\partial r}+\frac1{r^2}\frac{\partial^2 u}{\partial\theta^2}=\frac1{r^2}(x^2+y^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0$ #

36. 设 $u(x,y,z)=\frac{x-y+z}{x+y-z}$. 证明：(1) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$ (2) $x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+2xz\frac{\partial^2 u}{\partial x\partial z}+2yz\frac{\partial^2 u}{\partial y\partial z}=0$

证明：(1) $\frac{\partial u}{\partial x}=\frac{2(y-z)}{(x+y-z)^2}$，$\frac{\partial u}{\partial y}=\frac{-2x}{(x+y-z)^2}$，$\frac{\partial u}{\partial z}=\frac{2x}{(x+y-z)^2}\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$

(2) $\frac{\partial^2 u}{\partial x^2}=\frac{-4(y-z)}{(x+y-z)^3}$，$\frac{\partial^2 u}{\partial y^2}=\frac{4x}{(x+y-z)^3}$，$\frac{\partial^2 u}{\partial z^2}=\frac{4x}{(x+y-z)^3}$，$\frac{\partial^2 u}{\partial x\partial y}=\frac{2(x-y+z)}{(x+y-z)^3}$，$\frac{\partial^2 u}{\partial x\partial z}=\frac{2(-x+y-z)}{(x+y-z)^3}$，$\frac{\partial^2 u}{\partial y\partial z}=\frac{-4x}{(x+y-z)^3}$

$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+2xz\frac{\partial^2 u}{\partial x\partial z}+2yz\frac{\partial^2 u}{\partial y\partial z}=0$ #

37. 设 $x=2r-s$，$y=r+2s$，$f(x,y)$ 有二阶连续偏导数，求 $\frac{\partial^2 f(x,y)}{\partial r\partial s}$

解：$\frac{\partial^2 f(x,y)}{\partial r\partial s}=\frac{\partial}{\partial s}\left(f_1'\frac{\partial x}{\partial r}+f_2'\frac{\partial y}{\partial r}\right)=\frac{\partial}{\partial s}(2f_1'+f_2')=2\left(f_{11}''\frac{\partial x}{\partial s}+f_{12}''\frac{\partial y}{\partial s}\right)+f_{12}''\frac{\partial x}{\partial s}+f_{22}''\frac{\partial y}{\partial s}=-2f_{11}''+3f_{12}''+2f_{22}''$

38. 把 $f(x,y)=ax^2+2bxy+cy^2$ 写成 $F(x-1,y-1)$ 的形式

解：$F(s,t)=f(s+1,t+1)=as^2+2bst+ct^2+2(a+b)s+2(b+c)t+(a+2b+c)$

39. 求 e^{x+y} 在(0,0)处的Taylor公式

解：$e^{x+y}=e^{0+0}+\sum_{k=1}^{K}\frac1{k!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^k e^{x+y}\Big|_{(0,0)}+o\left((\sqrt{x^2+y^2})^K\right)=1+\sum_{k=1}^{K}\frac1{k!}(x+y)^k+o\left((x^2+y^2)^{K/2}\right)$

30. 求高阶偏导数 $\frac{\partial^{\sum_{i=1}^n m_i} f(x)}{\partial x_1^{m_1}\partial x_2^{m_2}\cdots\partial x_n^{m_n}}$：(1) $f(x_1,x_2,\cdots,x_n)=e^{\sum_{i=1}^n x_i}$ (2) $f(x_1,x_2,\cdots,x_n)=\ln(\sum_{i=1}^n a_i x_i)$

解：(1) $\frac{\partial^{\sum_{i=1}^n m_i} f(x)}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}}=e^{\sum_{i=1}^n x_i}$ (2) $\frac{\partial^{\sum_{i=1}^n m_i} f(x)}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}}=(-1)^{\sum_{i=1}^n m_i+1}\frac{\prod_{i=1}^n a_i^{m_i}(\sum_{i=1}^n m_i-1)!}{(\sum_{i=1}^n a_i x_i)^{\sum_{i=1}^n m_i}}$

31. 求二阶偏导数，其中f有二阶连续导数：(1) $z=f(x^2+y^2,xy)$ (2) $z=f(x_1+x_2+\cdots+x_n)$

解：(1) $\frac{\partial^2 z}{\partial x^2}=2f_1'+4x^2f_{11}''+4xyf_{12}''+y^2f_{22}''$，$\frac{\partial^2 z}{\partial x\partial y}=f_2'+4xyf_{11}''+2(x^2+y^2)f_{12}''+xyf_{22}''$

$\frac{\partial^2 z}{\partial y^2}=2f_1'+4y^2f_{11}''+4xyf_{12}''+x^2f_{22}''$

(2) $\frac{\partial^2 z}{\partial x_i\partial x_j}=f''(x_1+x_2+\cdots+x_n)$

32. 验证函数满足Laplace方程 $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$：(1) $z=\arctan\frac{y}{x}$ (2) $z=\ln\sqrt{x^2+y^2}$

证明：(1) $\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}(-\frac{y}{x^2+y^2})=\frac{2xy}{(x^2+y^2)^2}$，$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}(\frac{x}{x^2+y^2})=-\frac{2xy}{(x^2+y^2)^2}\Rightarrow\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$

(2) $\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}(\frac{x}{x^2+y^2})=\frac{y^2-x^2}{(x^2+y^2)^2}$，$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}(\frac{y}{x^2+y^2})=\frac{x^2-y^2}{(x^2+y^2)^2}\Rightarrow\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$ #

33. 验证 $u=(x_1^2+x_2^2+x_3^2+x_4^2)^{-1}$ 满足Laplace方程 $\sum_{i=1}^4\frac{\partial^2 u}{\partial x_i^2}=0$

证明：$\frac{\partial^2 u}{\partial x_i^2}=\frac{\partial}{\partial x_i}(-\frac{2x_i}{(x_1^2+x_2^2+x_3^2+x_4^2)^2})=2\cdot\frac{x_1^2+x_2^2+x_3^2+x_4^2-4x_i^2}{(x_1^2+x_2^2+x_3^2+x_4^2)^3}\Rightarrow\sum_{i=1}^4\frac{\partial^2 u}{\partial x_i^2}=0$ #

34. 设f(x)二次可微，证明 $F(x,t)=\frac12(f(x-ct)+f(x+ct))$（c为常数）满足 $\frac{\partial^2 F}{\partial t^2}=c^2\frac{\partial^2 F}{\partial x^2}$

证明：$\frac{\partial^2 F}{\partial t^2}=\frac{\partial}{\partial t}(\frac12(-cf'(x-ct)+cf'(x+ct)))=\frac12c^2(f''(x-ct)+f''(x+ct))$，$\frac{\partial^2 F}{\partial x^2}=\frac{\partial}{\partial x}(\frac12(f'(x-ct)+f'(x+ct)))$

$=\frac12(f''(x-ct)+f''(x+ct))$. 故 $\frac{\partial^2 F}{\partial t^2}=c^2\frac{\partial^2 F}{\partial t^2}$ #

35. 证明：在极坐标变换 $\begin{cases}x=r\cos\theta\\y=r\sin\theta\end{cases}$ 下，Laplace方程 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ 的形式为 $\frac{\partial^2 u}{\partial r^2}+\frac1r\cdot\frac{\partial u}{\partial r}+\frac1{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2}=0$

证明：$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta=\frac1r(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y})$

$\frac{\partial^2 u}{\partial r^2}=\frac1{r^2}(r(\cos\theta\frac{\partial u}{\partial x}+x(\cos\theta\frac{\partial^2 u}{\partial x^2}+\sin\theta\frac{\partial^2 u}{\partial x\partial y})+\sin\theta\frac{\partial u}{\partial y}+y(\cos\theta\frac{\partial^2 u}{\partial x\partial y}+\sin\theta\frac{\partial^2 u}{\partial y^2}))-x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y})$

$=\frac{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}}{r^2}$

$\frac{\partial u}{\partial\theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\theta}=-y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}$.

$\frac{\partial^2 u}{\partial\theta^2}=-x\frac{\partial u}{\partial x}-y(-y\frac{\partial^2 u}{\partial x^2}+x\frac{\partial^2 u}{\partial x\partial y})-y\frac{\partial u}{\partial y}+x(-y\frac{\partial^2 u}{\partial x\partial y}+x\frac{\partial^2 u}{\partial y^2})=-x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}+y^2\frac{\partial^2 u}{\partial x^2}+x^2\frac{\partial^2 u}{\partial y^2}-2xy\frac{\partial^2 u}{\partial x\partial y}$

故 $\frac{\partial^2 u}{\partial r^2}+\frac1r\frac{\partial u}{\partial r}+\frac1{r^2}\frac{\partial^2 u}{\partial\theta^2}=\frac1{r^2}(x^2+y^2)(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})=0$ #

36. 设 $u(x,y,z)=\frac{x-y+z}{x+y-z}$. 证明：(1) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$ (2) $x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+2xz\frac{\partial^2 u}{\partial x\partial z}+2yz\frac{\partial^2 u}{\partial y\partial z}=0$

证明：(1) $\frac{\partial u}{\partial x}=\frac{2(y-z)}{(x+y-z)^2}$，$\frac{\partial u}{\partial y}=\frac{-2x}{(x+y-z)^2}$，$\frac{\partial u}{\partial z}=\frac{2x}{(x+y-z)^2}\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$

(2) $\frac{\partial^2 u}{\partial x^2}=\frac{-4(y-z)}{(x+y-z)^3}$，$\frac{\partial^2 u}{\partial y^2}=\frac{4x}{(x+y-z)^3}$，$\frac{\partial^2 u}{\partial z^2}=\frac{4x}{(x+y-z)^3}$，$\frac{\partial^2 u}{\partial x\partial y}=\frac{2(x-y+z)}{(x+y-z)^3}$，$\frac{\partial^2 u}{\partial x\partial z}=\frac{2(-x+y-z)}{(x+y-z)^3}$，$\frac{\partial^2 u}{\partial y\partial z}=\frac{-4x}{(x+y-z)^3}$

$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+2xz\frac{\partial^2 u}{\partial x\partial z}+2yz\frac{\partial^2 u}{\partial y\partial z}=0$ #

37. 设 $x=2r-s$, $y=r+2s$, $f(x,y)$ 有二阶连续偏导数，求 $\frac{\partial^2 f(x,y)}{\partial r\partial s}$

解：$\frac{\partial^2 f(x,y)}{\partial r\partial s}=\frac{\partial}{\partial s}(f_1'\frac{\partial x}{\partial r}+f_2'\frac{\partial y}{\partial r})=\frac{\partial}{\partial s}(2f_1'+f_2')=2(f_{11}''\frac{\partial x}{\partial s}+f_{12}''\frac{\partial y}{\partial s})+f_{12}''\frac{\partial x}{\partial s}+f_{22}''\frac{\partial y}{\partial s}=-2f_{11}''+3f_{12}''+2f_{22}''$

38. 把 $f(x,y)=ax^2+2bxy+cy^2$ 写成 $F(x-1,y-1)$ 的形式

解：$F(s,t)=f(s+1,t+1)=as^2+2bst+ct^2+2(a+b)s+2(b+c)t+(a+2b+c)$

39. 求 e^{x+y} 在(0,0)处的Taylor公式

解：$e^{x+y}=e^{0+0}+\sum_{k=1}^K\frac1{k!}(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})^k e^{x+y}|_{(0,0)}+o((\sqrt{x^2+y^2})^K)=1+\sum_{k=1}^K\frac1{k!}(x+y)^k+o((x^2+y^2)^{K/2})$

40. 将下列函数在原点处展成Taylor公式(到四次项)：(1) $\frac{1+x+y+2xy}{1+x^2+y^2}$ (2) $\frac{x_1^2+\cdots+x_n^2}{1-(x_1+\cdots+x_n)}$

解：(1) $\frac{1+x+y+2xy}{1+x^2+y^2} = (1+x+y+2xy)\left(1-(x^2+y^2)+(x^2+y^2)^2+o((x^2+y^2)^2)\right)$

$= 1+x+y-x^2+2xy-y^2-x^3-x^2y-xy^2-y^3+x^4-2x^3y+2x^2y^2-2xy^3+y^4+o((x^2+y^2)^2)$ $((x,y)\to(0,0))$

(2) $\frac{x_1^2+\cdots+x_n^2}{1-(x_1+\cdots+x_n)} = (x_1^2+\cdots+x_n^2)\left(1+(x_1+\cdots+x_n)+(x_1+\cdots+x_n)^2+o((x_1+\cdots+x_n)^2)\right)$

$= (x_1^2+\cdots+x_n^2)+(x_1^2+\cdots+x_n^2)(x_1+\cdots+x_n)+(x_1^2+\cdots+x_n^2)(x_1+\cdots+x_n)^2+o((x_1^2+\cdots+x_n^2)^2)$ $((x_1,\cdots,x_n)\to(0,\cdots,0))$

41. Legendre多项式$P_n(x)$如下定义：$f(x,t)=\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n$. 证明：(1) $P_n(1)=1$ (2) $P_n(-1)=(-1)^n$

(3) $P_0(x)=1$, $P_1(x)=x$, $P_2(x)=\frac{1}{2}(3x^2-1)$

证明：(1) 令$x=1$. $\frac{1}{|1-t|}=\sum_{n=0}^{\infty}P_n(1)t^n$. 要使右边收敛，有$-1<t<1 \Longrightarrow \frac{1}{1-t}=\sum_{n=0}^{\infty}t^n=\sum_{n=0}^{\infty}P_n(1)t^n$

故 $P_n(1)=1$

(2) 令$x=-1$. $\frac{1}{|1+t|}=\sum_{n=0}^{\infty}P_n(-1)t^n$. $-1<t<1 \Longrightarrow \frac{1}{1+t}=\sum_{n=0}^{\infty}(-1)^nt^n=\sum_{n=0}^{\infty}P_n(-1)t^n \Longrightarrow P_n(-1)=(-1)^n$

(3) t的函数 $\underbrace{\frac{1}{\sqrt{1-2xt+t^2}}}_{g(t)=} = \sum_{n=0}^{\infty}\frac{g^{(n)}(0)}{n!}t^n \Longrightarrow P_n(x)=\frac{1}{n!}g^{(n)}(0)$. 特别，$P_0(x)=1$,

$P_1(x)=x$, $P_2(x)=\frac{1}{2}(3x^2-1)$ #

42. 设$f(x,y)=e^{xy}$. $k\in\mathbb{Z}_+$. 求$f(x,y)$在$(0,0)$处的所有k阶偏导数

解：设对x求偏导数m次，对y求偏导数k-m次. $\frac{\partial^m f(x,y)}{\partial x^m}=y^me^{xy}\triangleq f_1(x,y)$. $\frac{\partial f_1(x,y)}{\partial y}=e^{xy}(xy^m+my^{m-1})$

归纳易知 $\frac{\partial^p f_1(x,y)}{\partial y^p}=e^{xy}(P_1(x)\cdot xy^m+\cdots+P_p(x)\cdot xy^{m-p+1}+m\cdots(m-p+1)y^{m-p}) \Longrightarrow k-m=m$时，

$\frac{\partial^k f(x,y)}{\partial x^m\partial y^{k-m}}\Big|_{(0,0)}=m!$，否则 $\frac{\partial^k f(x,y)}{\partial x^m\partial y^{k-m}}\Big|_{(0,0)}=0$. 故 $\frac{\partial^k f(x,y)}{\partial x^m\partial y^{k-m}}\Big|_{(0,0)}=\begin{cases} m!, & m=\frac{k}{2} \\ 0, & \text{其他}\end{cases}$

43. 举例说明存在$(0,0)$某个邻域$U((0,0),\delta_0)$ $(\delta_0>0)$内的连续函数$z=F(x,y)$，满足$F(0,0)=0$和下列之一：

(1) $F'_y(0,0)$不存在 (2) $F'_y(0,0)$存在，$F'_y(0,0)=0$. 但$F(x,y)=0$在$U((0,0),\delta_0)$内唯一确定一个连续的隐函数$y=f(x)$ $(-\delta_0<x<\delta_0)$，使得$f(0)=0$，且$x\in(-\delta_0,\delta_0)$时，$F(x,f(x))=0$

证明：(1) $F(x,y)=|y|$ (2) $F(x,y)=y^3$ #

注：不管原点处F'_y如何，需使邻域内别处F'_y定号，这是定理证明中的要点

44. 证明 $x^2-2xy+z+xe^z=0$ 在$(1,1,0)$的某个邻域内唯一确定隐函数$z=f(x,y)$，并求$f(x,y)$在$(1,1)$处的Taylor公式(直到二次)

证明：令$F(x,y,z)=x^2-2xy+z+xe^z$，则$F(x,y,z)$有各阶连续偏导数. $F(1,1,0)=0$, $F'_z(1,1,0)=2\neq0$

$\Longrightarrow F(x,y,z)$在$(1,1,0)$某个邻域内唯一确定隐函数$z=f(x,y)$，且它有各阶连续偏导数. 待定系数：

$f(x,y)-f(1,1)=f(x,y)=a_1(x-1)+a_2(y-1)+b_{11}(x-1)^2+b_{12}(x-1)(y-1)+b_{22}(y-1)^2+o((x-1)^2+(y-1)^2)$

$(\sqrt{(x-1)^2+(y-1)^2}\to0)$. $e^z=1+z+\frac{z^2}{2}+o(|z|^2)=1+a_1(x-1)+a_2(y-1)+b_{11}(x-1)^2+b_{12}(x-1)(y-1)+b_{22}(y-1)^2$

$+\frac{1}{2}a_1^2(x-1)^2+\frac{1}{2}a_2^2(y-1)^2+a_1a_2(x-1)(y-1)+o((x-1)^2+(y-1)^2)$. 代入 $x^2-2xy+z+xe^z=0$. 得

$(a_1+b_{11}+1)(x-1)^2+(a_2+b_{12}-2)(x-1)(y-1)+b_{22}(y-1)^2+(2a_1+1)(x-1)+(2a_2-2)v+o((x-1)^2+(y-1)^2)=0$

$\Longrightarrow a_1=-\frac{1}{2}$, $a_2=-1$, $b_{11}=-\frac{1}{2}$, $b_{12}=1$, $b_{22}=0$

故 $f(x,y)=-\frac{1}{2}x^2+xy-\frac{1}{2}x+o((x-1)^2+(y-1)^2)$ $(\sqrt{(x-1)^2+(y-1)^2}\to0)$ #

45. 证明 $x+x^2+y^2+(x^2+y^2)z^2+\sin z=0$ 在$(0,0,0)$的某个邻域内唯一确定隐函数$z=f(x,y)$，并求$f(x,y)$在$(0,0)$处的所有二阶偏导数

40. 将下列函数在原点处展成Taylor公式(到四次项)：(1) $\frac{1+x+y+2xy}{1+x^2+y^2}$ (2) $\frac{x_1^2+\cdots+x_n^2}{1-(x_1+\cdots+x_n)}$

解：(1) $\frac{1+x+y+2xy}{1+x^2+y^2}=(1+x+y+2xy)\left(1-(x^2+y^2)+(x^2+y^2)^2+o((x^2+y^2)^2)\right)$

$=1+x+y-x^2+2xy-y^2-x^3-x^2y-xy^2-y^3+x^4-2x^3y+2x^2y^2-2xy^3+y^4+o((x^2+y^2)^2)\quad ((x,y)\to(0,0))$

(2) $\frac{x_1^2+\cdots+x_n^2}{1-(x_1+\cdots+x_n)}=(x_1^2+\cdots+x_n^2)\left(1+(x_1+\cdots+x_n)+(x_1+\cdots+x_n)^2+o((x_1+\cdots+x_n)^2)\right)$

$=(x_1^2+\cdots+x_n^2)+(x_1^2+\cdots+x_n^2)(x_1+\cdots+x_n)+(x_1^2+\cdots+x_n^2)(x_1+\cdots+x_n)^2+o((x_1^2+\cdots+x_n^2)^2)\quad ((x_1,\cdots,x_n)\to(0,\cdots,0))$

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证明：(1) 令$x=1$. $\frac{1}{|1-t|}=\sum_{n=0}^{\infty}P_n(1)t^n$. 要使右边收敛，有$-1<t<1\Rightarrow\frac{1}{1-t}=\sum_{n=0}^{\infty}t^n=\sum_{n=0}^{\infty}P_n(1)t^n$

故$P_n(1)=1$

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$P_1(x)=x$, $P_2(x)=\frac{1}{2}(3x^2-1)$ #

42. 设$f(x,y)=e^{xy}$. $k\in Z_+$. 求$f(x,y)$在$(0,0)$处的所有k阶偏导数

解：设对x求偏导数m次，对y求偏导数k-m次. $\frac{\partial f(x,y)}{\partial x^m}=y^me^{xy}\triangleq f_1(x,y)$. $\frac{\partial f_1(x,y)}{\partial y}=e^{xy}(xy^m+my^{m-1})$

归纳易知$\frac{\partial f_1(x,y)}{\partial y^p}=e^{xy}\left(P_1(x)\cdot xy^m+\cdots+P_p(x)\cdot xy^{m-p+1}+m\cdots(m-p+1)y^{m-p}\right)\Rightarrow k-m=m$时,

$\left.\frac{\partial f(x,y)}{\partial x^m\partial y^{k-m}}\right|_{(0,0)}=m!$, 否则$\left.\frac{\partial f(x,y)}{\partial x^m\partial y^{k-m}}\right|_{(0,0)}=0$. 故$\left.\frac{\partial f(x,y)}{\partial x^m\partial y^{k-m}}\right|_{(0,0)}=\begin{cases}m!, & m=\frac{k}{2}\\ 0, & \text{其他}\end{cases}$

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证明：(1) $F(x,y)=|y|$ (2) $F(x,y)=y^3$ #

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证明：令$F(x,y,z)=x^2-2xy+z+xe^z$，则$F(x,y,z)$有各阶连续偏导数. $F(1,1,0)=0$, $F_z'(1,1,0)=2\neq0$

$\Rightarrow F(x,y,z)$在$(1,1,0)$某个邻域内唯一确定隐函数$z=f(x,y)$，且它有各阶连续偏导数. 待定系数:

$f(x,y)-f(1,1)=f(x,y)=a_1(x-1)+a_2(y-1)+b_{11}(x-1)^2+b_{12}(x-1)(y-1)+b_{22}(y-1)^2+o((x-1)^2+(y-1)^2)$

$(\sqrt{(x-1)^2+(y-1)^2}\to0)$. $e^z=1+z+\frac{z^2}{2}+o(|z|^2)=1+a_1(x-1)+a_2(y-1)+b_{11}(x-1)^2+b_{12}(x-1)(y-1)+b_{22}(y-1)^2$
$+\frac{1}{2}a_1^2(x-1)^2+\frac{1}{2}a_2^2(y-1)^2+a_1a_2(x-1)(y-1)+o((x-1)^2+(y-1)^2)$. 代入$x^2-2xy+z+xe^z=0$, 得

$(a_1+b_{11}+1)(x-1)^2+(a_2+b_{12}-2)(x-1)(y-1)+b_{22}(y-1)^2+(2a_1+1)(x-1)+(2a_2-2)v+o((x-1)^2+(y-1)^2)=0$

$\Rightarrow a_1=-\frac{1}{2}$, $a_2=1$, $b_{11}=-\frac{1}{2}$, $b_{12}=1$, $b_{22}=0$

故$f(x,y)=-\frac{1}{2}x^2+xy-\frac{1}{2}x+o((x-1)^2+(y-1)^2)$ $(\sqrt{(x-1)^2+(y-1)^2}\to0)$ #

45. 证明$x+x^2+y^2+(x^2+y^2)z^2+\sin z=0$在$(0,0,0)$的某个邻域内唯一确定隐函数$z=f(x,y)$，并求$f(x,y)$在$(0,0)$处的所有三阶偏导数

证明：令 $F(x,y,z)=x+x^2+y+(x^2+y^2)z^2+\sin z$，则 $F(x,y,z)$ 有各阶连续偏导数，$F(0,0,0)=0$，$F'_z(0,0,0)=1\neq0$

$\Rightarrow F(x,y,z)=0$ 在 $(0,0,0)$ 的某个邻域内唯一确定隐函数 $z=f(x,y)$，且它有各阶连续偏导数.

$F(x,y,f(x,y))=0$. $F'_1+F'_3\frac{\partial f}{\partial x}=0\Rightarrow\frac{\partial f}{\partial x}=-\frac{F'_1}{F'_3}$ 同理 $\frac{\partial f}{\partial y}=-\frac{F'_2}{F'_3}$.

$F''_{11}+F''_{13}\frac{\partial f}{\partial x}+(F''_{13}+F''_{33}\frac{\partial f}{\partial x})\frac{\partial f}{\partial x}+F'_3\frac{\partial^2 f}{\partial x^2}=0$

$F'''_{111}+F'''_{113}\frac{\partial f}{\partial x}+(F'''_{113}+F'''_{133}\frac{\partial f}{\partial x})\frac{\partial f}{\partial x}+F''_{13}\frac{\partial^2 f}{\partial x^2}+\frac{\partial f}{\partial x}(F'''_{113}+F'''_{133}\frac{\partial f}{\partial x}+\frac{\partial f}{\partial x}(F'''_{133}+F'''_{333}\frac{\partial f}{\partial x})+F''_{33}\frac{\partial^2 f}{\partial x^2})$

$+(F''_{13}+F''_{33}\frac{\partial f}{\partial x})\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial x^2}(F''_{13}+F''_{33}\frac{\partial f}{\partial x})+F'_3\frac{\partial^3 f}{\partial x^3}=0\Rightarrow\frac{\partial^3 f(0,0)}{\partial x^3}=-1$. $\frac{\partial f(0,0)}{\partial x}=-1$, $\frac{\partial^2 f(0,0)}{\partial x^2}=-2$

$\frac{\partial f(0,0)}{\partial y}=0$. $F''_{22}+F''_{23}\frac{\partial f}{\partial y}+(F''_{23}+F''_{33}\frac{\partial f}{\partial y})\frac{\partial f}{\partial y}+F'_3\frac{\partial^2 f}{\partial y^2}=0$. $\frac{\partial^2 f(0,0)}{\partial y^2}=-2$

$F'''_{222}+F'''_{223}\frac{\partial f}{\partial y}+(F'''_{223}+F'''_{233}\frac{\partial f}{\partial y})\frac{\partial f}{\partial y}+F''_{23}\frac{\partial^2 f}{\partial y^2}+\frac{\partial f}{\partial y}(F'''_{223}+F'''_{233}\frac{\partial f}{\partial y}+\frac{\partial f}{\partial y}(F'''_{233}+F'''_{333}\frac{\partial f}{\partial y})+F''_{33}\frac{\partial^2 f}{\partial y^2})$

$+(F''_{23}+F''_{33}\frac{\partial f}{\partial y})\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial y^2}(F''_{23}+F''_{33}\frac{\partial f}{\partial y})+F'_3\frac{\partial^3 f}{\partial y^3}=0\Rightarrow\frac{\partial^3 f(0,0)}{\partial y^3}=0$

$F''_{12}+F''_{13}\frac{\partial f}{\partial y}+(F''_{23}+F''_{33}\frac{\partial f}{\partial y})\frac{\partial f}{\partial x}+F'_3\frac{\partial^2 f}{\partial x\partial y}=0\Rightarrow\frac{\partial^2 f(0,0)}{\partial x\partial y}=0$

$F'''_{112}+F'''_{113}\frac{\partial f}{\partial x}+(F'''_{123}+F'''_{133}\frac{\partial f}{\partial x})\frac{\partial f}{\partial y}+F''_{13}\frac{\partial^2 f}{\partial x\partial y}+(F'''_{123}+F'''_{233}\frac{\partial f}{\partial x}+(F'''_{133}+F'''_{333}\frac{\partial f}{\partial x})\frac{\partial f}{\partial y}+F''_{33}\frac{\partial^2 f}{\partial x\partial y})\frac{\partial f}{\partial x}$

$+(F''_{23}+F''_{33}\frac{\partial f}{\partial y})\frac{\partial^2 f}{\partial x^2}+(F''_{13}+F''_{33}\frac{\partial f}{\partial x})\frac{\partial^2 f}{\partial x\partial y}+F'_3\frac{\partial^3 f}{\partial x^2\partial y}=0\Rightarrow\frac{\partial^3 f(0,0)}{\partial x^2\partial y}=0$

$F'''_{122}+F'''_{123}\frac{\partial f}{\partial y}+(F'''_{123}+F'''_{133}\frac{\partial f}{\partial y})\frac{\partial f}{\partial y}+F''_{13}\frac{\partial^2 f}{\partial y^2}+\frac{\partial f}{\partial x}(F'''_{223}+F'''_{233}\frac{\partial f}{\partial y}+\frac{\partial f}{\partial y}(F'''_{233}+F'''_{333}\frac{\partial f}{\partial y})+F''_{33}\frac{\partial^2 f}{\partial y^2})$

$+\frac{\partial^2 f}{\partial x^2}(F''_{23}+F''_{33}\frac{\partial f}{\partial y})+(F''_{23}+F''_{33}\frac{\partial f}{\partial y})\frac{\partial^2 f}{\partial x\partial y}+F'_3\frac{\partial^3 f}{\partial x\partial y^2}=0\Rightarrow\frac{\partial^3 f(0,0)}{\partial x\partial y^2}=0$ #

46. 设 $z=F(x,y)$ 在区域 D 内有连续偏导数，处处成立 $F'_x(x,y)\neq0$，$F'_y(x,y)\neq0$. 证明：$\forall(x_0,y_0)\in D$，$F(x,y)=F(x_0,y_0)$ 在 (x_0,y_0) 某个邻域内确定的隐函数 $y=f(x)$ 与 $x=g(y)$ 互为反函数.

证明：显然 $F(x,y)=F(x_0,y_0)$ 在 (x_0,y_0) 某个邻域内确定隐函数 $y=f(x)$，$x=g(y)$.

$f'(x)=-\frac{F'_x(x,y)}{F'_y(x,y)}$ 因 D 内处处 $F'_x(x,y)\neq0$，$F'_y(x,y)\neq0$，从而 $F'_x(x,y)$，$F'_y(x,y)$ 保持符号 $\Rightarrow f(x)$ 严格单调. 于是有连续的反函数 $x=f^{-1}(y)$. $x_0=f^{-1}(y_0)=g(y_0)$. $(f^{-1})'(y)=\frac{1}{f'(x)}=-\frac{F'_y(x,y)}{F'_x(x,y)}=g'(y)$

故 $f^{-1}(y)=g(y)$ #

47. 求由下列方程确定的隐函数 $z=f(x,y)$ 的偏导数：(1) $F(x+y+z,xyz)=0$ (2) $F(x^2+y^2,x^2+y^2+z^2)=0$

解：(1) $F'_1(1+f'_x)+F'_2(yz+yf'_x\cdot x)=0\Rightarrow f'_x=-\frac{F'_1+yzF'_2}{F'_1+xyF'_2}$ 同理 $f'_y=-\frac{F'_1+xzF'_2}{F'_1+xyF'_2}$

(2) $F'_1\cdot2x+F'_2(2x+2z\cdot f'_x)=0\Rightarrow f'_x=-\frac{x(F'_1+F'_2)}{zF'_2}$ 同理 $f'_y=-\frac{y(F'_1+F'_2)}{zF'_2}$

48. 求由下列方程确定的隐函数的(偏)导数：(1) $x^3+y^3-3xy=0$，$\frac{dy}{dx}$，$\frac{d^2y}{dx^2}$

(2) $x+e^{yz}+z^2=0$，$\frac{\partial^2 z}{\partial x^2}$，$\frac{\partial^2 z}{\partial y^2}$，$\frac{\partial^2 z}{\partial x\partial y}$

解：(1) $F(x,y)=x^3+y^3-3xy$. $\frac{dy}{dx}=-\frac{F'_x(x,y)}{F'_y(x,y)}=\frac{x^2-y}{x-y^2}$. $\frac{d^2y}{dx^2}=\frac{d}{dx}\frac{x^2-y}{x-y^2}=2\left(\frac{x}{x-y^2}-\frac{x^2-y}{(x-y^2)^2}+\frac{y(x^2-y)^2}{(x-y^2)^3}\right)$

(2) $1+e^{yz}\cdot y\frac{\partial z}{\partial x}+2z\frac{\partial z}{\partial x}=0$. $\frac{\partial z}{\partial x}=-\frac{1}{ye^{yz}+2z}$. $y(e^{yz}\cdot y\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial x}+e^{yz}\cdot\frac{\partial^2 z}{\partial x^2})+2z\frac{\partial^2 z}{\partial x^2}$

$+2(\frac{\partial z}{\partial x})^2=0\Rightarrow\frac{\partial^2 z}{\partial x^2}=-\frac{y^2e^{yz}+2}{(ye^{yz}+2z)^3}$. $e^{yz}\cdot z+e^{yz}\cdot y\frac{\partial z}{\partial y}+2z\frac{\partial z}{\partial y}=0$. $\frac{\partial z}{\partial y}=-\frac{ze^{yz}}{ye^{yz}+2z}$

$ze^{yz}z+(e^{yz}+e^{yz}\cdot y\cdot z)\frac{\partial z}{\partial y}+ye^{yz}\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}(e^{yz}+ze^{yz}\cdot y+ye^{yz}\cdot y\frac{\partial z}{\partial y})+2z\frac{\partial^2 z}{\partial y^2}+2(\frac{\partial z}{\partial y})^2=0$

$\Rightarrow\frac{\partial^2 z}{\partial y^2}=-\frac{z^2e^{yz}}{ye^{yz}+2z}+\frac{2z(1+yz)e^{2yz}}{(ye^{yz}+2z)^2}-\frac{z^2(y^2e^{yz}+2)e^{2yz}}{(ye^{yz}+2z)^3}$

$ye^{yz}\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial x}(e^{yz}+yze^{yz}+y^2e^{yz}\frac{\partial z}{\partial y})+2z\frac{\partial^2 z}{\partial x\partial y}+2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}=0$

$\Rightarrow\frac{\partial^2 z}{\partial x\partial y}=\frac{(yz+1)e^{yz}}{(ye^{yz}+2z)^2}-\frac{ze^{yz}(y^2e^{yz}+2)}{(ye^{yz}+2z)^3}$

49. 设 $u=u(x,y)$ 由 $u=f(x,y,z,t)$，$g(y,z,t)=0$，$h(z,t)=0$ 确定. 求 $\frac{\partial u}{\partial x}$，$\frac{\partial u}{\partial y}$

解：$\begin{cases}du=f'_1dx+f'_2dy+f'_3dz+f'_4dt\\ g'_1dy+g'_2dz+g'_3dt=0\\ h'_1dz+h'_2dt=0\end{cases}\Rightarrow\begin{cases}dz=-\frac{g'_1h'_2}{g'_2h'_2-g'_3h'_1}dy\\ dt=\frac{g'_1h'_1}{g'_2h'_2-g'_3h'_1}dy\end{cases}$

$\Rightarrow du=f'_1dx+(f'_2-\frac{f'_3g'_1h'_2}{g'_2h'_2-g'_3h'_1}+\frac{f'_4g'_1h'_1}{g'_2h'_2-g'_3h'_1})dy\Rightarrow\frac{\partial u}{\partial x}=f'_1$，$\frac{\partial u}{\partial y}=f'_2-\frac{f'_3g'_1h'_2}{g'_2h'_2-g'_3h'_1}+\frac{f'_4g'_1h'_1}{g'_2h'_2-g'_3h'_1}$

50. 通过 $\begin{cases}u=x-2\sqrt{y}\\ v=x+2\sqrt{y}\end{cases}$ 化简 $\frac{\partial^2 z}{\partial x^2}-y\frac{\partial^2 z}{\partial y^2}-\frac{1}{2}\frac{\partial z}{\partial y}=0$

证明：令 $F(x,y,z)=x+x^2+y+(x^2+y^2)z^2+\sin z$，则 $F(x,y,z)$ 有各阶连续偏导数，$F(0,0,0)=0$，$F'_z(0,0,0)=1\neq0$

$\Rightarrow F(x,y,z)=0$ 在 $(0,0,0)$ 的某个邻域内唯一确定隐函数 $z=f(x,y)$，且它有各阶连续偏导数.

$F(x,y,f(x,y))=0$. $F'_1+F'_3\frac{\partial f}{\partial x}=0\Rightarrow \frac{\partial f}{\partial x}=-\frac{F'_1}{F'_3}$ 同理 $\frac{\partial f}{\partial y}=-\frac{F'_2}{F'_3}$.

$F''_{11}+F''_{13}\frac{\partial f}{\partial x}+\left(F''_{13}+F''_{33}\frac{\partial f}{\partial x}\right)\frac{\partial f}{\partial x}+F'_3\left(\frac{\partial^2 f}{\partial x^2}\right)=0$

$F'''_{111}+F'''_{113}\frac{\partial f}{\partial x}+\left(F'''_{113}+F'''_{133}\frac{\partial f}{\partial x}\right)\frac{\partial f}{\partial x}+F''_{13}\frac{\partial^2 f}{\partial x^2}+\frac{\partial f}{\partial x}\left(F'''_{113}+F'''_{133}\frac{\partial f}{\partial x}+\frac{\partial f}{\partial x}\left(F'''_{133}+F'''_{333}\frac{\partial f}{\partial x}\right)+F''_{33}\frac{\partial^2 f}{\partial x^2}\right)$

$+\left(F''_{13}+F''_{33}\frac{\partial f}{\partial x}\right)\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial x^2}\left(F''_{13}+F''_{33}\frac{\partial f}{\partial x}\right)+F'_3\frac{\partial^3 f}{\partial x^3}=0\Rightarrow \frac{\partial^3 f(0,0)}{\partial x^3}=-1$. $\frac{\partial f(0,0)}{\partial x}=-1$, $\frac{\partial^2 f(0,0)}{\partial x^2}=-2$

$\frac{\partial f(0,0)}{\partial y}=0$. $F''_{22}+F''_{23}\frac{\partial f}{\partial y}+\left(F''_{23}+F''_{33}\frac{\partial f}{\partial y}\right)\frac{\partial f}{\partial y}+F'_3\frac{\partial^2 f}{\partial y^2}=0$. $\frac{\partial^2 f(0,0)}{\partial y^2}=-2$

$F'''_{222}+F'''_{223}\frac{\partial f}{\partial y}+\left(F'''_{223}+F'''_{233}\frac{\partial f}{\partial y}\right)\frac{\partial f}{\partial y}+F''_{23}\frac{\partial^2 f}{\partial y^2}+\frac{\partial f}{\partial y}\left(F'''_{223}+F'''_{233}\frac{\partial f}{\partial y}+\frac{\partial f}{\partial y}\left(F'''_{233}+F'''_{333}\frac{\partial f}{\partial y}\right)+F''_{33}\frac{\partial^2 f}{\partial y^2}\right)$

$+\left(F''_{23}+F''_{33}\frac{\partial f}{\partial y}\right)\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial y^2}\left(F''_{23}+F''_{33}\frac{\partial f}{\partial y}\right)+F'_3\frac{\partial^3 f}{\partial y^3}=0\Rightarrow \frac{\partial^3 f(0,0)}{\partial y^3}=0$

$F''_{12}+F''_{13}\frac{\partial f}{\partial y}+\left(F''_{23}+F''_{33}\frac{\partial f}{\partial y}\right)\frac{\partial f}{\partial x}+F'_3\frac{\partial^2 f}{\partial x\partial y}=0\Rightarrow \frac{\partial^2 f(0,0)}{\partial x\partial y}=0$

$F'''_{112}+F'''_{113}\frac{\partial f}{\partial x}+\left(F'''_{113}+F'''_{133}\frac{\partial f}{\partial x}\right)\frac{\partial f}{\partial y}+F''_{13}\frac{\partial^2 f}{\partial x\partial y}+\left(F'''_{123}+F'''_{233}\frac{\partial f}{\partial x}+\left(F'''_{133}+F'''_{333}\frac{\partial f}{\partial x}\right)\frac{\partial f}{\partial y}+F''_{33}\frac{\partial^2 f}{\partial x\partial y}\right)\frac{\partial f}{\partial x}$

$+\left(F''_{23}+F''_{33}\frac{\partial f}{\partial y}\right)\frac{\partial^2 f}{\partial x^2}+\left(F''_{13}+F''_{33}\frac{\partial f}{\partial x}\right)\frac{\partial^2 f}{\partial x\partial y}+F'_3\frac{\partial^3 f}{\partial x^2\partial y}=0\Rightarrow \frac{\partial^3 f(0,0)}{\partial x^2\partial y}=0$

$F'''_{122}+F'''_{123}\frac{\partial f}{\partial y}+\left(F'''_{123}+F'''_{133}\frac{\partial f}{\partial y}\right)\frac{\partial f}{\partial y}+F''_{13}\frac{\partial^2 f}{\partial y^2}+\frac{\partial f}{\partial x}\left(F'''_{223}+F'''_{233}\frac{\partial f}{\partial y}+\frac{\partial f}{\partial y}\left(F'''_{233}+F'''_{333}\frac{\partial f}{\partial y}\right)+F''_{33}\frac{\partial^2 f}{\partial y^2}\right)$

$+\frac{\partial^2 f}{\partial x^2}\left(F''_{23}+F''_{33}\frac{\partial f}{\partial y}\right)+\left(F''_{23}+F''_{33}\frac{\partial f}{\partial y}\right)\frac{\partial^2 f}{\partial x\partial y}+F'_3\frac{\partial^3 f}{\partial x\partial y^2}=0\Rightarrow \frac{\partial^3 f(0,0)}{\partial x\partial y^2}=0$ #

46. 设 $z=F(x,y)$ 在区域 D 内有连续偏导数，处处成立 $F'_x(x,y)\neq0$，$F'_y(x,y)\neq0$. 证明：$\forall(x_0,y_0)\in D$，$F(x,y)=F(x_0,y_0)$ 在 (x_0,y_0) 某个邻域内确定的隐函数 $y=f(x)$ 与 $x=g(y)$ 互为反函数.

证明：显然 $F(x,y)=F(x_0,y_0)$ 在 (x_0,y_0) 某个邻域内确定隐函数 $y=f(x)$，$x=g(y)$.

$f'(x)=-\frac{F'_x(x,y)}{F'_y(x,y)}$ 因 D 内处处 $F'_x(x,y)\neq0$，$F'_y(x,y)\neq0$，从而 $F'_x(x,y)$，$F'_y(x,y)$ 保持符号 $\Rightarrow f(x)$ 严格单调，于是有连续的反函数 $x=f^{-1}(y)$. $x_0=f^{-1}(y_0)=g(y_0)$. $(f^{-1})'(y)=\frac{1}{f'(x)}=-\frac{F'_y(x,y)}{F'_x(x,y)}=g'(y)$

故 $f^{-1}(y)=g(y)$ #

47. 求由下列方程确定的隐函数 $z=f(x,y)$ 的偏导数：(1) $F(x+y+z,xyz)=0$ (2) $F(x^2+y^2,x^2+y^2+z^2)=0$

解：(1) $F'_1(1+f'_x)+F'_2(yz+yf'_x\cdot x)=0\Rightarrow f'_x=-\frac{F'_1+yzF'_2}{F'_1+xyF'_2}$. 同理 $f'_y=-\frac{F'_1+xzF'_2}{F'_1+xyF'_2}$

(2) $F'_1\cdot 2x+F'_2(2x+2z\cdot f'_x)=0\Rightarrow f'_x=-\frac{x(F'_1+F'_2)}{zF'_2}$ 同理 $f'_y=-\frac{y(F'_1+F'_2)}{zF'_2}$

48. 求由下列方程确定的隐函数的(偏)导数：(1) $x^3+y^3-3xy=0$，$\frac{dy}{dx}$，$\frac{d^2y}{dx^2}$

(2) $x+e^{yz}+z^2=0$，$\frac{\partial^2 z}{\partial x^2}$，$\frac{\partial^2 z}{\partial y^2}$，$\frac{\partial^2 z}{\partial x\partial y}$

解：(1) $F(x,y)=x^3+y^3-3xy$. $\frac{dy}{dx}=-\frac{F'_x(x,y)}{F'_y(x,y)}=\frac{x^2-y}{x-y^2}$. $\frac{d^2y}{dx^2}=\frac{d}{dx}\frac{x^2-y}{x-y^2}=2\left(\frac{x}{x-y^2}-\frac{x^2-y}{(x-y^2)^2}+\frac{y(x^2-y)^2}{(x-y^2)^3}\right)$

(2) $1+e^{yz}\cdot y\frac{\partial z}{\partial x}+2z\frac{\partial z}{\partial x}=0$. $\frac{\partial z}{\partial x}=-\frac{1}{ye^{yz}+2z}$. $y\left(e^{yz}\cdot y\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial x}+e^{yz}\cdot\frac{\partial^2 z}{\partial x^2}\right)+2z\frac{\partial^2 z}{\partial x^2}$

$+2\left(\frac{\partial z}{\partial x}\right)^2=0\Rightarrow \frac{\partial^2 z}{\partial x^2}=-\frac{y^2e^{yz}+2}{(ye^{yz}+2z)^3}$. $e^{yz}\cdot z+e^{yz}\cdot y\frac{\partial z}{\partial y}+2z\frac{\partial z}{\partial y}=0$. $\frac{\partial z}{\partial y}=-\frac{ze^{yz}}{ye^{yz}+2z}$

$ze^{yz}z+\left(e^{yz}+e^{yz}\cdot y\cdot z\right)\frac{\partial z}{\partial y}+ye^{yz}\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}\left(e^{yz}+ze^{yz}\cdot y+ye^{yz}\cdot y\frac{\partial z}{\partial y}\right)+2z\frac{\partial^2 z}{\partial y^2}+2\left(\frac{\partial z}{\partial y}\right)^2=0$

$\Rightarrow \frac{\partial^2 z}{\partial y^2}=-\frac{z^2e^{yz}}{ye^{yz}+2z}+\frac{2z(1+yz)e^{2yz}}{(ye^{yz}+2z)^2}-\frac{z^2(y^2e^{yz}+2)e^{2yz}}{(ye^{yz}+2z)^3}$

$ye^{yz}\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial x}\left(e^{yz}+yze^{yz}+y^2e^{yz}\frac{\partial z}{\partial y}\right)+2z\frac{\partial^2 z}{\partial x\partial y}+2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}=0$

$\Rightarrow \frac{\partial^2 z}{\partial x\partial y}=\frac{(yz+1)e^{yz}}{(ye^{yz}+2z)^2}-\frac{ze^{yz}(y^2e^{yz}+2)}{(ye^{yz}+2z)^3}$

49. 设 $u=u(x,y)$ 由 $u=f(x,y,z,t)$，$g(y,z,t)=0$，$h(z,t)=0$ 确定. 求 $\frac{\partial u}{\partial x}$，$\frac{\partial u}{\partial y}$

解：$\begin{cases} du=f'_1dx+f'_2dy+f'_3dz+f'_4dt \\ g'_1dy+g'_2dz+g'_3dt=0 \\ h'_1dz+h'_2dt=0 \end{cases}\Rightarrow \begin{cases} dz=-\frac{g'_1h'_2}{g'_2h'_2-g'_3h'_1}dy \\ dt=\frac{g'_1h'_1}{g'_2h'_2-g'_3h'_1}dy \end{cases}$

$\Rightarrow du=f'_1dx+\left(f'_2-\frac{f'_3g'_1h'_2}{g'_2h'_2-g'_3h'_1}+\frac{f'_4g'_1h'_1}{g'_2h'_2-g'_3h'_1}\right)dy\Rightarrow \frac{\partial u}{\partial x}=f'_1$，$\frac{\partial u}{\partial y}=f'_2-\frac{f'_3g'_1h'_2}{g'_2h'_2-g'_3h'_1}+\frac{f'_4g'_1h'_1}{g'_2h'_2-g'_3h'_1}$

50. 通过 $\begin{cases} u=x-2\sqrt{y} \\ v=x+2\sqrt{y} \end{cases}$ 化简 $\frac{\partial^2 z}{\partial x^2}-y\frac{\partial^2 z}{\partial y^2}-\frac{1}{2}\frac{\partial z}{\partial y}=0$

解：$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$. $\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}$. $\frac{\partial z}{\partial y}=-\frac{1}{\sqrt{y}}\frac{\partial z}{\partial u}+\frac{1}{\sqrt{y}}\frac{\partial z}{\partial v}$.

$\frac{\partial^2 z}{\partial y^2}=\frac{1}{y}\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}-2\frac{\partial^2 z}{\partial u\partial v}-\frac{1}{2\sqrt{y}}\left(\frac{\partial z}{\partial v}-\frac{\partial z}{\partial u}\right)\right)$. 原方程变为 $\frac{\partial^2 z}{\partial u\partial v}=0$

51. 设变换 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$，求 $\frac{\partial(r,\theta)}{\partial(x,y)}$.

解：$\frac{\partial(r,\theta)}{\partial(x,y)}=\left(\frac{\partial(x,y)}{\partial(r,\theta)}\right)^{-1}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}^{-1}=\frac{1}{r}=\frac{1}{\sqrt{x^2+y^2}}$

52. 设变换 $\begin{cases}x=\frac{u^2-v^2}{2}\\ y=uv\\ z=z\end{cases}$，求 $\frac{\partial(x,y,z)}{\partial(u,v,z)}$

解：$\frac{\partial(x,y,z)}{\partial(u,v,z)}=\begin{vmatrix}u & -v & 0\\ v & u & 0\\ 0 & 0 & 1\end{vmatrix}=u^2+v^2$

53. 设椭圆球坐标变换 $\begin{cases}x=ar\sin\varphi\cos\theta\\ y=br\sin\varphi\sin\theta\\ z=cr\cos\varphi\end{cases}$，求 $\frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)}$

解：$\frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)}=\begin{vmatrix}a\sin\varphi\cos\theta & ar\cos\varphi\cos\theta & -ar\sin\varphi\sin\theta\\ b\sin\varphi\sin\theta & br\cos\varphi\sin\theta & br\sin\varphi\cos\theta\\ c\cos\varphi & -cr\sin\varphi & 0\end{vmatrix}=abcr^2\begin{vmatrix}\sin\varphi\cos\theta & \cos\varphi\cos\theta & -\sin\varphi\sin\theta\\ \sin\varphi\sin\theta & \cos\varphi\sin\theta & \sin\varphi\cos\theta\\ \cos\varphi & -\sin\varphi & 0\end{vmatrix}$

$=abcr^2\sin\varphi$

54. 证明不存在 R^n 到 R^m $(m<n)$ 的 C^1 同胚映射.

证明：反设这样的 C^1 同胚 $f(x_1,\cdots,x_n)=(f_1,\cdots,f_m)$ 存在. 于是 $\exists\, i_1,\cdots,i_m$，使得 $\left.\frac{\partial(f_1,\cdots,f_m)}{\partial(x_{i_1},\cdots,x_{i_m})}\right|_{某个x_0=(x_1^0,\cdots,x_n^0)}\neq 0 \Longrightarrow$ 存在逆映射 $x_{i_1}=g_1(y_1,\cdots,y_m),\cdots,x_{i_m}=g_m(y_1,\cdots,y_m)$

这导致相同的 $(y_1,\cdots,y_m)$ 对应不同的 $(x_1,\cdots,x_n)$. 与 C^1 同胚矛盾 #

55. 求极值点：(1) $f(x,y)=x^3+y^3-3x-12y+1$ (2) $f(x,y)=xy\ln(x^2+y^2)$

解：(1) 令 $\frac{\partial f}{\partial x}=3x^2-3=0$, $\frac{\partial f}{\partial y}=3y^2-12=0 \Longrightarrow (x,y)=(\pm1,\pm2)$ Hessi 矩阵 $Hf(x,y)=\begin{pmatrix}6x & 0\\ 0 & 6y\end{pmatrix}$

$Hf(1,2)$ 正定，$Hf(-1,-2)$ 负定，$Hf(\pm1,\mp2)$ 不定. 故 $f(x,y)$ 极小值点 $(1,2)$. 极大值点 $(-1,-2)$

(2) 令 $\frac{\partial f}{\partial x}=y\left(\ln(x^2+y^2)+\frac{2x^2}{x^2+y^2}\right)=0$. $\frac{\partial f}{\partial y}=x\left(\ln(x^2+y^2)+\frac{2y^2}{x^2+y^2}\right)=0 \Longrightarrow (x,y)=(1,0),(0,1),\left(\pm\frac{1}{\sqrt{2e}},\pm\frac{1}{\sqrt{2e}}\right)$

Hessi 矩阵 $Hf(x,y)=\begin{pmatrix}2xy(x^2+4xy+2y^2)/(x^2+y^2)^2 & \ln(x^2+y^2)-\frac{4x^2y^2}{(x^2+y^2)^2}+2\\ \ln(x^2+y^2)-\frac{4x^2y^2}{(x^2+y^2)^2}+2 & 2xy(y^2+4xy+2x^2)/(x^2+y^2)^2\end{pmatrix}$

$Hf(1,0)$, $Hf(0,1)$ 不定. $Hf\left(\frac{1}{\sqrt{2e}},-\frac{1}{\sqrt{2e}}\right)$、$Hf\left(-\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}}\right)$ 负定，$Hf\left(\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}}\right)$. $Hf\left(-\frac{1}{\sqrt{2e}},-\frac{1}{\sqrt{2e}}\right)$

故 $f(x,y)$ 极小值点 $\left(\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}}\right),\left(-\frac{1}{\sqrt{2e}},-\frac{1}{\sqrt{2e}}\right)$. 极大值点 $\left(\frac{1}{\sqrt{2e}},-\frac{1}{\sqrt{2e}}\right),\left(-\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}}\right)$

56. 证明：(1) $(0,0)$ 是 $f(x,y)=(y-x^2)(y-3x^2)$ 的鞍点 (2) 当 $f(x,y)$ 定义域在过 $(0,0)$ 的任一直线时，它在 $(0,0)$ 取极小值.

证明：(1) $\frac{\partial f}{\partial x}=12x^3-8xy$. $\frac{\partial f}{\partial y}=2y-4x^2$. $Hf(x,y)=\begin{pmatrix}36x^2-8y & -8x\\ -8x & 2\end{pmatrix}$. $\mathrm{grad}f(0,0)=0$. $Hf(0,0)$ 不

故 $(0,0)$ 是 $f(x,y)$ 的鞍点

(2) 若直线是 $x=0$ 或 $y=0$，则结论显然. 不然设直线为 $y=kx$ $(k\neq0)$. $f(x,y)=g(x)=(kx-x^2)(kx-3x^2)$

$g''(x)=2k^2-24kx+36x^2$ $g''(0)=2k^2>0 \Longrightarrow x=0$ 是 $g(x)$ 的极小值点.

故 $f(x,y)$ 在 $(0,0)$ 取极小值 #

解: $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$, $\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}$, $\frac{\partial z}{\partial y}=-\frac{1}{\sqrt{y}}\frac{\partial z}{\partial u}+\frac{1}{\sqrt{y}}\frac{\partial z}{\partial v}$.

$\frac{\partial^2 z}{\partial y^2}=\frac{1}{y}\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}-2\frac{\partial^2 z}{\partial u\partial v}-\frac{1}{2\sqrt{y}}\left(\frac{\partial z}{\partial v}-\frac{\partial z}{\partial u}\right)\right)$. 原方程变为 $\frac{\partial^2 z}{\partial u\partial v}=0$

51. 设变换 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$, 求 $\frac{\partial(r,\theta)}{\partial(x,y)}$.

解: $\frac{\partial(r,\theta)}{\partial(x,y)}=\left(\frac{\partial(x,y)}{\partial(r,\theta)}\right)^{-1}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}^{-1}=\frac{1}{r}=\frac{1}{\sqrt{x^2+y^2}}$

52. 设变换 $\begin{cases}x=\frac{u^2-v^2}{2}\\ y=uv\\ z=z\end{cases}$, 求 $\frac{\partial(x,y,z)}{\partial(u,v,z)}$

解: $\frac{\partial(x,y,z)}{\partial(u,v,z)}=\begin{vmatrix}u & -v & 0\\ v & u & 0\\ 0 & 0 & 1\end{vmatrix}=u^2+v^2$

53. 设椭圆球坐标变换 $\begin{cases}x=ar\sin\varphi\cos\theta\\ y=br\sin\varphi\sin\theta\\ z=cr\cos\varphi\end{cases}$, 求 $\frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)}$

解: $\frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)}=\begin{vmatrix}a\sin\varphi\cos\theta & ar\cos\varphi\cos\theta & -ar\sin\varphi\sin\theta\\ b\sin\varphi\sin\theta & br\cos\varphi\sin\theta & br\sin\varphi\cos\theta\\ c\cos\varphi & -cr\sin\varphi & 0\end{vmatrix}=abcr^2\begin{vmatrix}\sin\varphi\cos\theta & \cos\varphi\cos\theta & -\sin\varphi\sin\theta\\ \sin\varphi\sin\theta & \cos\varphi\sin\theta & \sin\varphi\cos\theta\\ \cos\varphi & -\sin\varphi & 0\end{vmatrix}$

$=abcr^2\sin\varphi$

54. 证明不存在 R^n 到 R^m $(m<n)$ 的 C^1 同胚映射.

证明: 反设这样的 C^1 同胚 $f(x_1,\cdots,x_n)=(f_1,\cdots,f_m)$ 存在. 于是 $\exists\, i_1,\cdots,i_m$, 使得 $\left.\frac{\partial(f_1,\cdots,f_m)}{\partial(x_{i_1},\cdots,x_{i_m})}\right|_{某个x_0=(x_1^0,\cdots,x_n^0)}\neq 0$ $\Longrightarrow$ 存在逆映射 $x_{i_1}=g_1(y_1,\cdots,y_m),\cdots,x_{i_m}=g_m(y_1,\cdots,y_m)$

这导致相同的 $(y_1,\cdots,y_m)$ 对应不同的 $(x_1,\cdots,x_n)$. 与 C^1 同胚矛盾 #

55. 求极值点: (1) $f(x,y)=x^3+y^3-3x-12y+1$ (2) $f(x,y)=xy\ln(x^2+y^2)$

解: (1) 令 $\frac{\partial f}{\partial x}=3x^2-3=0$, $\frac{\partial f}{\partial y}=3y^2-12=0$ $\Longrightarrow (x,y)=(\pm1,\pm2)$. Hessi 矩阵 $Hf(x,y)=\begin{pmatrix}6x & 0\\ 0 & 6y\end{pmatrix}$

$Hf(1,2)$ 正定, $Hf(-1,-2)$ 负定, $Hf(\pm1,\mp2)$ 不定. 故 $f(x,y)$ 极小值点 $(1,2)$. 极大值点 $(-1,-2)$

(2) 令 $\frac{\partial f}{\partial x}=y\left(\ln(x^2+y^2)+\frac{2x^2}{x^2+y^2}\right)=0$. $\frac{\partial f}{\partial y}=x\left(\ln(x^2+y^2)+\frac{2y^2}{x^2+y^2}\right)=0$ $\Longrightarrow (x,y)=(1,0),(0,1),\left(\pm\frac{1}{\sqrt{2e}},\pm\frac{1}{\sqrt{2e}}\right)$

Hessi 矩阵 $Hf(x,y)=\begin{pmatrix}2xy(x^2+4xy+2y^2)/(x^2+y^2)^2 & \ln(x^2+y^2)-\frac{4x^2y^2}{(x^2+y^2)^2}+2\\ \ln(x^2+y^2)-\frac{4x^2y^2}{(x^2+y^2)^2}+2 & 2xy(y^2+4xy+2x^2)/(x^2+y^2)^2\end{pmatrix}$

$Hf(1,0)$, $Hf(0,1)$ 不定. $Hf\left(\frac{1}{\sqrt{2e}},-\frac{1}{\sqrt{2e}}\right)$, $Hf\left(-\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}}\right)$ 负定, $Hf\left(\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}}\right)$, $Hf\left(-\frac{1}{\sqrt{2e}},-\frac{1}{\sqrt{2e}}\right)$

故 $f(x,y)$ 极小值点 $\left(\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}}\right)$, $\left(-\frac{1}{\sqrt{2e}},-\frac{1}{\sqrt{2e}}\right)$. 极大值点 $\left(\frac{1}{\sqrt{2e}},-\frac{1}{\sqrt{2e}}\right)$, $\left(-\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}}\right)$

56. 证明: (1) $(0,0)$ 是 $f(x,y)=(y-x^2)(y-3x^2)$ 的鞍点 (2) 当 $f(x,y)$ 定义域为过 $(0,0)$ 的任一直线时包在 $(0,0)$ 取极小值.

证明: (1) $\frac{\partial f}{\partial x}=12x^3-8xy$, $\frac{\partial f}{\partial y}=2y-4x^2$. $Hf(x,y)=\begin{pmatrix}36x^2-8y & -8x\\ -8x & 2\end{pmatrix}$. $\operatorname{grad} f(0,0)=0$. $Hf(0,0)$ 不

故 $(0,0)$ 是 $f(x,y)$ 的鞍点

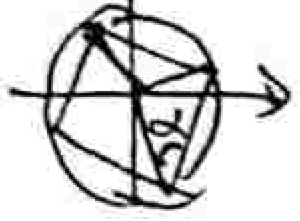
(2) 若直线是 $x=0$ 或 $y=0$ 则结论显然. 不然设直线为 $y=kx$. $f(x,y)=g(x)=(kx-x^2)(kx-3x^2)$

$g''(x)=2k^2-24kx+36x^2$ $g''(0)=2k^2>0 \Longrightarrow x=0$ 是 $g(x)$ 的极小值点

故 $f(x,y)$ 在 $(0,0)$ 取极小值 #

57. 设 $F(x,u,v)=0$ 和 $G(x,u,v)=0$ 确定可微函数组 $\begin{cases} u=u(x) \\ v=v(x) \end{cases}$. 求 $u=u(x)$ 的驻点满足的必要条件.

解: $u=u(x)$ 的驻点满足 $\frac{\partial u}{\partial x}=0$. 在 $F(x,u,v)=0$, $G(x,u,v)=0$ 两边对 x 求导:
$F_1'+F_2'\frac{\partial u}{\partial x}+F_3'\frac{\partial v}{\partial x}=0=G_1'+G_2'\frac{\partial u}{\partial x}+G_3'\frac{\partial v}{\partial x}$. 代入 $\frac{\partial u}{\partial x}=0$, 得 $(F_1'G_3'-F_3'G_1')\frac{\partial v}{\partial x}=0$
若 $\frac{\partial v}{\partial x}=0$, 则 $F_1'=G_1'=0$, 不成立. 故 $F_1'G_3'=F_3'G_1'$

58. 分别求 $\mathbb{R}^2$ 中单位圆内接三角形和内接长方形的最大面积
解: 三角形: 转化为求 $f(x,y,z)=\frac{1}{2}(\sin x+\sin y+\sin z)$ $(x,y,z>0)$ 在 $x+y+z=2\pi$ 下的条件极值
令 $F(x,y,z,\lambda)=\frac{1}{2}(\sin x+\sin y+\sin z)-\lambda(x+y+z-2\pi)$. 令 $F_x'=\frac{1}{2}\cos x-\lambda=0$. $F_y'=\frac{1}{2}\cos y-\lambda=0$
$F_z'=\frac{1}{2}\cos z-\lambda=0$, $F_\lambda'=x+y+z-2\pi=0 \Rightarrow x=y=z=\frac{2\pi}{3}$. $H_F(\frac{2\pi}{3},\frac{2\pi}{3},\frac{2\pi}{3})=\begin{pmatrix}-\frac{\sqrt{3}}{4} & 0 & 0\\ 0 & -\frac{\sqrt{3}}{4} & 0\\ 0 & 0 & -\frac{\sqrt{3}}{4}\end{pmatrix}$ 负定
故 $f(x,y,z)$ 有极大值 $f(\frac{2\pi}{3},\frac{2\pi}{3},\frac{2\pi}{3})=\frac{3\sqrt{3}}{4}$. 单位圆内接三角形最大面积为 $\frac{3\sqrt{3}}{4}$
长方形:  $S=g(\alpha)=2\sin 2\alpha\le 2$. 等号在 $\alpha=\frac{\pi}{4}$ 取得. 单位圆内接长方形最大面积为 2

59. 设 $u=u(x,y)$ 在 $\Delta=\{(x,y): x^2+y^2<1\}$ 的闭包上有二阶连续偏导数. 在 Δ 内满足 $u(x,y)=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$ 且在 $\partial\Delta$ 上 $u(x,y)\equiv 0$. 证明: 在 $\bar{\Delta}$ 上 $u(x,y)\equiv 0$
证明: 若不然, $\exists (x_1,y_1)\in\Delta$ 使得 $u(x_1,y_1)\ne 0$. 不妨设 $u(x_1,y_1)>0$, 从而 $u(x,y)$ 在 $\bar{\Delta}$ 上取到最大值, 设为 $u(x_0,y_0)$. $(x_0,y_0)\in\Delta$, 则 (x_0,y_0) 是极大值点 $\Rightarrow \frac{\partial^2 u}{\partial x^2}(x_0,y_0)\le 0$, $\frac{\partial^2 u}{\partial x^2}(x_0,y_0)\frac{\partial^2 u}{\partial y^2}(x_0,y_0)-\left(\frac{\partial^2 u}{\partial x\partial y}(x_0,y_0)\right)^2\ge 0 \Rightarrow \frac{\partial^2 u}{\partial y^2}(x_0,y_0)\le 0 \Rightarrow u(x_0,y_0)=\frac{\partial^2 u}{\partial x^2}(x_0,y_0)+\frac{\partial^2 u}{\partial y^2}(x_0,y_0)\le 0$. 矛盾 #

60. 体积为1的圆柱. 问如何设计底面半径和高使表面积最小?
解: 设底面半径为 r, 高为 h, 则 $\pi r^2 h=1$. 表面积 $S(r,h)=2\pi rh+2\pi r^2$
令 $f(r,h)=2\pi rh+2\pi r^2-\lambda(\pi r^2 h-1)$. 令 $\frac{\partial f}{\partial r}=2\pi h+4\pi r-2\pi\lambda rh=0$, $\frac{\partial f}{\partial h}=2\pi r-\lambda\pi r^2=0$
$\frac{\partial f}{\partial\lambda}=\pi r^2 h-1=0 \Rightarrow r=\sqrt[3]{\frac{1}{2\pi}}$, $h=\sqrt[3]{\frac{4}{\pi}}$. 此问题一定有最小值, 从而底面半径为 $\sqrt[3]{\frac{1}{2\pi}}$, 高为 $\sqrt[3]{\frac{4}{\pi}}$

61. 求 $x_0=(x_1^0,\dots,x_n^0)\in\mathbb{R}^n$ 到平面 $\sum_{i=1}^n a_i x_i=0$ 的距离
解: 设 $f(x_1,\dots,x_n)=\left(\sum_{i=1}^n (x_i-x_i^0)^2\right)^{1/2}$. 即求 f 在 $\sum_{i=1}^n a_i x_i=0$ 下的最小值. 令
$F(x_1,\dots,x_n,\lambda)=\left(\sum_{i=1}^n (x_i-x_i^0)^2\right)^{1/2}-\lambda\sum_{i=1}^n a_i x_i$. 令 $\frac{\partial F}{\partial x_i}=\frac{x_i-x_i^0}{\left(\sum_{i=1}^n (x_i-x_i^0)^2\right)^{1/2}}-\lambda a_i=0$.
$\frac{\partial F}{\partial\lambda}=\sum_{i=1}^n a_i x_i=0 \Rightarrow \lambda^2=\frac{1}{\sum_{i=1}^n a_i^2}$, $f(x_1,\dots,x_n)=\left|\sum_{i=1}^n a_i x_i^0\right|/\left(\sum_{i=1}^n a_i^2\right)^{1/2}$
此时 Hessi 矩阵是半正定 (除非 x_0 在平面上, 这时距离自然为0). 故距离为
$$\frac{\left|\sum_{i=1}^n a_i x_i^0\right|}{\sqrt{\sum_{i=1}^n a_i^2}}$$

62. 求原点到椭圆 $\begin{cases} z=x^2+y^2 \\ x+y+z=1 \end{cases}$ 的最小、最大距离
解: 令 $F(x,y,z)\triangleq(x^2+y^2+z^2)-\lambda(z-x^2-y^2)-\mu(x+y+z-1)$. $\frac{\partial F}{\partial x}=2(\lambda+1)x-\mu=0$.
$\frac{\partial F}{\partial y}=2(\lambda+1)y-\mu=0$. $\frac{\partial F}{\partial z}=2z-(\lambda+\mu)=0$. $\frac{\partial F}{\partial\lambda}=-z+x^2+y^2=0$. $\frac{\partial F}{\partial\mu}=-x-y-z+1=0$
$\Rightarrow (x,y,z)=\left(\frac{-1+\sqrt{3}}{2},\frac{-1+\sqrt{3}}{2},2-\sqrt{3}\right),\left(\frac{-1-\sqrt{3}}{2},\frac{-1-\sqrt{3}}{2},2+\sqrt{3}\right)$ 因为距离最值一定存在,
所以最小距离为 $\sqrt{9-5\sqrt{3}}$, 最大距离为 $\sqrt{9+5\sqrt{3}}$.

63. 求 $f(x,y,z)=4x^2+y^2+5z^2$ 在平面 $2x+3y+4z=12$ 上的最小值点
解: 令 $F(x,y,z)=4x^2+y^2+5z^2-\lambda(2x+3y+4z-12)$. $\frac{\partial F}{\partial x}=8x-2\lambda=0$. $\frac{\partial F}{\partial y}=2y-3\lambda=0$, $\frac{\partial F}{\partial z}=10z-4\lambda=0$, $\frac{\partial F}{\partial\lambda}=2x+3y+4z-12=0 \Rightarrow x=\frac{5}{11}$, $y=\frac{30}{11}$, $z=\frac{8}{11}$. 就是最小值点
故所求为 $(\frac{5}{11},\frac{30}{11},\frac{8}{11})$

57. 设 $F(x,u,v)=0$ 和 $G(x,u,v)=0$ 在确定可微函数组 $\begin{cases}u=u(x)\\ v=v(x)\end{cases}$，求 $u=u(x)$ 的驻点满足的必要条件.

解：$u=u(x)$ 的驻点满足 $\frac{\partial u}{\partial x}=0$. 在 $F(x,u,v)=0$, $G(x,u,v)=0$ 两边对 x 求导：

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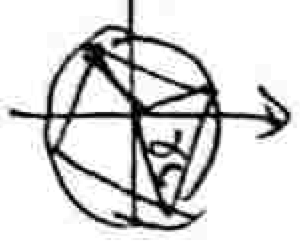
令 $F(x,y,z,\lambda)=\frac{1}{2}(\sin x+\sin y+\sin z)-\lambda(x+y+z-2\pi)$. 令 $F_x'=\frac{1}{2}\cos x-\lambda=0$. $F_y'=\frac{1}{2}\cos y-\lambda=0$

$F_z'=\frac{1}{2}\cos z-\lambda=0$, $F_\lambda'=x+y+z-2\pi=0 \Rightarrow x=y=z=\frac{2\pi}{3}$. $H_F(\frac{2\pi}{3},\frac{2\pi}{3},\frac{2\pi}{3})=\begin{pmatrix}-\frac{\sqrt3}{4}&0&0\\0&-\frac{\sqrt3}{4}&0\\0&0&-\frac{\sqrt3}{4}\end{pmatrix}$ 负定

故 $f(x,y,z)$ 有极大值 $f(\frac{2\pi}{3},\frac{2\pi}{3},\frac{2\pi}{3})=\frac{3\sqrt3}{4}$. 单位圆内接三角形最大面积为 $\frac{3\sqrt3}{4}$

长方形： $S=g(\alpha)=2\sin 2\alpha\le 2$. 等号在 $\alpha=\frac{\pi}{4}$ 取得. 单位圆内接长方形最大面积为2

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令 $f(r,h)=2\pi rh+2\pi r^2-\lambda(\pi r^2h-1)$. 令 $\frac{\partial f}{\partial r}=2\pi h+4\pi r-2\pi\lambda rh=0$, $\frac{\partial f}{\partial h}=2\pi r-\lambda\pi r^2=0$

$\frac{\partial f}{\partial\lambda}=\pi r^2h-1=0 \Rightarrow r=\sqrt[3]{\frac{1}{2\pi}}$, $h=\sqrt[3]{\frac{4}{\pi}}$. 此问题一定有最小值，从而底面半径为 $\sqrt[3]{\frac{1}{2\pi}}$，高为 $\sqrt[3]{\frac{4}{\pi}}$

61. 求 $x_0=(x_1^0,\cdots,x_n^0)\in\mathbb{R}^n$ 到平面 $\sum_{i=1}^n a_ix_i=0$ 的距离

解：设 $f(x_1,\cdots,x_n)=\left(\sum_{i=1}^n(x_i-x_i^0)^2\right)^{1/2}$. 即求 f 在 $\sum_{i=1}^n a_ix_i=0$ 下的最小值. 令

$F(x_1,\cdots,x_n,\lambda)=\left(\sum_{i=1}^n(x_i-x_i^0)^2\right)^{1/2}-\lambda\sum_{i=1}^n a_ix_i$. 令 $\frac{\partial F}{\partial x_i}=\frac{x_i-x_i^0}{\left(\sum_{i=1}^n(x_i-x_i^0)^2\right)^{1/2}}-\lambda a_i=0$.

$\frac{\partial F}{\partial\lambda}=\sum_{i=1}^n a_ix_i=0 \Rightarrow \lambda^2=\frac{1}{\sum_{i=1}^n a_i^2}$, $f(x_1,\cdots,x_n)=\left|\sum_{i=1}^n a_ix_i^0\right|/\left(\sum_{i=1}^n a_i^2\right)^{1/2}$

此时 Hessi 矩阵显然正定（除非 x_0 在平面上，这时距离自然为0）. 故距离为

$$\frac{\left|\sum_{i=1}^n a_ix_i^0\right|}{\sqrt{\sum_{i=1}^n a_i^2}}$$

62. 求原点到椭圆 $\begin{cases}z=x^2+y^2\\ x+y+z=1\end{cases}$ 的最小、最大距离

解：令 $F(x,y,z)\xlongequal{\Delta}(x^2+y^2+z^2)-\lambda(z-x^2-y^2)-\mu(x+y+z-1)$. $\frac{\partial F}{\partial x}=2(\lambda+1)x-\mu=0$.

$\frac{\partial F}{\partial y}=2(\lambda+1)y-\mu=0$. $\frac{\partial F}{\partial z}=2z-(\lambda+\mu)=0$. $\frac{\partial F}{\partial\lambda}=-z+x^2+y^2=0$. $\frac{\partial F}{\partial\mu}=-x-y-z+1=0$

$\Rightarrow (x,y,z)=\left(\frac{-1+\sqrt3}{2},\frac{-1+\sqrt3}{2},2-\sqrt3\right)$, $\left(\frac{-1-\sqrt3}{2},\frac{-1-\sqrt3}{2},2+\sqrt3\right)$ 因为距离最值一定存在.

所以最小距离为 $\sqrt{9-5\sqrt3}$，最大距离为 $\sqrt{9+5\sqrt3}$.

63. 求 $f(x,y,z)=4x^2+y^2+5z^2$ 在平面 $2x+3y+4z=12$ 上的最小值点

解：令 $F(x,y,z)=4x^2+y^2+5z^2-\lambda(2x+3y+4z-12)$. $\frac{\partial F}{\partial x}=8x-2\lambda=0$. $\frac{\partial F}{\partial y}=2y-3\lambda=0$, $\frac{\partial F}{\partial z}=10z-4\lambda=0$, $\frac{\partial F}{\partial\lambda}=2x+3y+4z-12=0 \Rightarrow x=\frac{5}{11}$, $y=\frac{30}{11}$, $z=\frac{8}{11}$. 显然是最小值点

故所求为 $\left(\frac{5}{11},\frac{30}{11},\frac{8}{11}\right)$

64. 求原点到曲线 $\begin{cases} xyz=1 \\ y=2x \end{cases}$ 的最短距离

解：令 $F(x,y,z,\lambda,\mu)=x^2+y^2+z^2-\lambda(xyz-1)-\mu(y-2x)$. $\frac{\partial F}{\partial x}=2x-\lambda yz+2\mu=0$, $\frac{\partial F}{\partial y}=2y-\lambda xz-\mu=0$

$\frac{\partial F}{\partial z}=2z-\lambda xy=0$. $\frac{\partial F}{\partial \lambda}=1-xyz=0$. $\frac{\partial F}{\partial \mu}=2x-y=0 \Longrightarrow x=\frac{1}{\sqrt[6]{10}}$, $y=\frac{2}{\sqrt[6]{10}}$, $z=\frac{\sqrt[3]{10}}{2}$

这些也是极小值点. 故最短距离为 $\sqrt[6]{\frac{675}{16}}$

65. 求曲线 $\begin{cases} x-y+4z=1 \\ 2x^2+4y^2=3 \end{cases}$ 上最高点与最低点的高度

解：$z=\frac{-x+y+1}{4}$. 只需求 $f(x,y)=\frac{-x+y+1}{4}$ 在 $2x^2+4y^2=3$ 下的最值. 设 $F(x,y,\lambda)=\frac{-x+y+1}{4}-\lambda(2x^2+4y^2-3)$

令 $\frac{\partial F}{\partial x}=-\frac{1}{4}-4\lambda x=0$, $\frac{\partial F}{\partial y}=\frac{1}{4}-8\lambda y=0$, $\frac{\partial F}{\partial \lambda}=-2x^2-4y^2+3=0 \Longrightarrow (x,y)=(1,-\frac{1}{2})$ 或 $(-1,\frac{1}{2})$

前者为极小值点. 后者为极大值点. 故最高点高度为 $\frac{5}{8}$, 最低点高度为 $-\frac{1}{8}$

66. 求 $f(x_1,\cdots,x_n)=\sum_{i=1}^n x_i$ 在球面 $x_1^2+\cdots+x_n^2=1$ 上的最大值，并证明 $\frac{1}{n}\sum_{i=1}^n x_i \le \left(\frac{1}{n}\sum_{i=1}^n x_i^2\right)^{1/2}$

解：令 $F(x_1,\cdots,x_n,\lambda)=\sum_{i=1}^n x_i-\lambda\left(\sum_{i=1}^n x_i^2-1\right)$, $\frac{\partial F}{\partial x_i}=1-2\lambda x_i=0$, $\frac{\partial F}{\partial \lambda}=1-\sum_{i=1}^n x_i^2=0 \Longrightarrow$

$x_1=\cdots=x_n=\pm\frac{1}{\sqrt{n}}$. 由 f 连续知 f 在球面上一定取到最值. 故最大值为 $\sqrt{n}$.

令 $y_i=\frac{x_i}{\sqrt{\sum_{i=1}^n x_i^2}}$, 则有 $y_1^2+\cdots+y_n^2=1 \Longrightarrow \sum_{i=1}^n y_i \le \sqrt{n} \Longrightarrow \frac{1}{n}\sum_{i=1}^n x_i \le \left(\frac{1}{n}\sum_{i=1}^n x_i^2\right)^{1/2}$

67. 设 $f(x)$ 在 $[0,1]$ 上可积. 求 $g(a,b,c)=\int_0^1(f(x)-ax^2-bx-c)^2dx$ $((a,b,c)\in\mathbb{R}^3)$ 的最小值点.

解：$g(a,b,c)=\frac{1}{5}a^2+\frac{1}{3}b^2+c^2+\frac{1}{2}ab+\frac{2}{3}ac+bc-\left(2\int_0^1 x^2f(x)dx\right)a-\left(2\int_0^1 xf(x)dx\right)b-\left(2\int_0^1 f(x)dx\right)c+\int_0^1 f^2(x)dx$. 令 $\frac{\partial g}{\partial a}=\frac{2}{5}a+\frac{1}{2}b+\frac{2}{3}c-2\int_0^1 x^2f(x)dx=0$

$\frac{\partial g}{\partial b}=\frac{2}{3}b+\frac{1}{2}a+c-2\int_0^1 xf(x)dx=0$, $\frac{\partial g}{\partial c}=2c+\frac{2}{3}a+b-2\int_0^1 f(x)dx=0 \Longrightarrow$

$a=\int_0^1 f(x)(180x^2-180x+30)dx$, $b=\int_0^1 f(x)(-180x^2+192x-36)dx$, $c=\int_0^1 f(x)(30x^2-36x+9)dx$

由 $\lim_{(a,b,c)\to(\infty,\infty,\infty)} g(a,b,c)=+\infty$ 知 $g(a,b,c)$ 取到最小值. 故 $g(a,b,c)$ 的最小值点为

$\left(\int_0^1 f(x)(180x^2-180x+30)dx,\ \int_0^1 f(x)(-180x^2+192x-36)dx;\ \int_0^1 f(x)(30x^2-36x+9)dx\right)$

68. 求 $z=\frac{1}{2}(x^k+y^k)$ 在 $x+y=c>0$ 下的极值，并证明 $\forall a,b\ge 0$. $\forall k\in\mathbb{N}$, $\left(\frac{a+b}{2}\right)^k \le \frac{a^k+b^k}{2}$

解：令 $F(x,y,\lambda)=\frac{1}{2}(x^k+y^k)-\lambda(x+y-c)$. $\frac{\partial F}{\partial x}=\frac{1}{2}kx^{k-1}-\lambda=0$. $\frac{\partial F}{\partial y}=\frac{1}{2}ky^{k-1}-\lambda=0$.

$\frac{\partial F}{\partial \lambda}=-(x+y-c)=0 \Longrightarrow x=y=\frac{c}{2}$. 极值为 $\left(\frac{c}{2}\right)^k$

$k\in\mathbb{N}$ 时，上述极值为极小，于是 $\frac{a^k+b^k}{2}\ge\left(\frac{a+b}{2}\right)^k$

69. 椭球面 $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$ 与平面 $x+y-z=0$ 的交线为一椭圆. 求该椭圆在该平面内所围区域的面积.

解：椭圆中心为 $(0,0,0)$, 来求长、短半轴. 只需求 $f(x,y,z)=x^2+y^2+z^2$ 在 $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$ 和 $x+y-z=0$ 下的最大、最小值. 令 $F(x,y,z,\lambda,\mu)=x^2+y^2+z^2-\lambda\left(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1\right)-\mu(x+y-z)=0$

$\frac{\partial F}{\partial x}=(2-\frac{\lambda}{2})x-\mu=0$. $\frac{\partial F}{\partial y}=(2-\frac{2}{5}\lambda)y-\mu=0$. $\frac{\partial F}{\partial z}=(2-\frac{2}{25}\lambda)z+\mu=0$. $\frac{\partial F}{\partial \lambda}=-\left(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1\right)=0$

$\frac{\partial F}{\partial \mu}=-(x+y-z)=0 \Longrightarrow (x^2,y^2,z^2)=\left(\frac{20}{19},\frac{45}{19},\frac{125}{19}\right)$, $\left(\frac{800}{323},\frac{1225}{646},\frac{25}{646}\right)$

由此长半轴长为 $\sqrt{10}$, 短半轴长为 $\sqrt{\frac{25}{17}}$, 面积为 $\pi\sqrt{\frac{250}{17}}$

64. 求原点到曲线 $\begin{cases} xyz=1 \\ y=2x \end{cases}$ 的最短距离

解：令 $\cancel{\ }F(x,y,z,\lambda,\mu)=x^2+y^2+z^2-\lambda(xyz-1)-\mu(y-2x)$. $\frac{\partial F}{\partial x}=2x-\lambda yz+2\mu=0$, $\frac{\partial F}{\partial y}=2y-\lambda xz-\mu=$
$\frac{\partial F}{\partial z}=2z-\lambda xy=0$. $\frac{\partial F}{\partial \lambda}=1-xyz=0$. $\frac{\partial F}{\partial \mu}=2x-y=0 \Longrightarrow x=\frac{1}{\sqrt[6]{10}}$, $y=\frac{2}{\sqrt[6]{10}}$, $z=\frac{\sqrt[3]{10}}{2}$

这点也是极小值点. 故最短距离为 $\sqrt[6]{\frac{675}{16}}$

65. 求曲线 $\begin{cases} x-y+4z=1 \\ 2x^2+4y^2=3 \end{cases}$ 上最高点与最低点的高度

解：$z=\frac{-x+y+1}{4}$. 只需求 $f(x,y)=\frac{-x+y+1}{4}$ 在 $2x^2+4y^2=3$ 下的最值. 设 $F(x,y,\lambda)=\frac{-x+y+1}{4}-\lambda(2x^2+4y^2-3)$

令 $\frac{\partial F}{\partial x}=-\frac{1}{4}-4\lambda x=0$, $\frac{\partial F}{\partial y}=\frac{1}{4}-8\lambda y=0$, $\frac{\partial F}{\partial \lambda}=-2x^2-4y^2+3=0 \Longrightarrow (x,y)=(1,-\frac{1}{2})$ 或 $(-1,\frac{1}{2})$

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解：令 $F(x_1,\cdots,x_n,\lambda)=\sum_{i=1}^n x_i-\lambda(\sum_{i=1}^n x_i^2-1)$, $\frac{\partial F}{\partial x_i}=1-2\lambda x_i=0$, $\frac{\partial F}{\partial \lambda}=1-\sum_{i=1}^n x_i^2=0 \Longrightarrow$
$x_1=\cdots=x_n=\pm\frac{1}{\sqrt{n}}$. 由 f 连续知 f 在球面上一定取到最值. 故最大值为 $\sqrt{n}$.

令 $y_i=\frac{x_i}{\sqrt{\sum_{i=1}^n x_i^2}}$, 则有 $y_1^2+\cdots+y_n^2=1 \Longrightarrow \sum_{i=1}^n y_i \le \sqrt{n} \Longrightarrow \frac{1}{n}\sum_{i=1}^n x_i \le (\frac{1}{n}\sum_{i=1}^n x_i^2)^{1/2}$

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$(2\int_0^1 xf(x)dx)b-(2\int_0^1 f(x)dx)c+\int_0^1 f^2(x)dx$. 令 $\frac{\partial g}{\partial a}=\frac{2}{5}a+\frac{1}{2}b+\frac{2}{3}c-2\int_0^1 x^2f(x)dx=0$
$\frac{\partial g}{\partial b}=\frac{2}{3}b+\frac{1}{2}a+c-2\int_0^1 xf(x)dx=0$, $\frac{\partial g}{\partial c}=2c+\frac{2}{3}a+b-2\int_0^1 f(x)dx=0 \Longrightarrow$
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由 $\lim_{(a,b,c)\to(\infty,\infty,\infty)} g(a,b,c)=+\infty$ 知 $g(a,b,c)$ 取到最小值. 故 $g(a,b,c)$ 的最小值点为
$(\int_0^1 f(x)(180x^2-180x+30)dx, \int_0^1 f(x)(-180x^2+192x-36)dx, \int_0^1 f(x)(30x^2-36x+9)dx)$

68. 求 $z=\frac{1}{2}(x^k+y^k)$ 在 $x+y=c>0$ 下的极值, 并证明 $\forall a,b\ge 0$, $\forall k\in\mathbb{N}$, $(\frac{a+b}{2})^k \le \frac{a^k+b^k}{2}$

解：令 $F(x,y,\lambda)=\frac{1}{2}(x^k+y^k)-\lambda(x+y-c)$. $\frac{\partial F}{\partial x}=\frac{1}{2}kx^{k-1}-\lambda=0$. $\frac{\partial F}{\partial y}=\frac{1}{2}ky^{k-1}-\lambda=0$.
$\frac{\partial F}{\partial \lambda}=-(x+y-c)=0 \Longrightarrow x=y=\frac{c}{2}$. 极值为 $(\frac{c}{2})^k$

$k\in\mathbb{N}$ 时, 上述极值为极小, 于是 $\frac{a^k+b^k}{2} \ge (\frac{a+b}{2})^k$

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解：椭圆中心为 $(0,0,0)$, 来求长、短半轴. 只需求 $f(x,y,z)=x^2+y^2+z^2$ 在 $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$ 和 $x+y-z=0$ 下的最大、最小值. 令 $F(x,y,z,\lambda,\mu)=x^2+y^2+z^2-\lambda(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1)-\mu(x+y-z)=0$
$\frac{\partial F}{\partial x}=(2-\frac{\lambda}{2})x-\mu=0$. $\frac{\partial F}{\partial y}=(2-\frac{2}{5}\lambda)y-\mu=0$. $\frac{\partial F}{\partial z}=(2-\frac{2}{25}\lambda)z+\mu=0$. $\frac{\partial F}{\partial \lambda}=-(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1)=0$
$\frac{\partial F}{\partial \mu}=-(x+y-z)=0 \Longrightarrow (x^2,y^2,z^2)=(\frac{20}{19},\frac{45}{19},\frac{125}{19})$, $(\frac{800}{323},\frac{1225}{646},\frac{25}{646})$

由此长半轴长为 $\sqrt{10}$, 短半轴长为 $\sqrt{\frac{25}{17}}$, 面积为 $\pi\sqrt{\frac{250}{17}}$

70. 设可微函数 $x=f(u,v)$, $y=g(u,v)$, $z=h(u,v)$ 满足 $F(x,y,z)=0$, 其中 F 是 C^1 的.

证明: $\frac{\partial(y,z)}{\partial(u,v)}dx+\frac{\partial(z,x)}{\partial(u,v)}dy+\frac{\partial(x,y)}{\partial(u,v)}dz=0$

证明: $dx=\frac{\partial f}{\partial u}du+\frac{\partial f}{\partial v}dv$, $dy=\frac{\partial g}{\partial u}du+\frac{\partial g}{\partial v}dv$, $dz=\frac{\partial h}{\partial u}du+\frac{\partial h}{\partial v}dv$.

$$\frac{\partial(y,z)}{\partial(u,v)}dx+\frac{\partial(z,x)}{\partial(u,v)}dy+\frac{\partial(x,y)}{\partial(u,v)}dz=\begin{vmatrix} dx & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ dy & \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ dz & \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}\end{vmatrix}=\begin{vmatrix} \frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv & \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv & \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}\end{vmatrix}=0 \quad \#$$

(也可这样理解: $\left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)}\right)$ 是曲面的法向量, 而 (dx,dy,dz) 是切向量)

71. 求曲线 $\begin{cases}3x^2y+y^2z+z=0\\ 2xz-x^2y-3=0\end{cases}$ 在 $(1,-1,1)$ 处的切线方程与法平面方程.

解: 令 $F(x,y,z)=3x^2y+y^2z+z$, $G(x,y,z)=2xz-x^2y-3$, 则 $\text{grad}\,F(1,-1,1)=(6xy,\ 3x^2+2yz,\ y^2)|_{(1,-1,1)}=(-6,1,1)$, $\text{grad}\,G(1,-1,1)=(2z-2xy,-x^2,2x)|_{(1,-1,1)}=(4,-1,2)$

从而切向量为 $\text{grad}\,F(1,-1,1)\times\text{grad}\,G(1,-1,1)=(3,16,2)$. 切线方程 $\frac{x-1}{3}=\frac{y+1}{16}=\frac{z-1}{2}$

法平面方程 $3x+16y+2z+11=0$

72. 求曲面的切平面方程与法线方程: (1) $x^2+y^2-z^2-4=0$, $(2,1,1)$ (2) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

解: (1) 法向量 $\vec{n}=\text{grad}\,(x^2+y^2-z^2-4)|_{(2,1,1)}=(2x,2y,-2z)|_{(2,1,1)}=(4,2,-2)$

切平面: $2x+y-z-4=0$. 法线: $\frac{x-2}{2}=\frac{y-1}{1}=\frac{z-1}{-1}$

(2) 设在 (x_0,y_0,z_0) 处求解. 法向量 $\vec{n}=\left(\frac{x_0}{a^2},\frac{y_0}{b^2},\frac{z_0}{c^2}\right)$

切平面: $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}+\frac{z_0z}{c^2}=1$. 法线: $\frac{a^2(x-x_0)}{x_0}=\frac{b^2(y-y_0)}{y_0}=\frac{c^2(z-z_0)}{z_0}$

73. 在曲面 $z=x^2-2xy-y^2-8x+4y$ 上求出所有的点, 在这些点处切平面水平

解: 令 $F(x,y,z)=x^2-2xy-y^2-8x+4y-z$. $\text{grad}\,F=(2x-2y-8,\ -2y-2x+4,\ -1)$

切平面水平 $\Longrightarrow 2x-2y-8=0$, $-2y-2x+4=0 \Longrightarrow x=3$, $y=-1$, $z=-14$

故所求为 $(3,-1,-14)$

74. $\mathbb{R}^3$ 中曲面在柱坐标 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\\ z=z\end{cases}$ 下方程为 $F(r,\theta,z)=0$, F 可微. 求 (r_0,θ_0,z_0) 处的切平面方程与法线方程.

解: $F'_r=F'_x\cos\theta+F'_y\sin\theta$. $F'_\theta=F'_x(-r\sin\theta)+F'_y(r\cos\theta)$

$\Longrightarrow F'_x=F'_r\cos\theta-\frac{1}{r}F'_\theta\sin\theta$, $F'_y=F'_r\sin\theta+\frac{1}{r}F'_\theta\cos\theta$

切平面: $(F'_r\cos\theta_0-\frac{1}{r_0}F'_\theta\sin\theta_0)(r\cos\theta-r_0\cos\theta_0)+(F'_r\sin\theta_0+\frac{1}{r_0}F'_\theta\cos\theta_0)(r\sin\theta-r_0\sin\theta_0)$
$+F'_z(z-z_0)=0$

法线: $\frac{r\cos\theta-r_0\cos\theta_0}{F'_r\cos\theta_0-\frac{1}{r_0}F'_\theta\sin\theta_0}=\frac{r\sin\theta-r_0\sin\theta_0}{F'_r\sin\theta_0+\frac{1}{r_0}F'_\theta\cos\theta_0}=\frac{z-z_0}{F'_z}$

75. 设曲面 S 由 $F(x,y,z)=0$ 给出, F 是区域 $D\subset\mathbb{R}^3$ 内的 C^1 函数. 在 $(x_0,y_0,z_0)\in D$ 处 $F'(x_0,y_0,z_0)\neq 0$. 证明该曲面 (x_0,y_0,z_0) 处切平面上过 (x_0,y_0,z_0) 的任一直线都是曲面上过 (x_0,y_0,z_0) 的某光滑曲线的切线.

证明: 对切平面上过 (x_0,y_0,z_0) 的任一直线, 过该直线作一平面与切平面正交. 该平面的方程可以写成 $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$, 这里 $(a,b,c)\cdot F'(x_0,y_0,z_0)=0$

于是 $\begin{cases}F(x,y,z)=0\\ a(x-x_0)+b(y-y_0)+c(z-z_0)=0\end{cases}$ 确定了曲面 $F(x,y,z)=0$ 上的一条曲线, 在 (x_0,y_0,z_0) 处切向量

$(a,b,c)\times F'(x_0,y_0,z_0)\neq 0$, 由 $F(x,y,z)\in C^1(D)$, 曲线在 (x_0,y_0,z_0) 附近的切向量都不为0, 且是连续的. 从而这是一条光滑曲线, 在 (x_0,y_0,z_0) 处切向量平行于直线的法向量

又直线过 (x_0,y_0,z_0), 故直线是曲线的切线 #

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证明: $\frac{\partial(y,z)}{\partial(u,v)}dx+\frac{\partial(z,x)}{\partial(u,v)}dy+\frac{\partial(x,y)}{\partial(u,v)}dz=0$

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切平面: $2x+y-z-4=0$. 法线: $\frac{x-2}{2}=\frac{y-1}{1}=\frac{z-1}{-1}$

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切平面: $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}+\frac{z_0z}{c^2}=1$. 法线: $\frac{a^2(x-x_0)}{x_0}=\frac{b^2(y-y_0)}{y_0}=\frac{c^2(z-z_0)}{z_0}$

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74. $\mathbb{R}^3$ 中曲面在柱坐标 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\\ z=z\end{cases}$ 下方程为 $F(r,\theta,z)=0$, F 可微. 求 (r_0,θ_0,z_0) 处的切平面方程与法线方程.

解: $F'_r=F'_x\cos\theta+F'_y\sin\theta$. $F'_\theta=F'_x(-r\sin\theta)+F'_y(r\cos\theta)$

$\Longrightarrow F'_x=F'_r\cos\theta-\frac{1}{r}F'_\theta\sin\theta$, $F'_y=F'_r\sin\theta+\frac{1}{r}F'_\theta\cos\theta$

切平面: $(F'_r\cos\theta_0-\frac{1}{r_0}F'_\theta\sin\theta_0)(r\cos\theta-r_0\cos\theta_0)+(F'_r\sin\theta_0+\frac{1}{r_0}F'_\theta\cos\theta_0)(r\sin\theta-r_0\sin\theta_0)$

$+F'_z(z-z_0)=0$

法线: $\frac{r\cos\theta-r_0\cos\theta_0}{F'_r\cos\theta_0-\frac{1}{r_0}F'_\theta\sin\theta_0}=\frac{r\sin\theta-r_0\sin\theta_0}{F'_r\sin\theta_0+\frac{1}{r_0}F'_\theta\cos\theta_0}=\frac{z-z_0}{F'_z}$

75. 设曲面 S 由 $F(x,y,z)=0$ 给出, F 是区域 $D\subset\mathbb{R}^3$ 内的 C^1 函数. 在 $(x_0,y_0,z_0)\in D$ 处 $F'(x_0,y_0,z_0)\neq0$. 证明该曲面 (x_0,y_0,z_0) 处切平面上过 (x_0,y_0,z_0) 的任一直线都是曲面上过 (x_0,y_0,z_0) 的某光滑曲线的切线.

证明: 对切平面上过 (x_0,y_0,z_0) 的任一直线, 过该直线作一平面与切平面正交. 该平面的方程可以写成 $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$, 这里 $(a,b,c)\cdot F'(x_0,y_0,z_0)=0$

于是 $\begin{cases}F(x,y,z)=0\\ a(x-x_0)+b(y-y_0)+c(z-z_0)=0\end{cases}$ 确定了曲面 $F(x,y,z)=0$ 上的一条曲线, 切向量 (在 (x_0,y_0,z_0) 处) $(a,b,c)\times F'(x_0,y_0,z_0)\neq0$, 由 $F(x,y,z)\in C^1(D)$, 曲线在 (x_0,y_0,z_0) 附近的切向量都不为0, 且是连续的. 从而这是一条光滑曲线, 切向量 (在 (x_0,y_0,z_0) 处) 平行于直线的法向量

又直线过 (x_0,y_0,z_0), 故直线是曲线的切线 #

76. 证明圆柱螺旋线 $\begin{cases} x=a\cos t \\ y=a\sin t \\ z=bt \end{cases}$ 上任一点处的切线与z轴的夹角为常数

证明：在 $t=t_0$ 处，切向量为 $(x'(t_0), y'(t_0), z'(t_0)) = (-a\sin t_0, a\cos t_0, b)$，与z轴夹角余弦值为 $\dfrac{b}{\sqrt{a^2+b^2}}$ 为常数 #

77. 求曲面 $z = xe^{x/y}$ 上每一点处的切平面方程，并证明任两点处切平面相交

解：$\forall (x_0,y_0,z_0)\in\mathbb{R}^3$，$z_0 = x_0e^{x_0/y_0}$，法向量 $\vec{n} = \left(e^{x_0/y_0}\left(1+\frac{x_0}{y_0}\right), -\frac{x_0^2}{y_0^2}e^{x_0/y_0}, -1\right)$

切平面方程 $e^{x_0/y_0}\left(1+\frac{x_0}{y_0}\right)(x-x_0) - \frac{x_0^2}{y_0^2}e^{x_0/y_0}(y-y_0) - (z-z_0) = 0$

任意点处切平面均过原点，从而相交 #

78. 求圆柱面 $x^2+y^2=1$ 与曲面 $z=xy$ 的夹角

解：在交点 (x_0,y_0,z_0)，圆柱面法向量 $\vec{n_1}=(2x_0,2y_0,0)$，曲面法向量 $\vec{n_2}=(y_0,x_0$

$\cos\theta = \dfrac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \dfrac{4x_0y_0}{2\cdot\sqrt{2}} = \sqrt{2}z_0$，故夹角为 $\arccos\sqrt{2}z_0$

79. 证明曲面 $F(x-az, y-bz)=0$ 上任一点处法线与一条固定直线垂直.

证明：(x_0,y_0,z_0) 处的法向量为 $(F_1'(x_0-az_0), F_2'(y_0-bz_0), -aF_1'(x_0-az_0)-bF_2'(x_0-az_0$

与 $(a,b,1)$ 垂直，所求固定直线为 $\dfrac{x}{a}=\dfrac{y}{b}=z$ #

80. 证明曲面 $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}$ $(a>0)$ 上任一点处切平面与三个坐标轴交点到原点距离之和为常数

证明：设 (x_0,y_0,z_0) 是曲面上任一点，法向量为 $\left(\frac{1}{\sqrt{x_0}}, \frac{1}{\sqrt{y_0}}, \frac{1}{\sqrt{z_0}}\right)$，切平面为

$\frac{1}{\sqrt{x_0}}(x-x_0) + \frac{1}{\sqrt{y_0}}(y-y_0) + \frac{1}{\sqrt{z_0}}(z-z_0) = 0$，与三坐标轴交点为 $(\sqrt{ax_0},0,0)$，$(0,\sqrt{ay_0},0)$，$(0,0,\sqrt{}$

到原点距离之和为 $\sqrt{ax_0}+\sqrt{ay_0}+\sqrt{az_0}=a$ #

81. 证明曲面 $xyz=a$ $(a>0)$ 上任一点处切平面与三个坐标平面所围体积为常数

证明：设 (x_0,y_0,z_0) 是曲面上任一点，法向量为 $\left(\frac{a}{x_0}, \frac{a}{y_0}, \frac{a}{z_0}\right)$，切平面为 $\frac{a}{x_0}(x-x_0)+\frac{a}{y_0}(y-y_0)$
$+\frac{a}{z_0}(z-z_0)=0$，与三坐标平面所围体积为 $\frac{1}{3}\cdot\frac{1}{2}|3x_0||3y_0||3z_0|$
$=\frac{9}{2}a$ #

82. 证明曲面 $x^2+4y+z^2=0$ 与 $x^2+y^2+z^2-6z+7=0$ 在 $(0,-1,2)$ 处相切

证明：第一个曲面在 $(0,-1,2)$ 处法向量为 $(0,4,4)$，第二个曲面在 $(0,-1,2)$ 处法向量为 $(0,-2,-2)$，两个切向量平行，从而二者在 $(0,-1,2)$ 处切平面相同，故两曲面在 $(0,-1,2)$ 处相切 #

76. 证明圆柱螺旋线 $\begin{cases} x=a\cos t \\ y=a\sin t \\ z=bt \end{cases}$ 上任一点处的切线与z轴的夹角为常数

证明：在 $t=t_0$ 处，切向量为 $(x'(t_0), y'(t_0), z'(t_0)) = (-a\sin t_0, a\cos t_0, b)$，与z轴夹角余弦值为 $\frac{b}{\sqrt{a^2+b^2}}$ 为常数 #

77. 求曲面 $z=xe^{x/y}$ 上每一点处的切平面方程，并证明任两点处切平面相交

解：$\forall (x_0,y_0,z_0)\in\mathbb{R}^3$，$z_0=x_0e^{x_0/y_0}$，法向量 $\vec{n}=\left(e^{x_0/y_0}(1+\frac{x_0}{y_0}), -\frac{x_0^2}{y_0^2}e^{x_0/y_0}, -1\right)$

切平面方程 $e^{x_0/y_0}(1+\frac{x_0}{y_0})(x-x_0) - \frac{x_0^2}{y_0^2}e^{x_0/y_0}(y-y_0) - (z-z_0)=0$

任意点处切平面均过原点，从而相交 #

78. 求圆柱面 $x^2+y^2=1$ 与曲面 $z=xy$ 的夹角

解：在交点 (x_0,y_0,z_0)，圆柱面法向量 $\vec{n}_1=(2x_0,2y_0,0)$，曲面法向量 $\vec{n}_2=(y_0,x_0$

$\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{4x_0y_0}{2\cdot\sqrt{2}} = \sqrt{2}z_0$. 故夹角为 $\arccos\sqrt{2}z_0$

79. 证明曲面 $F(x-az, y-bz)=0$ 上任一点处法线与一条固定直线垂直.

证明：(x_0,y_0,z_0) 处的法向量为 $(F_1'(x_0-az_0), F_2'(y_0-bz_0), -aF_1'(x_0-az_0)-bF_2'(x_0-az_0$

与 $(a,b,1)$ 垂直. 所求固定直线为 $\frac{x}{a}=\frac{y}{b}=z$ #

80. 证明曲面 $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}\ (a>0)$ 上任一点处切平面与三个坐标轴交点到原点距离之和为常数

证明：设 (x_0,y_0,z_0) 是曲面上任一点. 法向量为 $(\frac{1}{\sqrt{x_0}}, \frac{1}{\sqrt{y_0}}, \frac{1}{\sqrt{z_0}})$，切平面为

$\frac{1}{\sqrt{x_0}}(x-x_0)+\frac{1}{\sqrt{y_0}}(y-y_0)+\frac{1}{\sqrt{z_0}}(z-z_0)=0$. 与三坐标轴交点为 $(\sqrt{ax_0},0,0)$，$(0,\sqrt{ay_0},0)$，$(0,0,\sqrt{}$

到原点距离之和为 $\sqrt{ax_0}+\sqrt{ay_0}+\sqrt{az_0}=a$ #

81. 证明曲面 $xyz=a\ (a>0)$ 上任一点处切平面与三个坐标平面所围体积为常数

证明：设 (x_0,y_0,z_0) 是曲面上任一点，法向量为 $(\frac{a}{x_0}, \frac{a}{y_0}, \frac{a}{z_0})$. 切平面为 $\frac{a}{x_0}(x-x_0)+\frac{a}{y_0}(y-y_0)$
$+\frac{a}{z_0}(z-z_0)=0$. 与三坐标平面所围体积为 $\frac{1}{3}\cdot\frac{1}{2}|3x_0||3y_0||3z_0|$
$=\frac{9}{2}a$ #

82. 证明曲面 $x^2+4y+z^2=0$ 与 $x^2+y^2+z^2-6z+7=0$ 在 $(0,-1,2)$ 处相切

证明：第一个曲面在 $(0,-1,2)$ 处法向量为 $(0,4,4)$. 第二个曲面在 $(0,-1,2)$ 处法向量为 $(0,-2,-2)$. 两个切向量平行，从而二者在 $(0,-1,2)$ 处切平面相同. 故两曲面在 $(0,-1,2)$ 处相切 #

第十五章 重积分

1. 设 $\Omega\subset\mathbb{R}^2$ 是可求面积的有界闭区域，$z=h(x,y)$ 在 Ω 上连续非负，证明：$D=\{(x,y,z)\mid(x,y)\in\Omega,0\le z\le h(x,y)\}\subset\mathbb{R}^3$ 可求体积。

证明：$\forall\varepsilon>0$，我们来找一列长方体 $\{A_k\}_{k=1}^K$，使得 $\bigcup_{k=1}^K A_k\supset\partial D$ 且 $\sum_{k=1}^K V(A_k)<\varepsilon$.
因 Ω 是可求面积的有界闭区域，设 $\sigma(\Omega)=\sigma$，存在矩形族 $\{R_1,\dots,R_m\}$，使得 $\bigcup_{i=1}^m R_i\supset\Omega$ 且 $\sum_{i=1}^m\sigma(R_i)<2\sigma$. 以 R_i 为中径面作一高为 $\frac{\varepsilon}{6\sigma}$ 的长方体，记为 A_i^1. 由 $h(x,y)$ 在 Ω 上连续，设其最大值为 $M>0$. 由 $\sigma(\partial\Omega)=0$，存在矩形族 $\{S_1,\dots,S_l\}$ 使得 $\bigcup_{i=1}^l S_i\supset\partial\Omega$ 且 $\sum_{i=1}^l\sigma(S_i)<\frac{\varepsilon}{3M}$ 以 S_i 为底面向 z 轴正向，作高为 M 的长方体记为 A_i^2. 由 $h(x,y)$ 在 Ω 上一致连续，$\exists\delta>0$，当 $(x',y'),(x'',y'')\in\Omega$ 且 $|x'-x''|,|y'-y''|<\delta$ 时，$|f(x',y')-f(x'',y'')|<\frac{\varepsilon}{6\sigma}$ 对 x,y 轴分别分割，得一列矩形 $\{T_1,\dots,T_n\}$，满足：$\sum_{i=1}^n\sigma(T_n)<2\sigma$，$T_i$ 的长 $<\delta$ 将 T_i 扩张成高为 $\frac{\varepsilon}{6\sigma}$ 的长方体 A_i^3，可完全覆盖 $\{(x,y,h(x,y))\mid(x,y)\in T_i\}$，从而 $\bigcup_{i=1}^n A_i^3\supset\{(x,y,h(x,y))\mid(x,y)\in\Omega\}$. 令 $\{A_k\}_{k=1}^K=\bigcup_{i=1}^m A_i^1\cup(\bigcup_{i=1}^l A_i^2)\cup(\bigcup_{i=1}^n A_i^3)$，则 $\bigcup_{k=1}^K A_k\supset\partial D$，且 $\sum_{k=1}^K V(A_k)<\frac{\varepsilon}{6\sigma}\cdot2\sigma+\frac{\varepsilon}{3M}\cdot M+\frac{\varepsilon}{6\sigma}\cdot2\sigma=\varepsilon$. 由此 $V(\partial D)=0$. 故 D 可求体积 #

2. 设 $E=\{(x,y)\mid x,y\in\mathbb{Q}\}$，$D=[0,1]^2$. 证明：$D\cap E$ 不可求面积

证明：$\partial(D\cap E)=D$. $\sigma(\partial(D\cap E))=1\Rightarrow D\cap E$ 不可求面积 #

3. 设 $D\subset\mathbb{R}^2$ 是可求面积的有界区域，$f(x,y)$ 在 $\bar D$ 上有界，在 D 内连续，证明：$f(x,y)$ 在 $\bar D$ 上可积.

证明：设 $|f(x,y)|\le M\ ((x,y)\in\bar D)$. $\forall\varepsilon>0$，存在一列矩形 $R_1,\dots,R_m$ 使得 $R_i^\circ\cap R_j^\circ=\phi$ (i≠j)，$\partial D\subset\bigcup_{i=1}^m R_i^\circ$，$\sum_{i=1}^m\sigma(R_i)<\frac{\varepsilon}{4M}$，用 ω_i^1 表示 $f(x,y)$ 在 R_i 上的振幅，$\omega_i^1\le 2M$，$\sum_{i=1}^m\omega_i^1\sigma(R_i)<\frac{\varepsilon}{2}$.
$f(x,y)$ 在紧集 $E=\bar D\setminus(\bigcup R_i)^\circ$ 上连续，从而可积，存在一列矩形 $S_1,\dots,S_n$，使得 $S_i^\circ\cap S_j^\circ=\phi$ (i≠j)，$\bigcup_{i=1}^n S_i^\circ\supset E$，$\sum_{i=1}^n\omega_i^2\sigma(S_i)<\frac{\varepsilon}{2}$，$\omega_i^2$ 是 $f(x,y)$ 在 S_i 上的振幅. 由此，我们找到 $\bar D$ 的分割 $\{D_1,\dots,D_k\}$，使得 $\sum_{k=1}^k\omega_k\sigma(D_k)<\varepsilon$. 故 $f(x,y)$ 在 $\bar D$ 上可积 #

4. 设 $f(x)$ 在可求体积的有界闭区域 $\Omega\subset\mathbb{R}^n$ 上可积，$x=x(u)$ 是可求体积的有界闭区域 $D\subset\mathbb{R}^n$ 到可求体积的有界闭区域 $x(D)\subset\Omega$ 的 C^1 同胚. 证明：$f(x(u))$ 在 D 上可积

证明：$f(x)$ 在 Ω 上可积 $\Rightarrow f(x)$ 在 $x(D)$ 可积 $\Rightarrow f(x)$ 在 $x(D)$ 上的不连续点是 Jordan 零测集 $\Rightarrow f(x(u))$ 在 D 上的不连续点是 Jordan 零测集 $\Rightarrow f(x(u))$ 在 D 上可积 #

5. 设 $f(x,y)$, $g(x,y)$ 在可求面积的有界闭区域 $D\subset\mathbb{R}^2$ 上连续，$\forall(x,y)\in D$，$g(x,y)\ge0$. 证明：存在无穷多个 $(\xi,\eta)\in D^\circ$ 使得 $\iint_D f(x,y)g(x,y)dxdy=f(\xi,\eta)\iint_D g(x,y)dxdy$ (*)

证明：若 $\iint_D g(x,y)dxdy=0$，则由 $f(x,y)$ 连续，$\iint_D f(x,y)g(x,y)dxdy=0$，(*) 成为恒等式. 下设 $\iint_D g(x,y)dxdy>0$. 设 $f(x,y)$ 在 D 最大值为 $f(x_1,y_1)=M$，最小值为 $f(x_2,y_2)=m$. 若 $M=m$ 自然成立，设 $M>m$. 由积分中值公式，$\exists(\xi,\eta)\in D$ 使得 (*) 成立

第十五章 重积分

1. 设 $\Omega\subset\mathbb{R}^2$ 是可求面积的有界闭区域, $z=h(x,y)$ 在 Ω 上连续非负. 证明: $D=\{(x,y,z)\mid(x,y)\in\Omega,\ 0\le z\le h(x,y)\}\subset\mathbb{R}^3$ 可求体积.

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3. 设 $D\subset\mathbb{R}^2$ 是可求面积的有界区域, $f(x,y)$ 在 $\bar D$ 上有界, 在 D 内连续. 证明: $f(x,y)$ 在 $\bar D$ 上可积.

证明: 设 $|f(x,y)|\le M\ ((x,y)\in\bar D)$. $\forall\varepsilon>0$, 存在一列矩形 $R_1,\dots,R_m$ 使得 $R_i^\circ\cap R_j^\circ=\phi\ (i\ne j)$, $\partial D\subset\bigcup_{i=1}^m R_i$, $\sum_{i=1}^m\sigma(R_i)<\frac{\varepsilon}{4M}$. 用 ω_i^1 表示 $f(x,y)$ 在 R_i 上的振幅, $\omega_i^1\le 2M$, $\sum_{i=1}^m\omega_i^1\sigma(R_i)<\frac{\varepsilon}{2}$.

$f(x,y)$ 在紧集 $E=\bar D\setminus\left(\bigcup_{i=1}^m R_i\right)^\circ$ 上连续, 从而可积, 存在一列矩形 $S_1,\dots,S_n$, 使得 $S_i^\circ\cap S_j^\circ=\phi\ (i\ne j)$, $\bigcup_{i=1}^n S_i^\circ\supset E$, $\sum_{i=1}^n\omega_i^2\sigma(S_i)<\frac{\varepsilon}{2}$, ω_i^2 是 $f(x,y)$ 在 S_i 上的振幅. 由此, 我们找到 $\bar D$ 的分割 $\{D_1,\dots,D_k\}$, 使得 $\sum_{k=1}^k\omega_k\sigma(D_k)<\varepsilon$. 故 $f(x,y)$ 在 $\bar D$ 上可积 #

4. 设 $f(x)$ 在可求体积的有界闭区域 $\Omega\subset\mathbb{R}^n$ 上可积, $x=x(u)$ 是可求体积的有界闭区域 $D\subset\mathbb{R}^n$ 到可求体积的有界闭区域 $x(D)\subset\Omega$ 的 C^1 同胚. 证明: $f(x(u))$ 在 D 上可积

证明: $f(x)$ 在 Ω 上可积 $\implies f(x)$ 在 $x(D)$ 可积 $\implies f(x)$ 在 $x(D)$ 上的不连续点是 Jordan 零测集 $\implies f(x(u))$ 在 D 上的不连续点是 Jordan 零测集 $\implies f(x(u))$ 在 D 上可积 #

5. 设 $f(x,y)$, $g(x,y)$ 在可求面积的有界闭区域 $D\subset\mathbb{R}^2$ 上连续, $\forall(x,y)\in D$, $g(x,y)\ge 0$. 证明: 存在无穷多个 $(\xi,\eta)\in D^\circ$ 使得 $\iint_D f(x,y)g(x,y)\,dxdy=f(\xi,\eta)\iint_D g(x,y)\,dxdy$ (*)

证明: 若 $\iint_D g(x,y)\,dxdy=0$, 则由 $f(x,y)$ 连续, $\iint_D f(x,y)g(x,y)\,dxdy=0$, (*) 成为恒等式. 下设 $\iint_D g(x,y)\,dxdy>0$. 设 $f(x,y)$ 在 D 最大值为 $f(x_1,y_1)=M$, 最小值为 $f(x_2,y_2)=m$. 若 $M=m$ 自然成立, 设 $M>m$. 由积分中值公式, $\exists(\xi,\eta)\in D$ 使得(*)成立

若 $f(\xi,\eta)=M$ (或 m)，则 $\iint_D g(x,y)(f(x,y)-M)\,dxdy=0 \Longrightarrow$ 在 D° $g(x,y)>0$ 之上，有 f 恒等于 M。结论成立。下设 $m<f(\xi,\eta)<M$。由 f 在连接 $(x_1,y_1),(x_2,y_2)$ 的任意道路上存在 $(x,y)\in D^\circ$ 使得 $f(x,y)=f(\xi,\eta)$ #

6. 设 $f(x)$ 在 $U(x_0,\delta)\subset\mathbb{R}^n$ 连续，V_δ 为 $U(x_0,\delta)$ 的体积。证明：$\lim\limits_{\delta\to0}\frac{1}{V_\delta}\iint\cdots\int_{U(x_0,\delta)}f(x)\,dV=f(x_0)$

证明：由积分中值公式，$\iint\cdots\int_{U(x_0,\delta)}f(x)\,dV=f(\xi)\iint\cdots\int_{U(x_0,\delta)}dV=f(\xi)V_\delta$，这里 $\xi\in U(x_0,\delta)\Longrightarrow\frac{1}{V_\delta}\iint\cdots\int_{U(x_0,\delta)}f(x)\,dV=f(\xi)$。令 $\delta\to0$，由 $f(x)$ 的连续性，得 $\lim\limits_{\delta\to0}\frac{1}{V_\delta}\iint\cdots\int_{U(x_0,\delta)}f(x)\,dV=f(x_0)$ #

7. 设 $f(x,y)$ 在可求面积的有界闭区域 $D\subset\mathbb{R}^2$ 可积，证明 $u=F(x,y,z)=f(x,y)$ 在 $\Omega=\{(x,y,z)\mid(x,y)\in D,\ 1\le z\le2\}$ 上三重积分存在。反过来，设 $g(x,y,z)$ 在可求体积的有界闭区域 $\Omega\subset\mathbb{R}^3$ 上可积，$D=\{(x,y)\mid(x,y,0)\in\Omega\}$ 是可求~~体~~面积的有界闭区域，问，$g(x,y,0)$ 在 D 上二重积分一定存在吗？

证明：由 $f(x,y)$ 在 D 可积，$\forall\varepsilon>0$，$\exists$ D 的分割 $\{\Delta D_1,\ldots,\Delta D_K\}$，使得 $\sum\limits_{k=1}^{K}\omega_k(f)\Delta D_k<\varepsilon$

设 $\Delta\Omega_k=\{(x,y,z)\mid(x,y)\in\Delta D_k,\ 1\le z\le2\}$，则 $\{\Delta\Omega_1,\ldots,\Delta\Omega_K\}$ 是 Ω 的分割，且 $\sum\limits_{k=1}^{K}\omega_k(F)\Delta\Omega_k=\sum\limits_{k=1}^{K}\omega_k(f)\Delta D_k<\varepsilon$，故 $F(x,y,z)$ 在 Ω 上可积

问题的回答是否定的。如：取 $\Omega=[-1,1]^3$，$D=[-1,1]^2$。取 $g(x,y,z)=\begin{cases}0, & z\ne0\\ D(x,y), & z=0\end{cases}$

这里 $D(x,y)$ 是二维 Dirichlet 函数。则 $g(x,y,0)$ 在 D 不可积，而 $g(x,y,z)$ 在 Ω 显然可积 #

8. 设 $f(x)$ 在 $\mathbb{R}^n$ 有定义，在任意可求体积的有界闭区域上可积。证明：

$F(y)=\iint\cdots\int_{|x-y|\le1}f(x)\,dV$ 在 $\mathbb{R}^n$ 上连续

证明：任给 $y_0\in\mathbb{R}^n$，我们来证 $F(y)$ 在 y_0 处连续。由 $f(x)$ 在 $\{x\in\mathbb{R}^n\mid|x-y_0|\le3\}$ 上可积，设在此区间上 $|f(x)|\le M$。$\forall\varepsilon>0$，取 $\delta=\min\{\frac12,\frac{\varepsilon}{2\pi M}\}>0$，当 $|y-y_0|<\delta$ 时，有

$|F(y)-F(y_0)|\le2\iint\cdots\int_{\{|x-y_0|\le1\}\setminus\{|x-y|\le1\}}|f(x)|\,dV\le2\cdot M\cdot\pi\delta<\varepsilon$ #

9. 设 $f(x)$ 在 $[a,b]$ 上连续，$\forall x\in[a,b]$，$f(x)\ge\alpha>0$。记 $D=[a,b]^2$。证明：

$\iint_D\frac{f(x)}{f(y)}\,dxdy\ge(b-a)^2$

证明：$\iint_D\frac{f(x)}{f(y)}\,dxdy=\int_a^bf(x)\,dx\int_a^b\frac{1}{f(y)}\,dy\ge\left(\int_a^b\sqrt{f(x)}\cdot\sqrt{\frac{1}{f(x)}}\,dx\right)^2=(b-a)^2$ #

10. 计算：(1) $\iint_{[-2,2]\times[-1,1]}x^2y^3\,dxdy$ (2) $\int_0^{\sqrt3}dx\int_0^1\frac{8x}{(x^2+y^2+1)^2}\,dy$

(3) $\iiint_{[0,\ln2]\times[0,\ln3]\times[0,\ln4]}e^{x+y+z}\,dxdydz$

解：(1) 原式 $=4\iint_{[0,2]\times[0,1]}x^2y^3\,dxdy=4\int_0^2x^2dx\int_0^1y^3dy=\frac83$

(2) 原式 $=8\int_0^1dy\int_0^{\sqrt3}\frac{x}{(x^2+y^2+1)^2}\,dx=8\int_0^1dy\cdot\left(-\frac12\frac{1}{(x^2+y^2+1)}\right)\Big|_0^{\sqrt3}=4\int_0^1\left(\frac{1}{y^2+1}-\frac{1}{y^2+4}\right)dy$

$=4\left(\arctan y-\frac12\arctan\frac y2\right)\Big|_0^1=\pi-2\arctan\frac12$

(3) 原式 $=\int_0^{\ln2}e^xdx\int_0^{\ln3}e^ydy\int_0^{\ln4}e^zdz=1\times2\times3=6$

11. 改变积分顺序：(1) $\int_3^5dx\int_{-x}^{x^2}f(x,y)\,dy$ (2) $\int_{-1}^0dy\int_{-\sqrt{y+1}}^{\sqrt{y+1}}f(x,y)\,dx$ (3) $\int_0^6dx\int_0^{6-x}dy\int_{x+y}^6f(x,y,z)\,dz$

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解：(1) 原式 $=\int_{-5}^{-3}dy\int_{-y}^5f(x,y)\,dx+\int_{-3}^9dy\int_3^5f(x,y)\,dx+\int_9^{25}dy\int_{\sqrt y}^5f(x,y)\,dx$

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这里 $\xi\in U(x_0,\delta)\Longrightarrow\frac{1}{V_\delta}\iint\cdots\int_{U(x_0,\delta)}f(x)\,dV=f(\xi)$. 令 $\delta\to0$. 由 $f(x)$ 的连续性, 得

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解: (1) 原式 $=4\iint_{[0,2]\times[0,1]}x^2y^2\,dxdy=4\int_0^2x^2dx\int_0^1y^2dy=\frac{8}{3}$ ~~y³~~

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(4) $\int_0^1dy\int_{y^2}^{2-y}f(x,y)\,dx$ (5) $\int_{-1}^1dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dy\int_0^1f(x,y,z)\,dz$ (6) $\int_0^1dx_1\int_0^{1-x_1}dx_2\int_0^{1-x_1-x_2}dx_3\int_0^{1-x_1-x_2-x_3}f(x_1,x_2,x_3,x_4)\,dx_4$

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(6) 原式 $= \int_0^1 dx_4 \int_0^{1-x_4} dx_3 \int_0^{1-x_4-x_3} dx_2 \int_0^{1-x_4-x_3-x_2} f(x_1,x_2,x_3,x_4)\,dx_1$

(7) 原式 $= \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dz \int_{\sqrt{x^2+z^2}}^{1} f(x,y,z)\,dy$

(8) 原式 $= \int_{-2}^{2} dy \int_{-\sqrt{1-y^2/4}}^{\sqrt{1-y^2/4}} dx \int_0^{x^2+y^2/4} f(x,y,z)\,dz$

12. 计算：(1) $\iint_D (x^2+2y)\,dxdy$，D 由 $y=x^2$ 与 $y=\sqrt{x}$ 所围.

(2) $\iint_D \sin y^3\,dxdy$，D 由 $y=\sqrt{x}$，$y=2$，及 $x=0$ 所围　(3) $\iint_{1\le x^2+y^2\le 4} (x^2+x^4y)\,dxdy$

(4) $\iint_D x^2y^3\,dxdy$，D 由 $y^2=2x$，$x=\frac{1}{2}$ 所围

(5) $\iint_D (x+y)\,dxdy$，D 由 $y=e^x$，$y=1$，$x=0$，$x=1$ 所围

(6) $\iiint_D xyz\,dxdydz$，D 由 $x=0$，$x=1$，$y=0$，$y=1$，$z=2$，$z=\sqrt{x^2+y^2}$ 所围

(7) $\iiint_{x^2+y^2+z^2\le 1} x^2y^4\sin z\,dxdydz$

(8) $\iiint_{|x|+|y|+|z|\le 1} \cos x\cos y\cos z\,dxdydz$

(9) $\iiint_D (x^2+y^2)\,dxdydz$，D 由 $z=16(x^2+y^2)$，$z=4(x^2+y^2)$，$z=64$ 所围

(10) $\int\int\cdots\int_{[0,1]^n} \left(\sum_{i=1}^{n} x_i\right)^2 dx_1dx_2\cdots dx_n$

解：(1) 原式 $= \int_0^1 dx \int_{x^2}^{\sqrt{x}} (x^2+2y)\,dy = \int_0^1 (x^{5/2}+x-2x^4)\,dx = (\frac{2}{7}x^{7/2}+\frac{1}{2}x^2-\frac{2}{5}x^5)\Big|_0^1 = \frac{27}{70}$

(2) 原式 $= \int_0^2 dy \int_0^{y^2} \sin y^3\,dx = \int_0^2 y^2\sin y^3\,dy = -\frac{1}{3}\cos y^3\Big|_0^2 = \frac{1}{3}(1-\cos 8)$

(3) 原式 $= \iint_{\substack{1\le r\le 2\\0\le\theta\le 2\pi}} (r^2\cos^2\theta + r^5\cos^4\theta\sin\theta)\,r\,dr\,d\theta = \int_1^2 dr \int_0^{2\pi} r^3(\cos^2\theta + r^3\cos^4\theta\sin\theta)\,d\theta$

$= \int_1^2 dr\ r^3(\frac{\theta}{2}+\frac{1}{4}\sin 2\theta - \frac{1}{5}r^3\cos^5\theta)\Big|_{\theta=0}^{\theta=2\pi} = \int_1^2 \pi r^3\,dr = \frac{15}{4}\pi$

(4) 原式 $= \int_0^{\frac{1}{2}} dx \int_{-\sqrt{2x}}^{\sqrt{2x}} x^2y^3\,dy = \int_0^{\frac{1}{2}} dx\,(\frac{1}{4}x^2y^4)\Big|_{y=-\sqrt{2x}}^{y=\sqrt{2x}} = 0$

(5) 原式 $= \int_0^1 dx \int_1^{e^x} (x+y)\,dy = \int_0^1 dx\,(xy+\frac{1}{2}y^2)\Big|_{y=1}^{y=e^x} = \int_0^1 (xe^x+\frac{1}{2}e^{2x}-x-\frac{1}{2})\,dx$

$= (xe^x - e^x + \frac{1}{4}e^{2x} - \frac{1}{2}x^2 - \frac{1}{2}x)\Big|_0^1 = \frac{1}{4}e^2 - \frac{1}{4}$

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$= \frac{3}{8}$

(7) 原式 $= \cancel{\int_0^1 \sin z\,dz \int_0^{\sqrt{1-z^2}} y^4\,dy \int_0^{\sqrt{1-y^2-z^2}} x^2\,dx}\ \int_{-1}^{1} dz\cdot\sin z \iint_{x^2+y^2\le 1-z^2} x^2y^4\,dxdy$

$= \int_{-1}^{1} dz\,\sin z \int_{\substack{0\le r\le\sqrt{1-z^2}\\0\le\theta\le 2\pi}} r^2\cos^2\theta\, r^4\sin^4\theta\, r\,dr\,d\theta$

$= \int_{-1}^{1} dz\cdot\sin z \int_0^{\sqrt{1-z^2}} r^7 \int_0^{2\pi} \cos^2\theta\sin^4\theta\,d\theta = \int_{-1}^{1} \frac{\pi}{64}(1-z^2)^4\sin z\,dz$

$= 0$

(5) 原式 $= \int_{-1}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx \int_0^1 f(x,y,z)\,dz$

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解：(1) 原式 $= \int_0^1 dx \int_{x^2}^{\sqrt{x}} (x^2+2y)\,dy = \int_0^1 (x^{5/2}+x-2x^4)\,dx = (\frac{2}{7}x^{7/2}+\frac{1}{2}x^2-\frac{2}{5}x^5)\Big|_0^1 = \frac{27}{70}$

(2) 原式 $= \int_0^2 dy \int_0^{y^2} \sin y^3\,dx = \int_0^2 y^2\sin y^3\,dy = -\frac{1}{3}\cos y^3\Big|_0^2 = \frac{1}{3}(1-\cos 8)$

(3) 原式 $= \iint_{\substack{1\le r\le 2\\ 0\le\theta\le 2\pi}} (r^2\cos^2\theta + r^5\cos^4\theta\sin\theta)\,r\,dr\,d\theta = \int_1^2 dr \int_0^{2\pi} r^3(\cos^2\theta + r^3\cos^4\theta\sin\theta)\,d\theta$

$= \int_1^2 dr\; r^3\left(\frac{\theta}{2}+\frac{1}{4}\sin 2\theta - \frac{1}{5}r^3\cos^5\theta\right)\Big|_{\theta=0}^{\theta=2\pi} = \int_1^2 \pi r^3\,dr = \frac{15}{4}\pi$

(4) 原式 $= \int_0^{\frac{1}{2}} dx \int_{-\sqrt{2x}}^{\sqrt{2x}} x^2y^3\,dy = \int_0^{\frac{1}{2}} dx\,(\frac{1}{4}x^2y^4)\Big|_{y=-\sqrt{2x}}^{y=\sqrt{2x}} = 0$.

(5) 原式 $= \int_0^1 dx \int_1^{e^x} (x+y)\,dy = \int_0^1 dx\,(xy+\frac{1}{2}y^2)\Big|_{y=1}^{y=e^x} = \int_0^1 (xe^x+\frac{1}{2}e^{2x}-x-\frac{1}{2})\,dx$

$= (xe^x - e^x + \frac{1}{4}e^{2x} - \frac{1}{2}x^2 - \frac{1}{2}x)\Big|_0^1 = \frac{1}{4}e^2 - \frac{1}{4}$

(6) 原式 $= \int_0^1 dx \int_0^1 dy \int_{\sqrt{x^2+y^2}}^{2} xyz\,dz = \int_0^1 dx\int_0^1 dy \cdot \frac{1}{2}xy(4-x^2-y^2) = \int_0^1 (-\frac{1}{4}x^3+\frac{7}{8}x)\,dx$

$= \frac{3}{8}$

(7) 原式 $= \cancel{\int_0^1 \sin z\,dz \int_0^{\sqrt{1-z^2}} y^4\,dy \int_0^{\sqrt{1-y^2-z^2}} x^2\,dx}\ \int_{-1}^{1} dz\cdot\sin z \iint_{x^2+y^2\le 1-z^2} x^2y^4\,dxdy$

$= \int_{-1}^{1} dz\,\sin z \int_{\substack{0\le r\le\sqrt{1-z^2}\\ 0\le\theta\le 2\pi}} r^2\cos^2\theta\, r^4\sin^4\theta\; r\,dr\,d\theta$

$= \int_{-1}^{1} dz\cdot\sin z \int_0^{\sqrt{1-z^2}} r^7 \int_0^{2\pi} \cos^2\theta\sin^4\theta\,d\theta = \int_{-1}^{1} \frac{\pi}{64}(1-z^2)^4\sin z\,dz$.

$=0$

8\) 原式 $= 8\iiint\limits_{\substack{x+y+z\le 1\\ x,y,z\ge 0}} \cos x\cos y\cos z\,dx\,dy\,dz$

$= 8\int_0^1 dx\int_0^{1-x} dy\int_0^{1-x-y}\cos x\cos y\cos z\,dz$

$= 8\int_0^1 dx\int_0^{1-x}\cos x\cos y\sin(1-x-y)\,dy$

$= \cancel{8\int_0^1 \cdots \cos x\sin(1-x)\cdots dx}$

$= 4\int_0^1 (1-x)\cos x\sin(1-x)\,dx$

$= 4\int_0^1 x\sin x\cos(1-x)\,dx$

$= 2\int_0^1 x(\sin 1+\sin(2x-1))\,dx$

$= 2\sin 1-\cos 1$

81 原式 $= 8\iiint\limits_{\substack{x+y+z\le 1 \\ x,y,z\ge 0}} \cos x\cos y\cos z \, dx\, dy\, dz$

$$= 8\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \cos x \cos y \cos z \, dz$$

$$= 8\int_0^1 dx \int_0^{1-x} \cos x \cos y \sin(1-x-y) \, dy$$

$$= 4\int_0^1 (1-x)\cos x \sin(1-x)\, dx$$

$$= 4\int_0^1 x \sin x \cos(1-x)\, dx$$

$$= 2\int_0^1 x(\sin 1 + \sin(2x-1))\, dx$$

$$= 2\sin 1 - \cos 1$$

习题十五

10.

(1) $I=2\int_{-2}^{2}\int_0^1 x^2y^3\,dy\,dx$

$=\frac{1}{2}\int_{-2}^{2}x^2\,dx$

$=\frac{8}{3}$

(2) $I=\int_0^1 dy\int_0^{\sqrt3}\frac{4}{(x^2+y^2+1)^2}\,d(x^2+y^2+1)$

$=\int_0^1 dy\ \frac{-4}{x^2+y^2+1}\Big|_0^{\sqrt3}$

$=4\int_0^1\left(\frac{1}{y^2+1}-\frac{1}{y^2+4}\right)dy$

$=4\left(\arctan y-\frac{1}{2}\arctan\frac{y}{2}\right)\Big|_0^1$

$=\pi-2\arctan\frac{1}{2}$

(3) $I=\int_0^{\ln4}\int_0^{\ln3}\int_0^{\ln2}e^{x+y+z}\,dx\,dy\,dz$

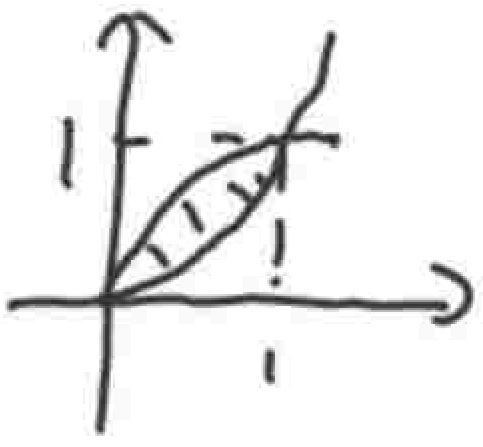
$=\int_0^{\ln4}\int_0^{\ln3}e^{y+z}\,dy\,dz$

$=\int_0^{\ln4}2e^{z}\,dz$

$=6$

12.

(1) $\int_0^1\int_{x^2}^{\sqrt x}(x^2+2y)\,dy\,dx$

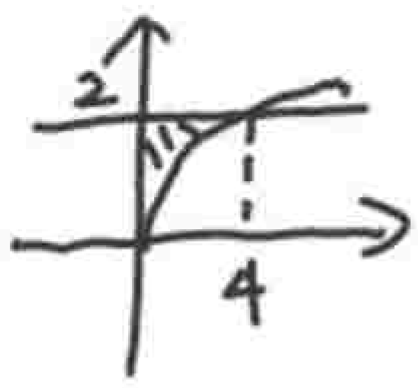


$=\int_0^1\left(x^{\frac{5}{2}}+x-2x^4\right)dx$

$=\frac{2}{7}+\frac{1}{2}-\frac{2}{5}$

$=\frac{27}{70}$

(2)



$I=\int_0^2\int_0^{y^2}\sin y^3\,dx\,dy$

$=\int_0^2 y^2\sin y^3\,dy$

$=\frac{1}{3}(-\cos y^3)\Big|_0^2$

$=\frac{1}{3}(1-\cos 8)$

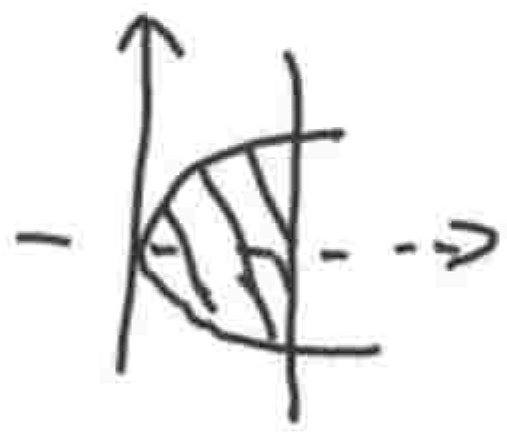
(3) 令 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$

$I=\int_0^{2\pi}\int_1^2(r^2\cos^2\theta+r^5\cos^4\theta\sin\theta)\,r\,dr\,d\theta$

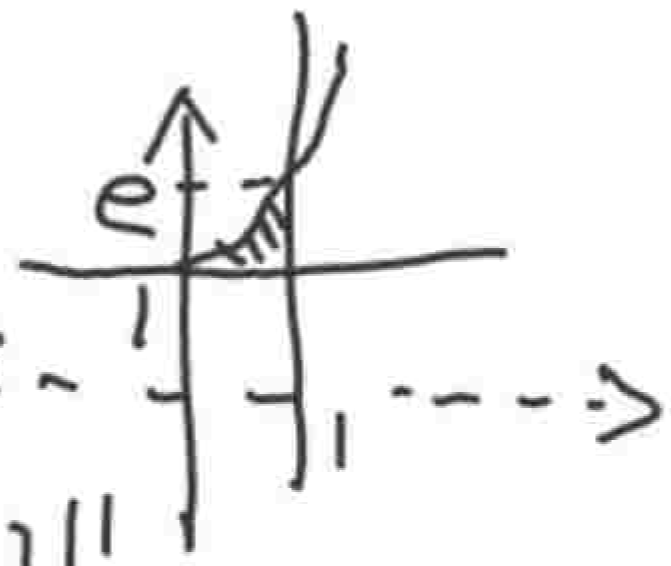
$=\int_0^{2\pi}\left(\frac{15}{4}\cos^2\theta+\frac{127}{7}\cos^4\theta\sin\theta\right)d\theta$

$=\frac{15}{4}\pi$

(4) 由对称性, $I=0$



(5) $I=\int_0^1\int_1^{e^x}(x+y)\,dy\,dx$



$=\int_0^1\left(\frac{1}{2}e^{2x}+xe^x-x-\frac{1}{2}\right)dx$

$=\left[\frac{1}{4}e^{2x}+(x-1)e^x-\frac{1}{2}x^2-\frac{1}{2}x\right]\Big|_0^1$

$=\frac{1}{4}e^2-\frac{1}{2}-\frac{1}{2}-\left(\frac{1}{4}-1\right)$

$=\frac{1}{4}(e^2-1)$

(6) $I=\int_0^1\int_0^1\int_{\sqrt{x^2+y^2}}^{2}xyz\,dz\,dy\,dx$

$=\int_0^1\int_0^1\frac{1}{2}xy(4-x^2-y^2)\,dy\,dx$

$=\frac{3}{8}$

习题十五

10.

(1) $I = 2\int_{-2}^{2}\int_0^1 x^2y^3\,dy\,dx$

$= \frac{1}{2}\int_{-2}^{2} x^2\,dx$

$= \frac{8}{3}$

(2) $I = \int_0^1 dy \int_0^{\sqrt{3}} \frac{4}{(x^2+y^2+1)^2}\, d(x^2+y^2+1)$

$= \int_0^1 dy\, \frac{-4}{x^2+y^2+1}\Big|_0^{\sqrt{3}}$

$= 4\int_0^1 \left(\frac{1}{y^2+1} - \frac{1}{y^2+4}\right)dy$

$= 4\left(\arctan y - \frac{1}{2}\arctan\frac{y}{2}\right)\Big|_0^1$

$= \pi - 2\arctan\frac{1}{2}$

(3) $I = \int_0^{\ln 4}\int_0^{\ln 3}\int_0^{\ln 2} e^{x+y+z}\,dx\,dy\,dz$

$= \int_0^{\ln 4}\int_0^{\ln 3} e^{y+z}\,dy\,dz$

$= \int_0^{\ln 4} 2e^{z}\,dz$

$= 6$

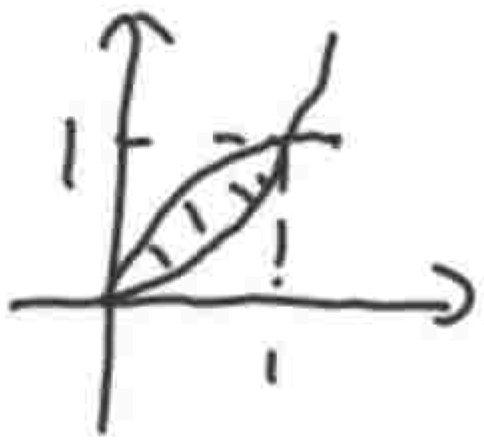
12.

(1) $\int_0^1\int_{x^2}^{\sqrt{x}} (x^2+2y)\,dy\,dx$

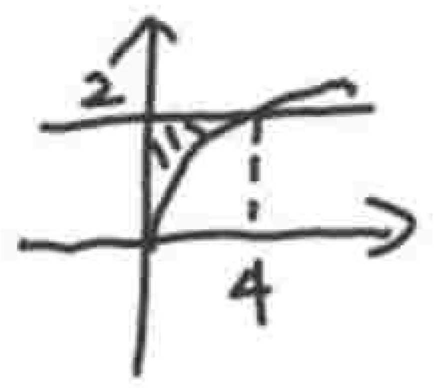
$= \int_0^1 \left(x^{\frac{5}{2}} + x - 2x^4\right)dx$

$= \frac{2}{7} + \frac{1}{2} - \frac{2}{5}$

$= \frac{27}{70}$



(2)



$I = \int_0^2\int_0^{y^2} \sin y^3\,dx\,dy$

$= \int_0^2 y^2 \sin y^3\,dy$

$= \frac{1}{3}(-\cos y^3)\Big|_0^2$

$= \frac{1}{3}(1-\cos 8)$

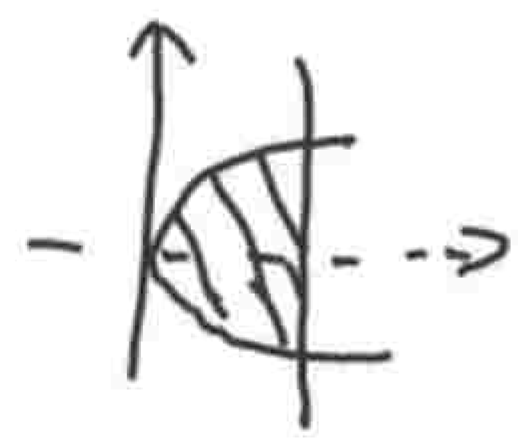
(3) 令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$I = \int_0^{2\pi}\int_1^2 (r^2\cos^2\theta + r^5\cos^4\theta\sin\theta)\,r\,dr\,d\theta$

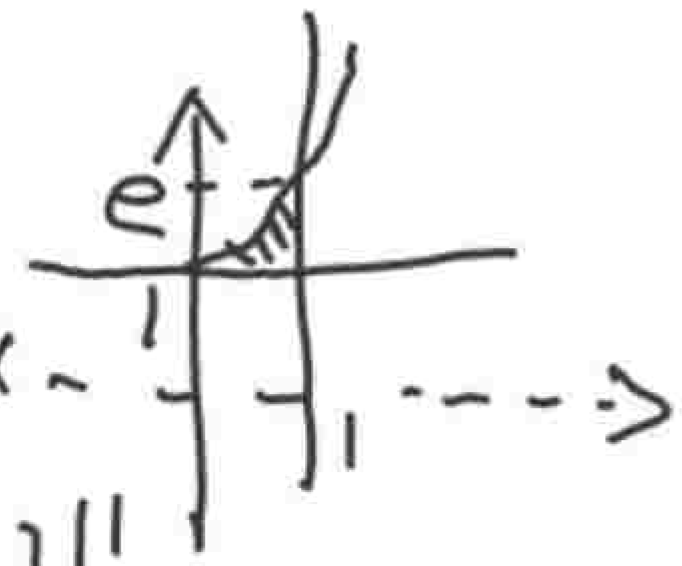
$= \int_0^{2\pi} \left(\frac{15}{4}\cos^2\theta + \frac{127}{7}\cos^4\theta\sin\theta\right)d\theta$

$= \frac{15}{4}\pi$

(4) 由对称性, $I = 0$



(5) $I = \int_0^1\int_1^{e^x} (x+y)\,dy\,dx$



$= \int_0^1 \left(\frac{1}{2}e^{2x} + xe^x - x - \frac{1}{2}\right)dx$

$= \left[\frac{1}{4}e^{2x} + (x+1)e^x - \frac{1}{2}x^2 - \frac{1}{2}x\right]\Big|_0^1$

$= \frac{1}{4}e^2 - \frac{1}{2} - \frac{1}{2} - \left(\frac{1}{4} - 1\right)$

$= \frac{1}{4}(e^2 - 1)$

(6) $I = \int_0^1\int_0^1\int_{\sqrt{x^2+y^2}}^{2} xyz\,dz\,dy\,dx$

$= \int_0^1\int_0^1 \frac{1}{2}xy(4 - x^2 - y^2)\,dy\,dx$

$= \frac{3}{8}$

(7) 由对称性, $I=0$

(8) $I = 8\int_0^1 \int_0^{1-z} \int_0^{1-y-z} \cos x \cos y \cos z \, dx dy dz$

$= 8\int_0^1 \int_0^{1-z} \sin(1-y-z) \cos y \cos z \, dy dz$

$= 4\int_0^1 \int_0^{1-z} [\sin(1-z)+\sin(1-2y-z)] \cos z \, dy dz$

$= 4\int_0^1 (1-z)\sin(1-z)\cos z \, dz$

$= 4\int_0^1 x \sin x \cos(1-x) \, dx$

$= 2\int_0^1 x[\sin 1 + \sin(2x-1)] \, dx$

$= 2\sin 1 - \cos 1$

(9) $I = \int_0^{64} \iint_{D_z} (x^2+y^2) \, dx dy \quad (D_z = \{(x,y) \mid \frac{z}{16} \le x^2+y^2 \le \frac{z}{4}\})$

$= \int_0^{64} \int_0^{2\pi} \int_{\frac{\sqrt{z}}{4}}^{\frac{\sqrt{z}}{2}} r^3 \, dr d\theta dz \quad$ (令 $x = r\cos\theta,\ y = r\sin\theta$)

$= \int_0^{64} \int_0^{2\pi} \frac{1}{4}\left(\frac{z^2}{16} - \frac{z^2}{256}\right) d\theta dz$

$= \frac{15\pi}{512} \int_0^{64} z^2 \, dz$

$= \frac{5\pi}{512} \cdot 2^{18}$

$= 2560\pi$

(7) 由对称性，$I=0$

(8) $I=8\int_0^1\int_0^{1-z}\int_0^{1-y-z}\cos x\cos y\cos z\,dx\,dy\,dz$

$=8\int_0^1\int_0^{1-z}\sin(1-y-z)\cos y\cos z\,dy\,dz$

$=4\int_0^1\int_0^{1-z}[\sin(1-z)+\sin(1-2y-z)]\cos z\,dy\,dz$

$=4\int_0^1(1-z)\sin(1-z)\cos z\,dz$

$=4\int_0^1 x\sin x\cos(1-x)\,dx$

$=2\int_0^1 x[\sin 1+\sin(2x-1)]\,dx$

$=2\sin 1-\cos 1$

(9) $I=\int_0^{64}\iint_{D_z}(x^2+y^2)\,dx\,dy \quad \left(D_z=\left\{(x,y)\,\middle|\,\frac{z}{16}\le x^2+y^2\le\frac{z}{4}\right\}\right)$

$=\int_0^{64}\int_0^{2\pi}\int_{\frac{\sqrt{z}}{4}}^{\frac{\sqrt{z}}{2}}r^3\,dr\,d\theta\,dz \quad$ (令 $x=r\cos\theta,\ y=r\sin\theta$)

$=\int_0^{64}\int_0^{2\pi}\frac{1}{4}\left(\frac{z^2}{16}-\frac{z^2}{256}\right)d\theta\,dz$

$=\frac{15\pi}{512}\int_0^{64}z^2\,dz$

$=\frac{5\pi}{512}\cdot 2^{18}$

$=2560\pi$

(10) $J_n \triangleq \int_0^1 \cdots \int_0^1 (x_1+\cdots+x_n)\,dx_1\cdots dx_n$

$= \int_0^1 \cdots \int_0^1 \left(\frac{1}{2}+x_2+\cdots+x_n\right)dx_2\cdots dx_n$

$= \cdots$

$= \frac{n}{2}$

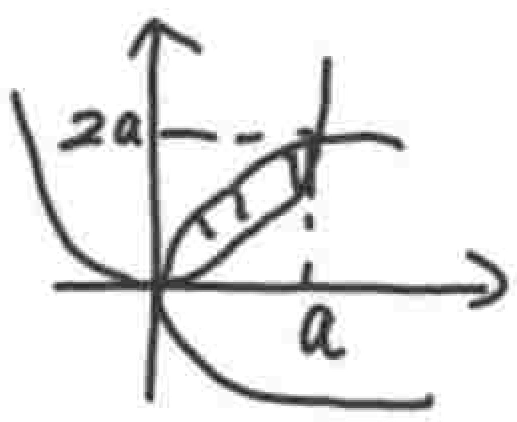
$K \triangleq \int_0^1 \cdots \int_0^1 dx_1\cdots dx_n = 1$

$I_n = \int_0^1 \cdots \int_0^1 \left(\sum_{i=1}^{n} x_i\right)^2 dx_1\cdots dx_n$

$= \int_0^1 \cdots \int_0^1 \left(\left(\sum_{i=2}^{n} x_i\right)^2 + \sum_{i=2}^{n} x_i + \frac{1}{3}\right) dx_2\cdots dx_n$

$= I_{n-1} + J_{n-1} + \frac{1}{3}K$

$I_n = I_1 + \sum_{i=1}^{n-1} J_i + \frac{n-1}{3} = \frac{1}{4}n^2 + \frac{1}{12}n$

13. 

$S = \int_0^a \left(\sqrt{4ax} - \frac{2}{a}x^2\right)dx$

$= \left(\frac{2}{3}\sqrt{4a}\cdot x^{\frac{3}{2}} - \frac{2}{3a}x^3\right)\Big|_0^a$

$= \frac{2}{3}a^2$

(10) $J_n \triangleq \int_0^1 \cdots \int_0^1 (x_1+\cdots+x_n)\,dx_1\cdots dx_n$

$= \int_0^1 \cdots \int_0^1 \left(\frac{1}{2}+x_2+\cdots+x_n\right) dx_2\cdots dx_n$

$= \cdots$

$= \frac{n}{2}$

$K \triangleq \int_0^1 \cdots \int_0^1 dx_1\cdots dx_n = 1$

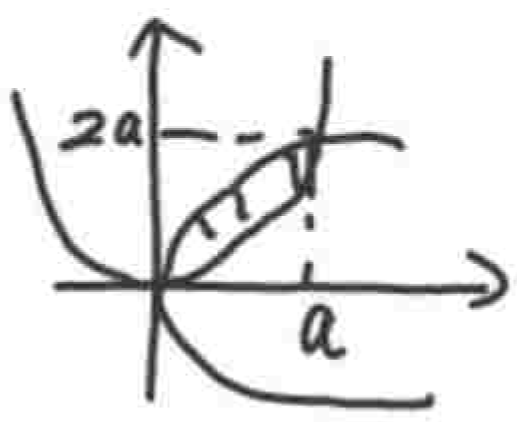
$I_n = \int_0^1 \cdots \int_0^1 \left(\sum_{i=1}^{n} x_i\right)^2 dx_1\cdots dx_n$

$= \int_0^1 \cdots \int_0^1 \left(\left(\sum_{i=2}^{n} x_i\right)^2 + \sum_{i=2}^{n} x_i + \frac{1}{3}\right) dx_2\cdots dx_n$

$= I_{n-1} + J_{n-1} + \frac{1}{3}K$

$I_n = I_1 + \sum_{i=1}^{n-1} J_i + \frac{n-1}{3} = \frac{1}{4}n^2 + \frac{1}{12}n$

13.



$S = \int_0^a \left(\sqrt{4ax} - \frac{2}{a}x^2\right) dx$

$= \left(\frac{2}{3}\sqrt{4a}\cdot x^{\frac{3}{2}} - \frac{2}{3a}x^3\right)\Big|_0^a$

$= \frac{2}{3}a^2$

14. 令 $\begin{cases} x=\sqrt{2}rt \\ y=\sqrt{3}r(\cos\theta-t) \\ z=\sqrt{2}r\sin\theta \end{cases}$

$t\in[0,\cos\theta],\ \theta\in[0,\frac{\pi}{2}],\ r\in[0,1]$.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,t)}=\begin{vmatrix} \sqrt{2}t & \sqrt{3}(\cos\theta-t) & \sqrt{2}\sin\theta \\ 0 & -\sqrt{3}r\sin\theta & \sqrt{2}r\cos\theta \\ \sqrt{2}r & -\sqrt{3}r & 0 \end{vmatrix}$$

$$=\sqrt{2}t\cdot\sqrt{3}r\cdot\sqrt{2}r\cos\theta$$

$$+\sqrt{2}r\cdot[\sqrt{6}r\cos\theta(\cos\theta-t)+\sqrt{6}r\sin^2\theta]$$

$$=2\sqrt{3}r^2t\cos\theta+2\sqrt{3}r^2-2\sqrt{3}r^2t\cos\theta$$

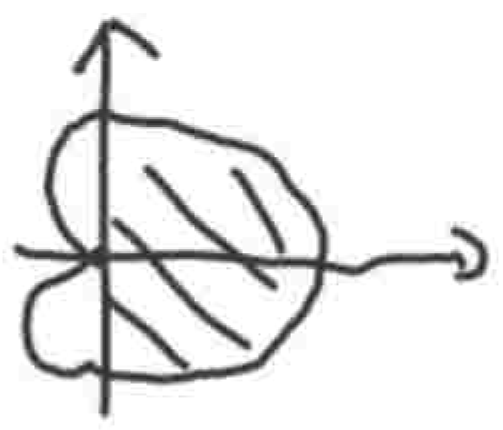
$$=2\sqrt{3}r^2$$

$$V=\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta}\int_0^1 2\sqrt{3}r^2\,dr\,dt\,d\theta$$

$$=\frac{2}{3}\sqrt{3}\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta$$

$$=\frac{2}{3}\sqrt{3}$$

17. (1) 由奇偶性，显然 $I=0$



(2) 令 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$

$$I=\int_0^{\frac{\pi}{2}}\int_0^1\frac{1}{\sqrt{4-r^2}}r\,dr\,d\theta$$

$$=\frac{1}{2}(2-\sqrt{3})\pi$$

14. 令 $\begin{cases} x=\sqrt{2}rt \\ y=\sqrt{3}r(\cos\theta-t) \\ z=\sqrt{2}r\sin\theta \end{cases}$

$t\in[0,\cos\theta]$, $\theta\in[0,\frac{\pi}{2}]$, $r\in[0,1]$.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,t)}=\begin{vmatrix} \sqrt{2}t & \sqrt{3}(\cos\theta-t) & \sqrt{2}\sin\theta \\ 0 & -\sqrt{3}r\sin\theta & \sqrt{2}r\cos\theta \\ \sqrt{2}r & -\sqrt{3}r & 0 \end{vmatrix}$$

$$=\sqrt{2}t\cdot\sqrt{3}r\cdot\sqrt{2}r\cos\theta$$

$$+\sqrt{2}r\cdot[\sqrt{6}r\cos\theta(\cos\theta-t)+\sqrt{6}r\sin^2\theta]$$

$$=2\sqrt{3}r^2t\cos\theta+2\sqrt{3}r^2-2\sqrt{3}r^2t\cos\theta$$

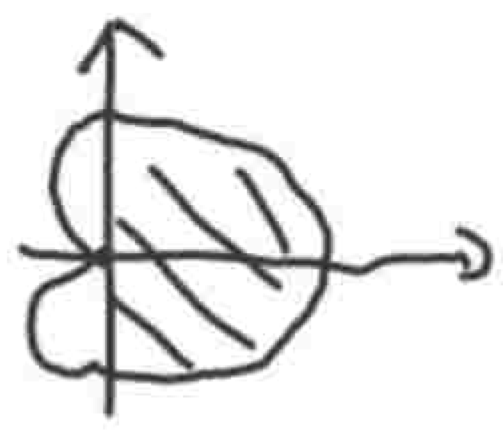
$$=2\sqrt{3}r^2$$

$$V=\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta}\int_0^1 2\sqrt{3}r^2\,dr\,dt\,d\theta$$

$$=\frac{2}{3}\sqrt{3}\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta$$

$$=\frac{2}{3}\sqrt{3}$$

17. (1) 由奇偶性，显然 $I=0$

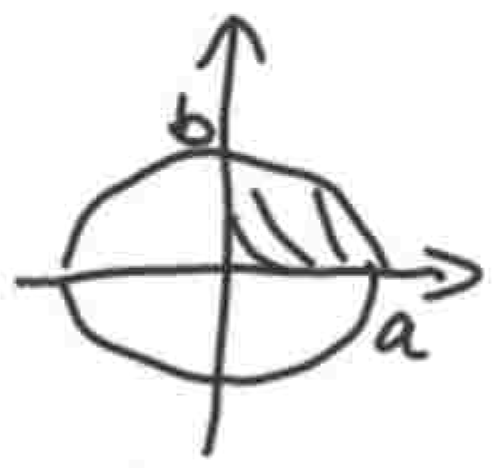


(2) 令 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$

$$I=\int_0^{\frac{\pi}{2}}\int_0^1\frac{1}{\sqrt{4-r^2}}r\,dr\,d\theta$$

$$=\frac{1}{2}(2-\sqrt{3})\pi$$

(3) 不妨设 $a,b>0$



$$I=\int_0^b dy\int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\,dx$$

$$=\int_0^b \frac{1}{2}\cdot\frac{a^2}{b^2}(b^2-y^2)y\,dy$$

$$=\frac{a^2}{2b^2}\left(\frac{1}{2}b^2y^2-\frac{1}{4}y^4\right)\Big|_0^b$$

$$=\frac{1}{8}a^2b^2$$

(4) 令 $\begin{cases}u=x^2-y^2\\ v=xy\end{cases}$

$$\frac{\partial(x,y)}{\partial(u,v)}=1\Big/\frac{\partial(u,v)}{\partial(x,y)}$$

$$=1\Big/\begin{vmatrix}2x & -2y\\ y & x\end{vmatrix}=\frac{1}{2(x^2+y^2)}$$

$$I=\int_1^9\int_2^4\frac{1}{2}\,dv\,du=8$$

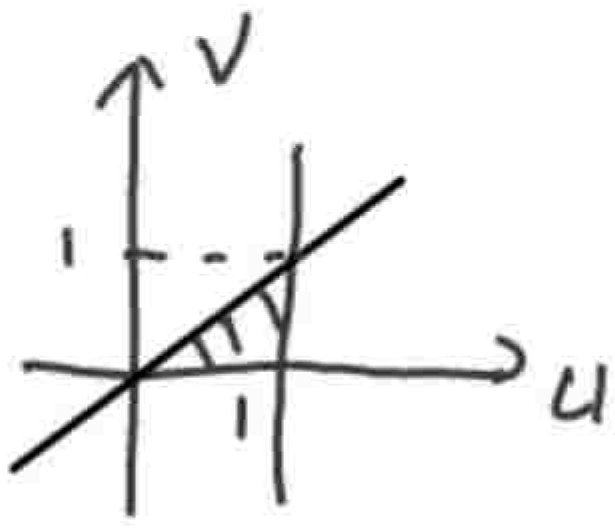
(5) 令 $\begin{cases}u=x+y\\ v=y\end{cases}\Rightarrow\begin{cases}x=u-v\\ y=v\end{cases}$

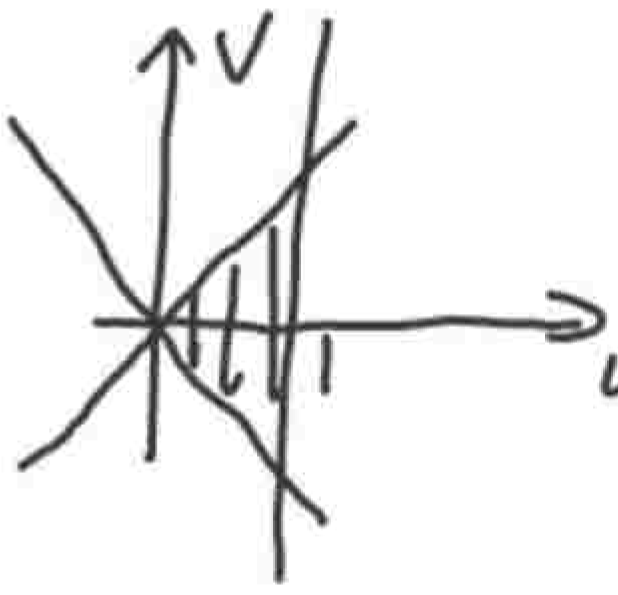
$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}1 & -1\\ 0 & 1\end{vmatrix}=1$$

$$I=\int_0^1\int_0^v e^{\frac{v}{u}}\,dv\,du$$

$$=\int_0^1 u(e-1)\,du$$

$$=\frac{1}{2}(e-1)$$





(6) 令 $\begin{cases}u=x+y\\ v=x-y\end{cases}\Rightarrow\begin{cases}x=(u+v)/2\\ y=(u-v)/2\end{cases}$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{vmatrix}=-\frac{1}{2}$$

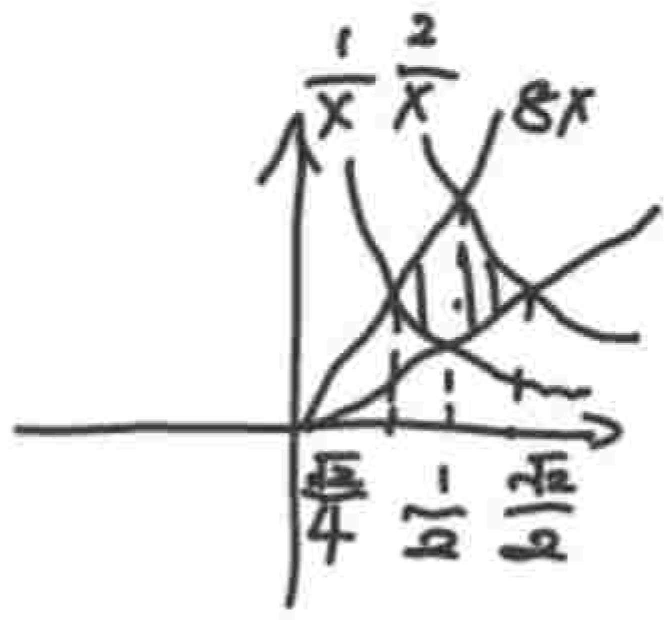
$$I=\int_0^1\int_{-u}^{u}\frac{1}{2}\cos\frac{v}{u}\,dv\,du$$

$$=\int_0^1\int_0^u\cos\frac{v}{u}\,dv\,du$$

$$=\sin 1\int_0^1 u\,du$$

$$=\frac{1}{2}\sin 1$$

(7)



$$I_1=\int_{\frac{\sqrt2}{4}}^{\frac{1}{2}}dx\int_{\frac{1}{x}}^{8x}\left(x^2+\frac{1}{4}y^2\right)dy$$

$$=\int_{\frac{\sqrt2}{4}}^{\frac{1}{2}}\left(\frac{152}{3}x^3-x-\frac{1}{12x^3}\right)dx$$

$$=\frac{35}{96}$$

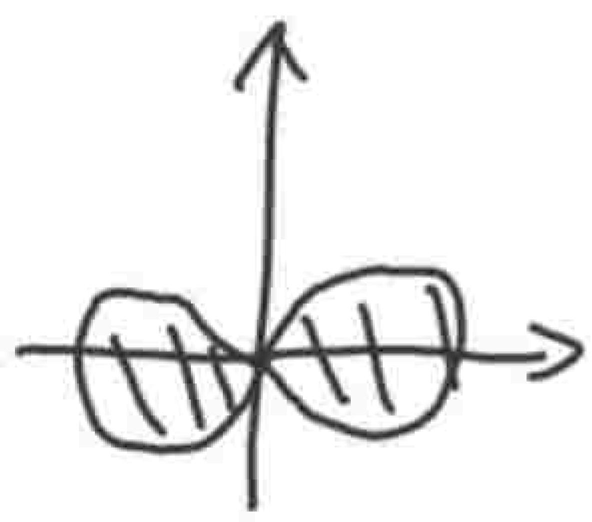
$$I_2=\int_{\frac{1}{2}}^{\frac{\sqrt2}{2}}dx\int_{4x}^{\frac{2}{x}}\left(x^2+\frac{1}{4}y^2\right)dy$$

$$=\frac{23}{48}$$

$$I=I_1+I_2=\frac{27}{32}$$

18. 令 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$

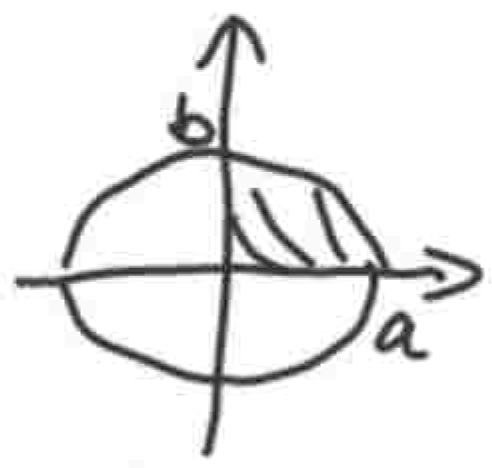
$$I=4\int_0^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos2\theta}} r\,dr\,d\theta$$



$$=2a^2\int_0^{\frac{\pi}{4}}\cos2\theta\,d\theta$$

$$=a^2$$

(3) 不妨设 $a,b>0$

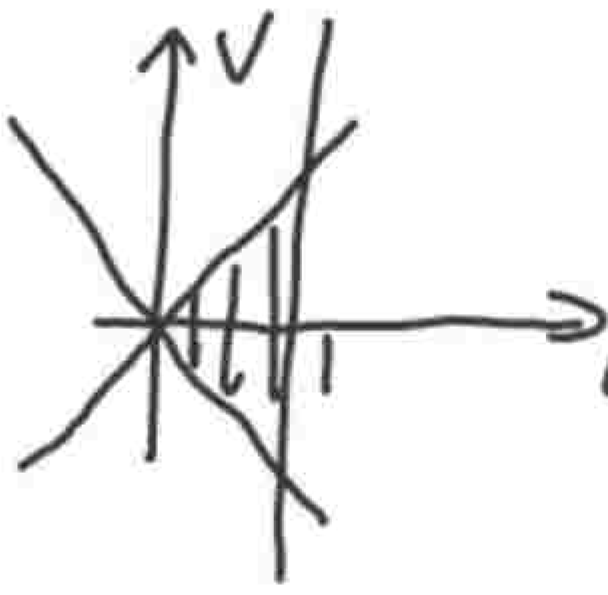


$$I=\int_0^b dy\int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\,dx$$

$$=\int_0^b \frac{1}{2}\cdot\frac{a^2}{b^2}(b^2-y^2)y\,dy$$

$$=\frac{a^2}{2b^2}\left(\frac{1}{2}b^2y^2-\frac{1}{4}y^4\right)\Big|_0^b$$

$$=\frac{1}{8}a^2b^2$$



(4) 令 $\begin{cases} u=x^2-y^2 \\ v=xy \end{cases}$

$$\frac{\partial(x,y)}{\partial(u,v)}=1\Big/\frac{\partial(u,v)}{\partial(x,y)}$$

$$=1\Big/\begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix}=\frac{1}{2(x^2+y^2)}$$

$$I=\int_1^9\int_2^4\frac{1}{2}\,dv\,du=8$$

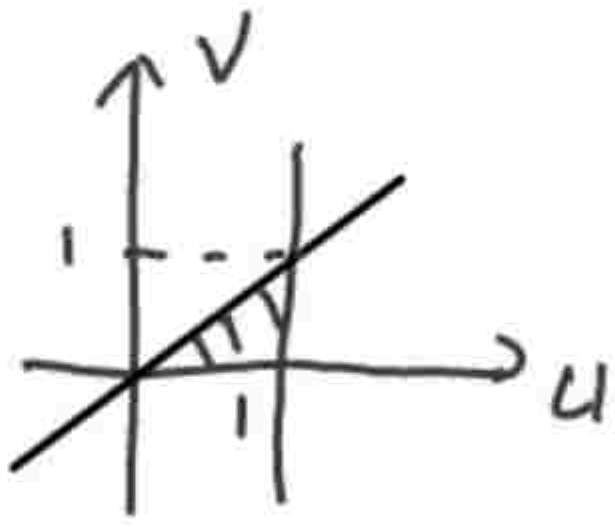
(5) 令 $\begin{cases} u=x+y \\ v=y \end{cases} \Rightarrow \begin{cases} x=u-v \\ y=v \end{cases}$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}=1$$

$$I=\int_0^1\int_0^v e^{\frac{v}{u}}\,dv\,du$$

$$=\int_0^1 u(e-1)\,du$$

$$=\frac{1}{2}(e-1)$$



(6) 令 $\begin{cases} u=x+y \\ v=x-y \end{cases} \Rightarrow \begin{cases} x=(u+v)/2 \\ y=(u-v)/2 \end{cases}$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}=-\frac{1}{2}$$

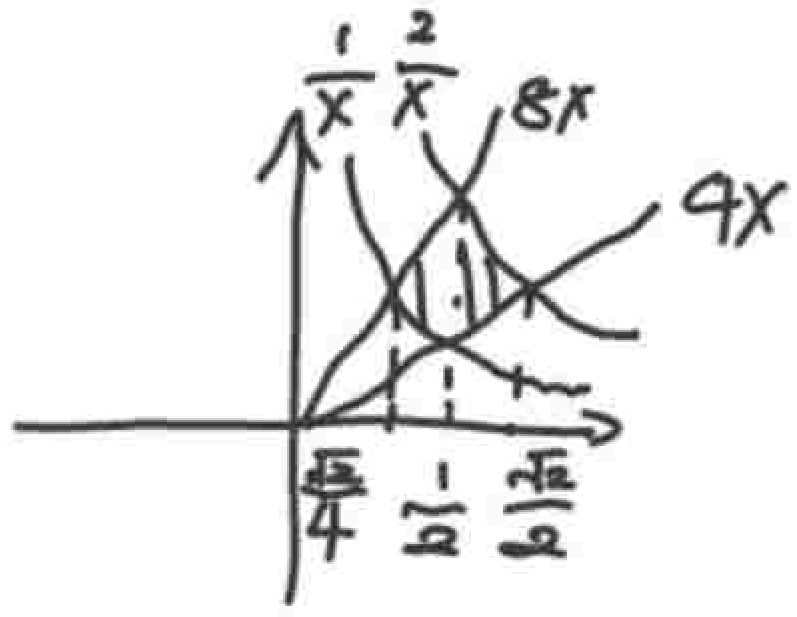
$$I=\int_0^1\int_{-u}^{u}\frac{1}{2}\cos\frac{v}{u}\,dv\,du$$

$$=\int_0^1\int_0^u\cos\frac{v}{u}\,dv\,du$$

$$=\sin 1\int_0^1 u\,du$$

$$=\frac{1}{2}\sin 1$$

(7)



$$I_1=\int_{\frac{\sqrt{2}}{4}}^{\frac{1}{2}}dx\int_{\frac{1}{x}}^{8x}\left(x^2+\frac{1}{4}y^2\right)dy$$

$$=\int_{\frac{\sqrt{2}}{4}}^{\frac{1}{2}}\left(\frac{152}{3}x^3-x-\frac{1}{12x^3}\right)dx$$

$$=\frac{35}{96}$$

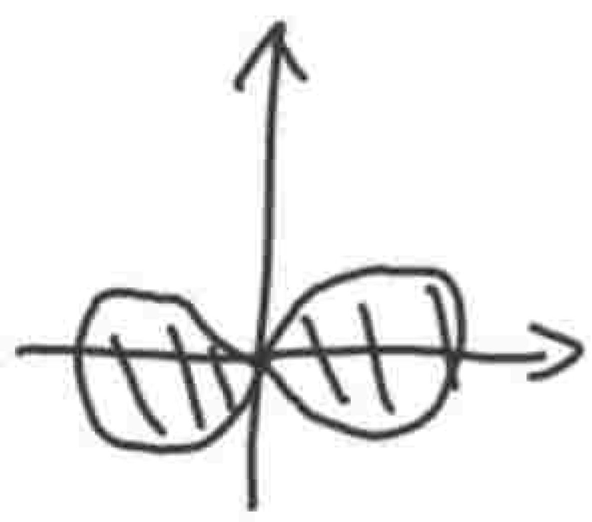
$$I_2=\int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}}dx\int_{4x}^{\frac{2}{x}}\left(x^2+\frac{1}{4}y^2\right)dy$$

$$=\frac{23}{48}$$

$$I=I_1+I_2=\frac{27}{32}$$

18. 令 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$

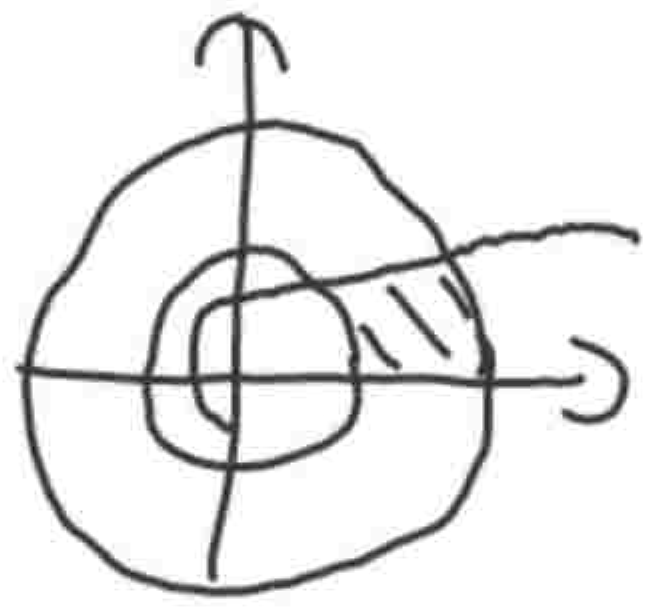
$$I=4\int_0^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$



$$=2a^2\int_0^{\frac{\pi}{4}}\cos 2\theta\,d\theta$$

$$=a^2$$

19.



$$S_1 = \int_1^2 \int_0^{\frac{1}{r}} r\,d\theta\,dr$$
$$= \int_1^2 dr$$
$$= 1$$

$$S_2 = 3\pi - 1$$

20. 哪里围成了立体?

21. $\iint_D y^2 \rho(x,y)\,dS$

22. $$m = \int_0^1 dy \int_0^y dx \int_0^{xy} (1+2z)\,dz$$
$$= \int_0^1 dy \int_0^y (x^2y^2 + xy)\,dx$$
$$= \int_0^1 \left(\frac{1}{3}y^5 + \frac{1}{2}y^3\right) dy$$
$$= \frac{1}{18} + \frac{1}{8}$$
$$= \frac{13}{72}$$

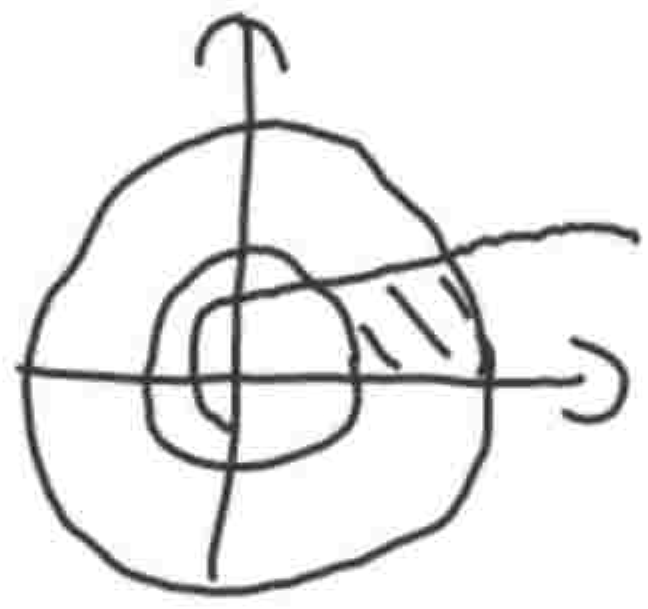
$$r_x = \frac{1}{m}\int_0^1 dy \int_0^y dx \int_0^{xy} x(1+2z)\,dz$$
$$= \frac{1}{m}\int_0^1 dy \int_0^y (x^3y^2 + x^2y)\,dx$$
$$= \frac{1}{m}\int_0^1 \left(\frac{1}{4}y^6 + \frac{1}{3}y^4\right) dy$$
$$= \frac{72}{13}\left(\frac{1}{28} + \frac{1}{15}\right)$$
$$= \frac{258}{455}$$

$$r_y = \frac{1}{m}\int_0^1 dy \int_0^y dx \int_0^{xy} y(1+2z)\,dz$$
$$= \frac{1}{m}\int_0^1 \left(\frac{1}{3}y^6 + \frac{1}{2}y^4\right) dy$$
$$= \frac{72}{13}\left(\frac{1}{21} + \frac{1}{10}\right)$$
$$= \frac{372}{455}$$

$$r_z = \frac{1}{m}\int_0^1 dy \int_0^y dx \int_0^{xy} z(1+2z)\,dz$$
$$= \frac{1}{m}\int_0^1 dy \int_0^y \left(\frac{2}{3}x^3y^3 + \frac{1}{2}x^2y^2\right) dx$$
$$= \frac{1}{m}\int_0^1 \left(\frac{1}{6}y^7 + \frac{1}{6}y^5\right) dy$$
$$= \frac{72}{13}\left(\frac{1}{48} + \frac{1}{36}\right)$$
$$= \frac{7}{26}$$

故质心 $\vec{r} = \left(\frac{258}{455}, \frac{372}{455}, \frac{7}{26}\right)$

19.



$$S_1=\int_1^2\int_0^{\frac{1}{r}} r\,d\theta\,dr$$
$$=\int_1^2 dr$$
$$=1$$

$$S_2=3\pi-1$$

20. 哪里围成了立体？

21. $\iint_D y^2\rho(x,y)\,ds$

22. $$m=\int_0^1 dy\int_0^y dx\int_0^{xy}(1+2z)\,dz$$
$$=\int_0^1 dy\int_0^y (x^2y^2+xy)\,dx$$
$$=\int_0^1\left(\frac{1}{3}y^5+\frac{1}{2}y^3\right)dy$$
$$=\frac{1}{18}+\frac{1}{8}$$
$$=\frac{13}{72}$$

$$r_x=\frac{1}{m}\int_0^1 dy\int_0^y dx\int_0^{xy} x(1+2z)\,dz$$
$$=\frac{1}{m}\int_0^1 dy\int_0^y (x^3y^2+x^2y)\,dx$$
$$=\frac{1}{m}\int_0^1\left(\frac{1}{4}y^6+\frac{1}{3}y^4\right)dy$$
$$=\frac{72}{13}\left(\frac{1}{28}+\frac{1}{15}\right)$$
$$=\frac{258}{455}$$

$$r_y=\frac{1}{m}\int_0^1 dy\int_0^y dx\int_0^{xy} y(1+2z)\,dz$$
$$=\frac{1}{m}\int_0^1\left(\frac{1}{3}y^6+\frac{1}{2}y^4\right)dy$$
$$=\frac{72}{13}\left(\frac{1}{21}+\frac{1}{10}\right)$$
$$=\frac{372}{455}$$

$$r_z=\frac{1}{m}\int_0^1 dy\int_0^y dx\int_0^{xy} z(1+2z)\,dz$$
$$=\frac{1}{m}\int_0^1 dy\int_0^y\left(\frac{2}{3}x^3y^3+\frac{1}{2}x^2y^2\right)dx$$
$$=\frac{1}{m}\int_0^1\left(\frac{1}{6}y^7+\frac{1}{6}y^5\right)dy$$
$$=\frac{72}{13}\left(\frac{1}{48}+\frac{1}{36}\right)$$
$$=\frac{7}{26}$$

故质心 $\vec{r}=\left(\frac{258}{455},\frac{372}{455},\frac{7}{26}\right)$

23. 令 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases}$

$$V=\int_0^{\frac{\pi}{2}}d\theta\int_{\cos\theta}^{2\cos\theta}dr\int_0^{r^2}r\,dz$$

$$=\int_0^{\frac{\pi}{2}}d\theta\int_{\cos\theta}^{2\cos\theta}r^3dr$$

$$=\int_0^{\frac{\pi}{2}}\frac{15}{4}\cos^4\theta\,d\theta$$

$$=\frac{45}{64}\pi$$

(3) 令 $\begin{cases} x=r\sin\varphi\cos\theta \\ y=r\sin\varphi\sin\theta \\ z=r\cos\varphi \end{cases}$

$$I=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{2\cos\varphi}\frac{1}{r}\cdot r^2\sin\varphi\,dr\,d\theta\,d\varphi$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}2\sin\varphi\cos^2\varphi\,d\theta\,d\varphi$$

$$=\pi\int_0^{\frac{\pi}{2}}\sin\varphi\cos^2\varphi\,d\varphi$$

$$=\frac{1}{3}\pi$$

24. (1)

令 $\begin{cases} x=r\sin\varphi\cos\theta \\ y=\sqrt{2}\,r\sin\varphi\sin\theta \\ z=\sqrt{3}\,r\cos\varphi \end{cases}$

$$I=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1\sqrt{6}\,r^3\sin^2\varphi\cos\varphi\sin\theta\cos\theta\;\sqrt{6}\,r^2\sin\varphi\,dr\,d\theta\,d\varphi$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin^3\varphi\cos\varphi\cdot\frac{1}{2}\sin2\theta\,d\theta\,d\varphi$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^3\varphi\cos\varphi\,d\varphi$$

$$=\frac{1}{8}$$

(2) 令 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases}$

满足 $r^2\le z\le 12-2r^2,\ 0\le\theta\le 2\pi$

$$I=\int_0^2dr\int_0^{2\pi}d\theta\int_{r^2}^{12-2r^2}r^2\cdot r\,dz$$

$$=2\pi\int_0^2(12r^3-3r^5)dr$$

$$=2\pi\left(3r^4-\frac{1}{2}r^6\right)\Big|_0^2$$

$$=32\pi$$

(4) 令 $\begin{cases} u=y+z-x \\ v=x+z-y \\ w=x+y-z \end{cases}\Longrightarrow\begin{cases} x=(v+w)/2 \\ y=(u+w)/2 \\ z=(u+v)/2 \end{cases}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{vmatrix}=\frac{1}{4}$$

$$I=\int_0^1\int_0^1\int_0^1\frac{1}{4}uvw\,du\,dv\,dw$$

$$=\frac{1}{32}$$

(5) 令 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases}$

$$I=\int_3^4dr\int_0^{2\pi}d\theta\int_0^r r\cdot r\,dz$$

$$=2\pi\int_3^4r^3dr$$

$$=\frac{175}{2}\pi$$

23. 令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$$V = \int_0^{\frac{\pi}{2}} d\theta \int_{\cos\theta}^{2\cos\theta} dr \int_0^{r^2} r\,dz$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_{\cos\theta}^{2\cos\theta} r^3 dr$$

$$= \int_0^{\frac{\pi}{2}} \frac{15}{4}\cos^4\theta\, d\theta$$

$$= \frac{45}{64}\pi$$

24.(1)

令 $\begin{cases} x = r\sin\varphi\cos\theta \\ y = \sqrt{2}\, r\sin\varphi\sin\theta \\ z = \sqrt{3}\, r\cos\varphi \end{cases}$

$$I = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \sqrt{6}\, r^3 \sin^2\varphi\cos\varphi\sin\theta\cos\theta \cdot \sqrt{6}\, r^2 \sin\varphi\, dr\, d\theta\, d\varphi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin^3\varphi\cos\varphi \cdot \frac{1}{2}\sin 2\theta\, d\theta\, d\varphi$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^3\varphi\cos\varphi\, d\varphi$$

$$= \frac{1}{8}$$

(2) 令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

满足 $r^2 \le z \le 12 - 2r^2,\ 0 \le \theta \le 2\pi$

$$I = \int_0^2 dr \int_0^{2\pi} d\theta \int_{r^2}^{12-2r^2} r^2 \cdot r\, dz$$

$$= 2\pi \int_0^2 (12r^3 - 3r^5)\, dr$$

$$= 2\pi \left(3r^4 - \frac{1}{2}r^6\right)\Big|_0^2$$

$$= 32\pi$$

(3) 令 $\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$

$$I = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{2\cos\varphi} \frac{1}{r} \cdot r^2 \sin\varphi\, dr\, d\theta\, d\varphi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} 2\sin\varphi\cos^2\varphi\, d\theta\, d\varphi$$

$$= \pi \int_0^{\frac{\pi}{2}} \sin\varphi\cos^2\varphi\, d\varphi$$

$$= \frac{1}{3}\pi$$

(4) 令 $\begin{cases} u = y + z - x \\ v = x + z - y \\ w = x + y - z \end{cases} \Rightarrow \begin{cases} x = (v+w)/2 \\ y = (u+w)/2 \\ z = (u+v)/2 \end{cases}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{vmatrix} = \frac{1}{4}$$

$$I = \int_0^1\int_0^1\int_0^1 \frac{1}{4} uvw\, du\, dv\, dw$$

$$= \frac{1}{32}$$

(5) 令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$$I = \int_3^4 dr \int_0^{2\pi} d\theta \int_0^r r \cdot r\, dz$$

$$= 2\pi \int_3^4 r^3 dr$$

$$= \frac{175}{2}\pi$$

(6) 令 $\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = 1 + r\cos\varphi \end{cases}$

$$I = \int_0^{\pi} d\varphi \int_0^{2\pi} d\theta \int_0^1 (1+r\cos\varphi)(r^2+2+2r\cos\varphi-1)r^2\sin\varphi\, dr$$

$$= \int_0^{\pi} d\varphi \int_0^{2\pi} d\theta \int_0^1 \sin\varphi(\cos\varphi r^5 + (2\cos^2\varphi+1)r^4 + 3\cos\varphi r^3 + r^2)\,dr$$

$$= \int_0^{\pi} d\varphi \int_0^{2\pi} \sin\varphi\left(\frac{1}{6}\cos\varphi + \frac{2}{5}\cos^2\varphi + \frac{1}{5} + \frac{3}{4}\cos\varphi + \frac{1}{3}\right) d\theta$$

$$= 2\pi \int_0^{\pi} \sin\varphi\left(\frac{2}{5}\cos^2\varphi + \frac{11}{12}\cos\varphi + \frac{8}{15}\right) d\varphi$$

$$= 2\pi\left(\frac{4}{15} + \frac{16}{15}\right)$$

$$= \frac{8}{3}\pi$$

(7) 令 $\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$

$$I = \int_0^{\pi} d\varphi \int_0^{2\pi} d\theta \int_0^3 r^3 \cdot r^2 \sin\varphi\, dr$$

$$= \int_0^{\pi} d\varphi \int_0^{2\pi} \frac{243}{2} \sin\varphi\, d\theta$$

$$= \int_0^{\pi} 243\pi \sin\varphi\, d\varphi$$

$$= 486\pi$$

25. 令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$$V = \int_1^2 dr \int_0^{2\pi} d\theta \int_0^{12-r^2} r\, dz$$

$$= 2\pi \int_1^2 (12r - r^3)\, dr$$

$$= 2\pi\left(6r^2 - \frac{1}{4}r^4\right)\Big|_1^2$$

$$= \frac{57}{2}\pi$$

27. (1) 记 $D_R = \{(x,y,z) \mid x^2+y^2+z^2 \le R^2\}$.

$$I = \lim_{R\to+\infty} \iiint_{D_R} e^{-(x^2+y^2+z^2)}\, dx\,dy\,dz$$

$$= \lim_{R\to+\infty} \int_0^R dr \int_0^{2\pi} d\theta \int_0^{\pi} e^{-r^2} r^2 \sin\varphi\, d\varphi$$

$$= \lim_{R\to+\infty} \int_0^R 4\pi r^2 e^{-r^2}\, dr$$

$$= 4\pi \int_0^{+\infty} r^2 e^{-r^2}\, dr$$

$$= -2\pi \int_0^{+\infty} r\, de^{-r^2}$$

$$= -2\pi r e^{-r^2}\Big|_0^{+\infty} + 2\pi \int_0^{+\infty} e^{-r^2}\, dr$$

$$= \pi\sqrt{\pi} \quad (\text{Gauss积分})$$

(16) 令 $\begin{cases} x=r\sin\varphi\cos\theta \\ y=r\sin\varphi\sin\theta \\ z=1+r\cos\varphi \end{cases}$

$$I=\int_0^{\pi}d\varphi\int_0^{2\pi}d\theta\int_0^1(1+r\cos\varphi)(r^2+2+2r\cos\varphi-1)r^2\sin\varphi\,dr$$

$$=\int_0^{\pi}d\varphi\int_0^{2\pi}d\theta\int_0^1\sin\varphi(\cos\varphi r^5+(2\cos^2\varphi+1)r^4+3\cos\varphi r^3+r^2)dr$$

$$=\int_0^{\pi}d\varphi\int_0^{2\pi}\sin\varphi\left(\frac{1}{6}\cos\varphi+\frac{2}{5}\cos^2\varphi+\frac{1}{5}+\frac{3}{4}\cos\varphi+\frac{1}{3}\right)d\theta$$

$$=2\pi\int_0^{\pi}\sin\varphi\left(\frac{2}{5}\cos^2\varphi+\frac{11}{12}\cos\varphi+\frac{8}{15}\right)d\varphi$$

$$=2\pi\left(\frac{4}{15}+\frac{16}{15}\right)$$

$$=\frac{8}{3}\pi$$

(17) 令 $\begin{cases} x=r\sin\varphi\cos\theta \\ y=r\sin\varphi\sin\theta \\ z=r\cos\varphi \end{cases}$

$$I=\int_0^{\pi}d\varphi\int_0^{2\pi}d\theta\int_0^3 r^3\cdot r^2\sin\varphi\,dr$$

$$=\int_0^{\pi}d\varphi\int_0^{2\pi}\frac{243}{2}\sin\varphi\,d\theta$$

$$=\int_0^{\pi}243\pi\sin\varphi\,d\varphi$$

$$=486\pi$$

25. 令 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases}$

$$V=\int_1^2dr\int_0^{2\pi}d\theta\int_0^{12-r^2}r\,dz$$

$$=2\pi\int_1^2(12r-r^3)dr$$

$$=2\pi\left(6r^2-\frac{1}{4}r^4\right)\Big|_1^2$$

$$=\frac{57}{2}\pi$$

27. (1) 记 $D_R=\{(x,y,z)\mid x^2+y^2+z^2\le R^2\}$.

$$I=\lim_{R\to+\infty}\iiint_{D_R}e^{-(x^2+y^2+z^2)}dxdydz$$

$$=\lim_{R\to+\infty}\int_0^R dr\int_0^{2\pi}d\theta\int_0^{\pi}e^{-r^2}r^2\sin\varphi\,d\varphi$$

$$=\lim_{R\to+\infty}\int_0^R 4\pi r^2e^{-r^2}dr$$

$$=4\pi\int_0^{+\infty}r^2e^{-r^2}dr$$

$$=-2\pi\int_0^{+\infty}r\,de^{-r^2}$$

$$=-2\pi re^{-r^2}\Big|_0^{+\infty}+2\pi\int_0^{+\infty}e^{-r^2}dr$$

$$=\pi\sqrt{\pi}\ \text{(Gauss积分)}$$

(2) $I=\lim\limits_{R\to+\infty}\int_0^R dr\int_0^{2\pi}d\theta\int_0^{\pi}\frac{r^2\sin\varphi}{(1+r^2)^2}d\varphi$

$=\lim\limits_{R\to+\infty}4\pi\int_0^R\frac{r^2}{(1+r^2)^2}dr$

$=4\pi\int_0^{+\infty}\frac{r^2}{(1+r^2)^2}dr$

$=2\pi\left(\arctan r-\frac{r}{r^2+1}\right)\Big|_0^{+\infty}$

$=\pi^2$

(3) $I=\lim\limits_{R\to1^-}\int_0^R dr\int_0^{2\pi}d\theta\int_0^{\pi}\frac{r^2\sin\varphi}{\sqrt{r^2-r\cos\varphi+\frac{1}{4}}}d\varphi$

$=\int_0^1 dr\int_0^{2\pi}d\theta\cdot r\sqrt{4r^2-4r\cos\varphi+1}\Big|_0^{\pi}$

$=\int_0^1 dr\int_0^{2\pi}d\theta\cdot r(2r+1-|2r-1|)$

$=\int_0^{\frac{1}{2}}8\pi r^2dr+\int_{\frac{1}{2}}^1 4\pi r dr$

$=\frac{8}{3}\pi r^3\Big|_0^{\frac{1}{2}}+2\pi r^2\Big|_{\frac{1}{2}}^1$

$=\frac{1}{3}\pi+\frac{3}{2}\pi$

$=\frac{11}{6}\pi$

(2) $I=\lim\limits_{R\to+\infty}\int_0^R dr\int_0^{2\pi}d\theta\int_0^{\pi}\frac{r^2\sin\varphi}{(1+r^2)^2}d\varphi$

$=\lim\limits_{R\to+\infty}4\pi\int_0^R\frac{r^2}{(1+r^2)^2}dr$

$=4\pi\int_0^{+\infty}\frac{r^2}{(1+r^2)^2}dr$

$=2\pi\left(\arctan r-\frac{r}{r^2+1}\right)\Big|_0^{+\infty}$

$=\pi^2$

(3) $I=\lim\limits_{R\to1^-}\int_0^R dr\int_0^{2\pi}d\theta\int_0^{\pi}\frac{r^2\sin\varphi}{\sqrt{r^2-r\cos\varphi+\frac{1}{4}}}d\varphi$

$=\int_0^1 dr\int_0^{2\pi}d\theta\cdot r\sqrt{4r^2-4r\cos\varphi+1}\Big|_0^{\pi}$

$=\int_0^1 dr\int_0^{2\pi}d\theta\cdot r(2r+1-|2r-1|)$

$=\int_0^{\frac{1}{2}}8\pi r^2dr+\int_{\frac{1}{2}}^1 4\pi r\,dr$

$=\frac{8}{3}\pi r^3\Big|_0^{\frac{1}{2}}+2\pi r^2\Big|_{\frac{1}{2}}^1$

$=\frac{1}{3}\pi+\frac{3}{2}\pi$

$=\frac{11}{6}\pi$

28. 令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

满足 $0 \leq z \leq \frac{1}{(1+r\cos\theta+3r\sin\theta)^3} \triangleq z_0$

$$V = \int_0^{+\infty} dr \int_0^{\frac{\pi}{2}} d\theta \int_0^{z_0} r\,dz$$

$$= \int_0^{+\infty} dr \int_0^{\frac{\pi}{2}} \frac{r\,d\theta}{(1+r\cos\theta+3r\sin\theta)^3}$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{+\infty} \frac{r\,dr}{(1+r\cos\theta+3r\sin\theta)^3}$$

$$= \int_0^{\frac{\pi}{2}} d\theta \left(-\frac{2ar+1}{2a^2(ar+1)^2}\right)\Big|_0^{+\infty} \quad (a = \cos\theta+3\sin\theta)$$

$$= \int_0^{\frac{\pi}{2}} \frac{d\theta}{2(\cos\theta+3\sin\theta)^2}$$

$$= \frac{\sin\theta}{6\sin\theta+2\cos\theta}\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{6}$$

28. 令 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases}$

满足 $0\leq z\leq \frac{1}{(1+r\cos\theta+3r\sin\theta)^3} \triangleq z_0$

$$V=\int_0^{+\infty}dr\int_0^{\frac{\pi}{2}}d\theta\int_0^{z_0}r\,dz$$

$$=\int_0^{+\infty}dr\int_0^{\frac{\pi}{2}}\frac{r\,d\theta}{(1+r\cos\theta+3r\sin\theta)^3}$$

$$=\int_0^{\frac{\pi}{2}}d\theta\int_0^{+\infty}\frac{r\,dr}{(1+r\cos\theta+3r\sin\theta)^3}$$

$$=\int_0^{\frac{\pi}{2}}d\theta\left(-\frac{2ar+1}{2a^2(ar+1)^2}\right)\Big|_0^{+\infty}\quad(a=\cos\theta+3\sin\theta)$$

$$=\int_0^{\frac{\pi}{2}}\frac{d\theta}{2(\cos\theta+3\sin\theta)^2}$$

$$=\frac{\sin\theta}{6\sin\theta+2\cos\theta}\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{1}{6}$$

习题十六

1. $\begin{cases} x_0=\int_\Gamma x\rho ds / \int_\Gamma \rho ds \\ y_0=\int_\Gamma y\rho ds / \int_\Gamma \rho ds \\ z_0=\int_\Gamma z\rho ds / \int_\Gamma \rho ds \end{cases}$

2. (1) 由对称性，$\int_\Gamma x ds=0$

(2) 由对称性，$\int_\Gamma xy ds=0$

(3) $\int_\Gamma x^2 ds=\int_0^{2\pi}\cos^2\theta d\theta=\pi$

(4) $\int_\Gamma |x| ds=4\int_0^{\frac{\pi}{2}}\cos\theta d\theta=4$

3.

(1) 由对称性，$\int_\Gamma x ds=0$

(2) $\int_\Gamma xy ds=\frac{1}{3}\int_\Gamma (xy+yz+zx)ds$

$=\frac{1}{6}\int_\Gamma [(x+y+z)^2-(x^2+y^2+z^2)]ds$

$=-\frac{1}{6}\int_\Gamma ds$

$=-\frac{1}{3}\pi$

(3) $\int_\Gamma x^2 ds=\frac{1}{3}\int_\Gamma (x^2+y^2+z^2)ds$

$=\frac{1}{3}\int_\Gamma ds$

$=\frac{2}{3}\pi$

4. (1) (a) $I=\int_0^{2\pi}\cos t\sin t\, dt=0$

(b) $I=\int_0^{2\pi}\cos t\sin t(t+1)\cdot\sqrt{2}dt$

$=-\frac{\sqrt{2}}{4}\int_0^{2\pi}(t+1)d\cos 2t$

$=-\frac{\sqrt{2}}{4}(t+1)\cos 2t\Big|_0^{2\pi}+\frac{\sqrt{2}}{4}\int_0^{2\pi}\cos 2t dt$

$=-\frac{\sqrt{2}}{4}(2\pi+1-1)$

$=-\frac{\pi}{\sqrt{2}}$

(2) 令 $\begin{cases} x=a\cos^3\theta, \\ y=a\sin^3\theta, \end{cases}$ $\theta\in[0,2\pi]$.

$I=\int_0^{2\pi}(\sin^4\theta+\cos^4\theta)a^{\frac{4}{3}}\cdot 3a|\sin\theta\cos\theta|d\theta$

$=12a^{\frac{7}{3}}\int_0^{\frac{\pi}{2}}(\sin^4\theta+\cos^4\theta)\sin\theta\cos\theta d\theta$

$=6a^{\frac{7}{3}}\int_0^{\frac{\pi}{2}}(1-\frac{1}{2}\sin^2 2\theta)\sin 2\theta d\theta$

$=-a^{\frac{7}{3}}(\frac{15}{8}\cos 2x+\frac{1}{8}\cos 6x)\Big|_0^{\frac{\pi}{2}}$

$=4a^{\frac{7}{3}}$

(3) $I=\int_0^1\frac{1}{2}t^3\sqrt{t^2+1}dt$

$=\frac{1}{30}(x^2+1)^{\frac{3}{2}}(3x^2-2)\Big|_0^1$

$=\frac{1}{15}(\sqrt{2}+1)$

(4) 设 $\begin{cases} x_1=t, \\ \cdots \\ x_n=t, \end{cases}$ $t\in[0,1]$.

$I=\int_0^1 nt\sqrt{n}dt=\frac{1}{2}n\sqrt{n}$

习题十六

1. $\begin{cases} x_0=\int_\Gamma x\rho ds/\int_\Gamma \rho ds \\ y_0=\int_\Gamma y\rho ds/\int_\Gamma \rho ds \\ z_0=\int_\Gamma z\rho ds/\int_\Gamma \rho ds \end{cases}$

2. (1) 由对称性，$\int_\Gamma x ds=0$

(2) 由对称性，$\int_\Gamma xy ds=0$

(3) $\int_\Gamma x^2 ds=\int_0^{2\pi}\cos^2\theta d\theta=\pi$

(4) $\int_\Gamma |x| ds=4\int_0^{\frac{\pi}{2}}\cos\theta d\theta=4$

3.

(1) 由对称性，$\int_\Gamma x ds=0$

(2) $\int_\Gamma xy ds=\frac{1}{3}\int_\Gamma (xy+yz+zx)ds$

$=\frac{1}{6}\int_\Gamma [(x+y+z)^2-(x^2+y^2+z^2)]ds$

$=-\frac{1}{6}\int_\Gamma ds$

$=-\frac{1}{3}\pi$

(3) $\int_\Gamma x^2 ds=\frac{1}{3}\int_\Gamma (x^2+y^2+z^2)ds$

$=\frac{1}{3}\int_\Gamma ds$

$=\frac{2}{3}\pi$

4. (1) (a) $I=\int_0^{2\pi}\cos t\sin t\, dt=0$

(b) $I=\int_0^{2\pi}\cos t\sin t(t+1)\cdot\sqrt{2}dt$

$=-\frac{\sqrt{2}}{4}\int_0^{2\pi}(t+1)d\cos 2t$

$=-\frac{\sqrt{2}}{4}(t+1)\cos 2t\Big|_0^{2\pi}+\frac{\sqrt{2}}{4}\int_0^{2\pi}\cos 2t dt$

$=-\frac{\sqrt{2}}{4}(2\pi+1-1)$

$=-\frac{\pi}{\sqrt{2}}$

(2) 令 $\begin{cases} x=a\cos^3\theta, \\ y=a\sin^3\theta, \end{cases}$ $\theta\in[0,2\pi]$.

$I=\int_0^{2\pi}(\sin^4\theta+\cos^4\theta)a^{\frac{4}{3}}\cdot 3a|\sin\theta\cos\theta|d\theta$

$=12a^{\frac{7}{3}}\int_0^{\frac{\pi}{2}}(\sin^4\theta+\cos^4\theta)\sin\theta\cos\theta d\theta$

$=6a^{\frac{7}{3}}\int_0^{\frac{\pi}{2}}(1-\frac{1}{2}\sin^2 2\theta)\sin 2\theta d\theta$

$=-a^{\frac{7}{3}}(\frac{15}{8}\cos 2x+\frac{1}{8}\cos(6x))\Big|_0^{\frac{\pi}{2}}$

$=4a^{\frac{7}{3}}$

(3) $I=\int_0^1\frac{1}{2}t^3\sqrt{t^2+1}dt$

$=\frac{1}{30}(x^2+1)^{\frac{3}{2}}(3x^2-2)\Big|_0^1$

$=\frac{1}{15}(\sqrt{2}+1)$

(4) 设 $\begin{cases} x_1=t, \\ \cdots \\ x_n=t, \end{cases}$ $t\in[0,1]$.

$I=\int_0^1 nt\sqrt{n}dt=\frac{1}{2}n\sqrt{n}$

5. 令 $\begin{cases} x=\cos\theta, \\ y=\sin\theta, \end{cases}$ $\theta\in[0,2\pi]$.

$$\int_L x^2 ds = \int_0^{2\pi} \cos^2\theta \, d\theta = \pi$$

$$\int_L ds = 2\pi$$

平均值为 $\frac{1}{2}$

6. $$\int_\Gamma f(x,y,z)\,ds = \int_\alpha^\beta f(x(t),y(t),z(t))\sqrt{x'(t)^2+\cdots}\,dt$$

(定积分第一中值定理) $= f(x(t_0),y(t_0),z(t_0))\int_\alpha^\beta \sqrt{x'(t)^2+\cdots}\,dt$

$$= f(x(t_0),y(t_0),z(t_0))\,L$$

取 $\xi = x(t_0)$, $\eta = y(t_0)$, $\zeta = z(t_0)$ 即可.

7. (1) (a)

令 $(x,y,z)=(t,t,t)$, $t\in[0,1]$

$$I = \int_0^1 (-6t^2+2t+1-4t^4)\,dt$$

$$= -2+1+1-\frac{4}{5}$$

$$= -\frac{4}{5}$$

(b) $$I = \int_0^1 1\,dt + \int_0^1 2t\,dt + \int_0^1 (3t^2-6)\,dt$$

$$= 1+1+1-6$$

$$= -3$$

5. 令 $\begin{cases} x=\cos\theta, \\ y=\sin\theta, \end{cases}$ $\theta\in[0,2\pi]$.

$$\int_L x^2 ds = \int_0^{2\pi} \cos^2\theta \, d\theta = \pi$$

$$\int_L ds = 2\pi$$

平均值为 $\frac{1}{2}$

6. $\int_\Gamma f(x,y,z)\,ds = \int_\alpha^\beta f(x(t),y(t),z(t))\sqrt{x'(t)^2+\cdots}\,dt$

(定积分第一中值定理) $= f(x(t_0),y(t_0),z(t_0))\int_\alpha^\beta \sqrt{x'(t)^2+\cdots}\,dt$

$= f(x(t_0),y(t_0),z(t_0))L$

取 $\xi = x(t_0)$, $\eta = y(t_0)$, $\zeta = z(t_0)$ 即可.

7. (1) (a)

令 $(x,y,z)=(t,t,t)$, $t\in[0,1]$

$$I = \int_0^1 (-6t^2+2t+1-4t^4)\,dt$$

$$= -2+1+1-\frac{4}{5}$$

$$= -\frac{4}{5}$$

(b) $I = \int_0^1 1\,dt + \int_0^1 2t\,dt + \int_0^1 (3t^2-6)\,dt$

$$=1+1+1-6$$

$$=-3$$

(2)(a) 令 $(x,y,z)=(\cos t,\sin t,0)$, $t\in[0,2\pi]$

$$I=\int_0^{2\pi}(\sin t(-\sin t)+\cos t\cos t)\,dt$$

$$=\int_0^{2\pi}\cos 2t\,dt$$

$$=0$$

(b) $$I=\int_0^{2\pi}[(\sin t+t)(-\sin t)+(\cos t+t)\cos t+(\cos t+\sin t)]\,dt$$

$$=\int_0^{2\pi}t(\cos t-\sin t)\,dt$$

$$=\int_0^{2\pi}t\,d\sin t+\int_0^{2\pi}t\,d\cos t$$

$$=t(\sin t+\cos t)\Big|_0^{2\pi}-\int_0^{2\pi}(\sin t+\cos t)\,dt$$

$$=2\pi$$

(3) $$I=\int_0^1[e^x(x^2+x)+2x+1]\,dx$$

$$=2+\int_0^1(x^2+x)\,de^x$$

$$=2+(x^2+x)e^x\Big|_0^1-\int_0^1(2x+1)\,de^x$$

$$=2+2e-(2x+1)e^x\Big|_0^1+\int_0^1 2e^x\,dx$$

$$=2+2e-3e+1+2e-2$$

$$=e+1$$

(4) 令 $(x,y)=(a\cos t,b\sin t)$, $t\in[0,\frac{\pi}{2}]$

$$I=\int_0^{\frac{\pi}{2}}[(a\cos t+b\sin t)(-a\sin t)+(a\cos t-b\sin t)(b\cos t)]\,dt$$

$$=\int_0^{\frac{\pi}{2}}\left(ab\cos 2t-\frac{a^2+b^2}{2}\sin 2t\right)dt$$

$$=\left(\frac{ab}{2}\sin 2t+\frac{a^2+b^2}{4}\cos 2t\right)\Big|_0^{\frac{\pi}{2}}$$

$$=-\frac{1}{2}(a^2+b^2)$$

(2)(a) 令 $(x,y,z)=(\cos t,\sin t,0)$, $t\in[0,2\pi]$

$$I=\int_0^{2\pi}(\sin t(-\sin t)+\cos t\cos t)\,dt$$

$$=\int_0^{2\pi}\cos 2t\,dt$$

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(b) $I=\int_0^{2\pi}[(\sin t+t)(-\sin t)+(\cos t+t)\cos t+(\cos t+\sin t)]\,dt$

$$=\int_0^{2\pi}t(\cos t-\sin t)\,dt$$

$$=\int_0^{2\pi}t\,d\sin t+\int_0^{2\pi}t\,d\cos t$$

$$=t(\sin t+\cos t)\Big|_0^{2\pi}-\int_0^{2\pi}(\sin t+\cos t)\,dt$$

$$=2\pi$$

(3) $I=\int_0^1[e^x(x^2+x)+2x+1]\,dx$

$$=2+\int_0^1(x^2+x)\,de^x$$

$$=2+(x^2+x)e^x\Big|_0^1-\int_0^1(2x+1)\,de^x$$

$$=2+2e-(2x+1)e^x\Big|_0^1+\int_0^1 2e^x\,dx$$

$$=2+2e-3e+1+2e-2$$

$$=e+1$$

(4) 令 $(x,y)=(a\cos t,b\sin t)$, $t\in[0,\frac{\pi}{2}]$

$$I=\int_0^{\frac{\pi}{2}}[(a\cos t+b\sin t)(-a\sin t)+(a\cos t-b\sin t)(b\cos t)]\,dt$$

$$=\int_0^{\frac{\pi}{2}}\left(ab\cos 2t-\frac{a^2+b^2}{2}\sin 2t\right)dt$$

$$=\left(\frac{ab}{2}\sin 2t+\frac{a^2+b^2}{4}\cos 2t\right)\Big|_0^{\frac{\pi}{2}}$$

$$=-\frac{1}{2}(a^2+b^2)$$

(5) 令 $\begin{cases} x=\frac{1}{\sqrt{2}}\cos t, \\ y=\sin t, \\ z=1-\frac{1}{\sqrt{2}}\cos t, \end{cases}$ $t\in[0,2\pi]$

$$I=\int_0^{2\pi}\Big[\sin t\left(-\tfrac{1}{\sqrt{2}}\sin t\right)+\left(1-\tfrac{1}{\sqrt{2}}\cos t\right)\cos t+\tfrac{1}{\sqrt{2}}\cos t\left(1+\tfrac{1}{\sqrt{2}}\sin t\right)\Big]dt$$

计算得 $I=-\sqrt{2}\pi$

8. (a) 令 $(x,y)=(\cos t,\sin t)$, $t\in[0,2\pi]$

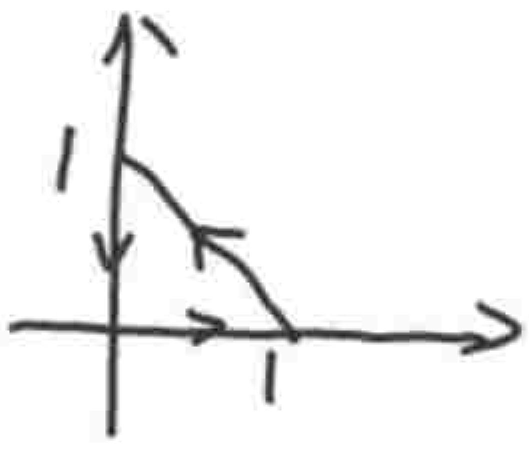
(1) $I=\int_0^{2\pi}dt=2\pi$

(2) $I=\int_0^{2\pi}\cos 2t\,dt=0$

(b) (1) 由对称性知 $I=0$

(2) 由对称性知 $I=0$

9.



$$I=\int_0^1 e^x dx+\int_0^1\left(e^{1-2y}-e\right)dy+\int_1^0 e^{-y}dy$$

$$=e-1-e-\frac{1}{2}\left(\frac{1}{e}-e\right)+\frac{1}{e}-1$$

$$=\frac{1}{2e}+\frac{e}{2}-2$$

10. 记 $\vec{f}=(P,Q,R)$.

$$\left|\int_\Gamma P\,dx+Q\,dy+R\,dz\right|$$

$$=\left|\int_\Gamma \vec{f}\cdot d\vec{r}\right|$$

$$\le\int_\Gamma\left|\vec{f}\cdot d\vec{r}\right|$$

$$\le\int_\Gamma|\vec{f}||d\vec{r}|$$

$$=\int_\Gamma\sqrt{P^2+Q^2+R^2}\,dr$$

$$\le M\int_\Gamma dr$$

$$=ML$$

11. 由上一题结论，有

$$\int_{\Gamma_R}\frac{z\,dx+x\,dy+y\,dz}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\le\int_{\Gamma_R}\frac{(z^2+x^2+y^2)^{\frac{1}{2}}dr}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$=\frac{1}{R^2}\int_{\Gamma_R}dr$$

$$\le\frac{1}{R^2}\cdot 2\pi R$$

$$=\frac{2\pi}{R}$$

$$\to 0\ (R\to+\infty).$$

(5) 令 $\begin{cases} x=\frac{1}{\sqrt{2}}\cos t, \\ y=\sin t, \\ z=1-\frac{1}{\sqrt{2}}\cos t, \end{cases}$ $t\in[0,2\pi]$

$$I=\int_0^{2\pi}\Big[\sin t\big(-\tfrac{1}{\sqrt{2}}\sin t\big)+\big(1-\tfrac{1}{\sqrt{2}}\cos t\big)\cos t+\tfrac{1}{\sqrt{2}}\cos t\big(1+\tfrac{1}{\sqrt{2}}\sin t\big)\Big]dt$$

计算可知 $I=-\sqrt{2}\pi$

8. (a) 令 $(x,y)=(\cos t,\sin t)$, $t\in[0,2\pi]$

(1) $I=\int_0^{2\pi}dt=2\pi$

(2) $I=\int_0^{2\pi}\cos 2t\,dt=0$

(b) (1) 由对称性知 $I=0$

(2) 由对称性知 $I=0$

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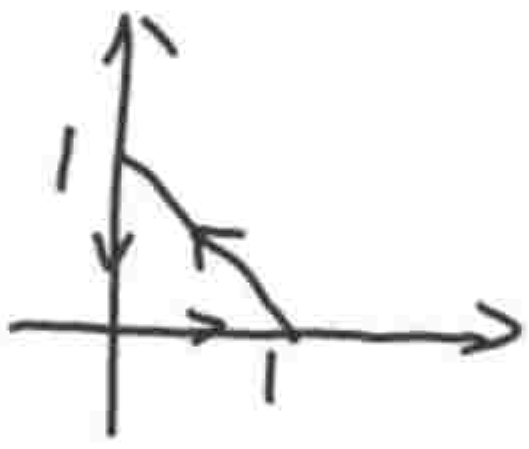


$$I=\int_0^1 e^x dx+\int_0^1\left(e^{1-2y}-e\right)dy+\int_1^0 e^{-y}dy$$

$$=e-1-e-\frac{1}{2}\left(\frac{1}{e}-e\right)+\frac{1}{e}-1$$

$$=\frac{1}{2e}+\frac{e}{2}-2$$

10. 记 $\vec{f}=(P,Q,R)$.

$$\left|\int_\Gamma Pdx+Qdy+Rdz\right|$$

$$=\left|\int_\Gamma \vec{f}\cdot d\vec{r}\right|$$

$$\le\int_\Gamma\left|\vec{f}\cdot d\vec{r}\right|$$

$$\le\int_\Gamma|\vec{f}||d\vec{r}|$$

$$=\int_\Gamma\sqrt{P^2+Q^2+R^2}\,dr$$

$$\le M\int_\Gamma dr$$

$$=ML$$

11. 由上一题结论，有

$$\int_{\Gamma_R}\frac{z\,dx+x\,dy+y\,dz}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\le\int_{\Gamma_R}\frac{(z^2+x^2+y^2)^{\frac{1}{2}}dr}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$=\frac{1}{R^2}\int_{\Gamma_R}dr$$

$$\le\frac{1}{R^2}\cdot 2\pi R$$

$$=\frac{2\pi}{R}$$

$$\to 0\ (R\to+\infty).$$

12. (1)

$$S=\int_0^{\sqrt{2}}\int_0^{\sqrt{2}}\sqrt{1+4x^2+4y^2}\,dx\,dy$$

令 $(x,y)=(r\cos\theta, r\sin\theta)$.

$$S=\int_0^{2\pi}\int_0^{\sqrt{2}}\sqrt{4r^2+1}\,r\,dr\,d\theta$$

$$=\frac{13}{3}\pi.$$

(2) $$S=2\int_0^{\frac{\pi}{6}}\int_0^{2\pi}\sin\varphi\,d\theta\,d\varphi$$

$$=(4-2\sqrt{3})\pi$$

(3) $$z'_x=\frac{6x}{2\sqrt{3(x^2+y^2)}}=\frac{\sqrt{3}x}{\sqrt{x^2+y^2}}$$

$$z'_y=\frac{\sqrt{3}y}{\sqrt{x^2+y^2}}$$

$$S=\iint_D\sqrt{1+z_x'^2+z_y'^2}\,dx\,dy$$

$$=2\iint_D dx\,dy$$

联立 $z^2=3(x^2+y^2)$, $x+y+z=2$. 有

$$\frac{1}{6}(x+1-\frac{y}{2})^2+\frac{1}{2}(\frac{y}{2}+1)^2=1.$$

令 $$\begin{cases}\sqrt{6}r\cos\theta=x+1-\frac{y}{2}\\ \sqrt{2}r\sin\theta=\frac{y}{2}+1\end{cases}$$

$$\Rightarrow\begin{cases}x=\sqrt{6}r\cos\theta+\sqrt{2}r\sin\theta-2\\ y=2\sqrt{2}r\sin\theta-2\end{cases}$$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|=4\sqrt{3}r$$

$$S=2\int_0^1 dr\int_0^{2\pi}4\sqrt{3}r\,d\theta$$

$$=8\sqrt{3}\pi$$

(4) $z=\sqrt{1-x^2}$, $z'_x=-\frac{x}{\sqrt{1-x^2}}$

$$S=48\int_0^{\frac{\sqrt{2}}{2}}dx\int_0^x\frac{1}{\sqrt{1-x^2}}dy$$

$$=48\int_0^{\frac{\sqrt{2}}{2}}\frac{x}{\sqrt{1-x^2}}dx$$

$$=-48\sqrt{1-x^2}\Big|_0^{\frac{\sqrt{2}}{2}}$$

$$=48-24\sqrt{2}$$

(5) $z=-\frac{1}{c}(ax+by+d)$

$$\sqrt{1+z_x'^2+z_y'^2}=(1+\frac{a^2}{c^2}+\frac{b^2}{c^2})^{\frac{1}{2}}$$

$$=\frac{1}{|c|}\sqrt{a^2+b^2+c^2}$$

$$S=\iint_D\frac{1}{|c|}\sqrt{a^2+b^2+c^2}\,dx\,dy$$

$$=\frac{\pi}{|c|}\sqrt{a^2+b^2+c^2}$$

13. 令 $P=(r\cos\theta, r\sin\theta, 1-r)$

$$P'_r=(\cos\theta,\sin\theta,-1)$$

$$P'_\theta=(-r\sin\theta, r\cos\theta, 0)$$

$$E=2,\ F=0,\ G=r^2$$

$$S=\int_0^1\int_0^{2\pi}\sqrt{2}r\,d\theta\,dr=\sqrt{2}\pi$$

12. (1)

$$S=\int_0^{\sqrt{2}}\int_0^{\sqrt{2}}\sqrt{1+4x^2+4y^2}\,dx\,dy$$

令 $(x,y)=(r\cos\theta, r\sin\theta)$.

$$S=\int_0^{2\pi}\int_0^{\sqrt{2}}\sqrt{4r^2+1}\,r\,dr\,d\theta$$

$$=\frac{13}{3}\pi.$$

(2) $$S=2\int_0^{\frac{\pi}{6}}\int_0^{2\pi}\sin\varphi\,d\theta\,d\varphi$$

$$=(4-2\sqrt{3})\pi$$

(3) $$z'_x=\frac{6x}{2\sqrt{3(x^2+y^2)}}=\frac{\sqrt{3}x}{\sqrt{x^2+y^2}}$$

$$z'_y=\frac{\sqrt{3}y}{\sqrt{x^2+y^2}}$$

$$S=\iint_D\sqrt{1+z_x'^2+z_y'^2}\,dx\,dy$$

$$=2\iint_D dx\,dy$$

联立 $z^2=3(x^2+y^2)$, $x+y+z=2$, 有

$$\frac{1}{6}\left(x+1-\frac{y}{2}\right)^2+\frac{1}{2}\left(\frac{y}{2}+1\right)^2=1.$$

令 $$\begin{cases}\sqrt{6}r\cos\theta=x+1-\frac{y}{2}\\ \sqrt{2}r\sin\theta=\frac{y}{2}+1\end{cases}$$

$$\Rightarrow\begin{cases}x=\sqrt{6}r\cos\theta+\sqrt{2}r\sin\theta-2\\ y=2\sqrt{2}r\sin\theta-2\end{cases}$$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|=4\sqrt{3}r$$

$$S=2\int_0^1 dr\int_0^{2\pi}4\sqrt{3}r\,d\theta$$

$$=8\sqrt{3}\pi$$

(4) $z=\sqrt{1-x^2}$, $z'_x=-\frac{x}{\sqrt{1-x^2}}$

$$S=48\int_0^{\frac{\sqrt{2}}{2}}dx\int_0^x\frac{1}{\sqrt{1-x^2}}dy$$

$$=48\int_0^{\frac{\sqrt{2}}{2}}\frac{x}{\sqrt{1-x^2}}dx$$

$$=-48\sqrt{1-x^2}\Big|_0^{\frac{\sqrt{2}}{2}}$$

$$=48-24\sqrt{2}$$

(5) $z=-\frac{1}{c}(ax+by+d)$

$$\sqrt{1+z_x'^2+z_y'^2}=\left(1+\frac{a^2}{c^2}+\frac{b^2}{c^2}\right)^{\frac{1}{2}}$$

$$=\frac{1}{|c|}\sqrt{a^2+b^2+c^2}$$

$$S=\iint_D\frac{1}{|c|}\sqrt{a^2+b^2+c^2}\,dx\,dy$$

$$=\frac{\pi}{|c|}\sqrt{a^2+b^2+c^2}$$

13. 令 $P=(r\cos\theta, r\sin\theta, 1-r)$

$$P'_r=(\cos\theta,\sin\theta,-1)$$

$$P'_\theta=(-r\sin\theta, r\cos\theta, 0)$$

$$E=2,\ F=0,\ G=r^2$$

$$S=\int_0^1\int_0^{2\pi}\sqrt{2}r\,d\theta\,dr=\sqrt{2}\pi$$

14. (1)

令 $\begin{cases} x=\sin\varphi\cos\theta \\ y=\sin\varphi\sin\theta \\ z=\cos\varphi \end{cases}$

$\varphi\in[0,\frac{\pi}{2}]$, $\theta\in[0,2\pi]$

记 $\vec{r}=(x,y,z)$

$\vec{r}'_\varphi=(\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)$

$\vec{r}'_\theta=(-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0)$

$E=\vec{r}'_\varphi\cdot\vec{r}'_\varphi=1$

$F=\vec{r}'_\varphi\cdot\vec{r}'_\theta=0$

$G=\vec{r}'_\theta\cdot\vec{r}'_\theta=\sin^2\varphi$

$I=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\cos^3\varphi\sin\varphi\, d\varphi\, d\theta$

$=\int_0^{2\pi}d\theta\left(-\frac{1}{4}\cos^4\varphi\right)\Big|_0^{\frac{\pi}{2}}$

$=\frac{1}{4}\int_0^{2\pi}d\theta$

$=\frac{\pi}{2}$

(2) 令 $\begin{cases} x=\cos\theta \\ y=\sin\theta \\ z=z \end{cases}$

$P'_\theta=(-\sin\theta, \cos\theta, 0)$

$P'_z=(0, 0, 1)$

$EG-F^2=1$

$I=\int_0^{2\pi}d\theta\int_0^1\cos^2\theta\sin^2\theta\, dz$

$=\frac{1}{4}\int_0^{2\pi}\sin^2 2\theta\, d\theta$

$=\frac{1}{8}\int_0^{2\pi}(1-\cos4\theta)\, d\theta$

$=\frac{\pi}{4}$

(3) 令 $\begin{cases} x=\sin\varphi\cos\theta \\ y=\sin\varphi\sin\theta \\ z=\cos\varphi \end{cases}$

$P'_\varphi=(\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)$

$P'_\theta=(-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0)$

$E=1$, $F=0$, $G=\sin^2\varphi$

$I=\int_0^{2\pi}d\theta\int_0^{\pi}\sin^5\varphi\sin^2\theta\cos^2\theta\, d\varphi$

$=\frac{4}{15}\int_0^{2\pi}\sin^2 2\theta\, d\theta$

$=\frac{4}{15}\pi$

(4) 令 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=r \end{cases}$

$P'_r=(\cos\theta, \sin\theta, 1)$

$P'_\theta=(-r\sin\theta, r\cos\theta, 0)$

$E=2$, $F=0$, $G=r^2$

$I=\int_0^{2\pi}d\theta\int_1^{1+h}r^3\sin\theta\cos\theta\cdot\sqrt{2}r\, dr$

$=0$

(5) 记 $\vec{r}=(x,y,z)$

$\vec{r}'_u=(\cos v, \sin v, 0)$

$\vec{r}'_v=(-u\sin v, u\cos v, 1)$

$E=1$, $F=0$, $G=u^2+1$

$I=\int_0^1\int_0^{2\pi}v^2\sqrt{u^2+1}\, dv\, du$

$=\frac{8}{3}\pi^3\int_0^1\sqrt{u^2+1}\, du$

$=\frac{4}{3}\pi^3[\sqrt{2}+\ln(\sqrt{2}+1)]$

14.(1)

令 $\begin{cases} x = \sin\varphi\cos\theta \\ y = \sin\varphi\sin\theta \\ z = \cos\varphi \end{cases}$

$\varphi \in [0, \frac{\pi}{2}]$, $\theta \in [0, 2\pi]$

记 $\vec{r} = (x, y, z)$

$\vec{r}'_\varphi = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)$

$\vec{r}'_\theta = (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0)$

$E = \vec{r}'_\varphi \cdot \vec{r}'_\varphi = 1$

$F = \vec{r}'_\varphi \cdot \vec{r}'_\theta = 0$

$G = \vec{r}'_\theta \cdot \vec{r}'_\theta = \sin^2\varphi$

$I = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \cos^3\varphi \sin\varphi \, d\varphi \, d\theta$

$= \int_0^{2\pi} d\theta \left(-\frac{1}{4}\cos^4\varphi\right)\Big|_0^{\frac{\pi}{2}}$

$= \frac{1}{4}\int_0^{2\pi} d\theta$

$= \frac{\pi}{2}$

(2) 令 $\begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = z \end{cases}$

$r'_\theta = (-\sin\theta, \cos\theta, 0)$

$r'_z = (0, 0, 1)$

$EG - F^2 = 1$

$I = \int_0^{2\pi} d\theta \int_0^1 \cos^2\theta \sin^2\theta \, dz$

$= \frac{1}{4}\int_0^{2\pi} \sin^2 2\theta \, d\theta$

$= \frac{1}{8}\int_0^{2\pi} (1 - \cos 4\theta) \, d\theta$

$= \frac{\pi}{4}$

(3) 令 $\begin{cases} x = \sin\varphi\cos\theta \\ y = \sin\varphi\sin\theta \\ z = \cos\varphi \end{cases}$

$r'_\varphi = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)$

$r'_\theta = (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0)$

$E = 1$, $F = 0$, $G = \sin^2\varphi$

$I = \int_0^{2\pi} d\theta \int_0^{\pi} \sin^5\varphi \sin^2\theta \cos^2\theta \, d\varphi$

$= \frac{4}{15}\int_0^{2\pi} \sin^2 2\theta \, d\theta$

$= \frac{4}{15}\pi$

(4) 令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r \end{cases}$

$r'_r = (\cos\theta, \sin\theta, 1)$

$r'_\theta = (-r\sin\theta, r\cos\theta, 0)$

$E = 2$, $F = 0$, $G = r^2$

$I = \int_0^{2\pi} d\theta \int_1^{1+h} r^3 \sin\theta\cos\theta \cdot \sqrt{2} r \, dr$

$= 0$

(5) 记 $\vec{r} = (x, y, z)$

$\vec{r}'_u = (\cos v, \sin v, 0)$

$\vec{r}'_v = (-u\sin v, u\cos v, 1)$

$E = 1$, $F = 0$, $G = u^2 + 1$

$I = \int_0^1 \int_0^{2\pi} v^2 \sqrt{u^2+1} \, dv \, du$

$= \frac{8}{3}\pi^3 \int_0^1 \sqrt{u^2+1} \, du$

$= \frac{4}{3}\pi^3 [\sqrt{2} + \ln(\sqrt{2}+1)]$

15. 令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r^2 \end{cases}$

满足 $r^2 \le r\sin\theta$，即 $0 \le r \le \sin\theta$.

$P'_r = (\cos\theta, \sin\theta, 2r)$

$P'_\theta = (-r\sin\theta, r\cos\theta, 0)$

$E = 4r^2+1,\ F = 0,\ G = r^2$

$\sqrt{EG-F^2} = r\sqrt{4r^2+1}$

$I = \int_0^\pi d\theta \int_0^{\sin\theta} \sqrt{1+4r^2} \cdot r\sqrt{4r^2+1}\, dr$

$= \int_0^\pi (\sin^4\theta + \frac{1}{2}\sin^2\theta)\, d\theta$

$= \frac{5}{8}\pi$

16. 设 $\begin{cases} x = \sin\varphi\cos\theta \\ y = \sin\varphi\sin\theta \\ z = \cos\varphi \end{cases}$

$\varphi \in [0, \alpha],\ \theta \in [0, 2\pi]$

设质心坐标为 z_0

$I_1 \triangleq \int_0^\alpha \int_0^{2\pi} \cos\varphi \sin\varphi\, d\theta\, d\varphi$

$= \pi \int_0^\alpha \sin 2\varphi\, d\varphi$

$= \frac{\pi}{2}(1-\cos 2\alpha)$

$I_2 \triangleq \int_0^\alpha \int_0^{2\pi} \sin\varphi\, d\theta\, d\varphi$

$= 2\pi \int_0^\alpha \sin\varphi\, d\varphi$

$= 2\pi(1-\cos\alpha)$

$z_0 = I_1/I_2 = \frac{1-\cos 2\alpha}{4(1-\cos\alpha)} = \cos^2\frac{\alpha}{2}$

故质心坐标为 $(0, 0, \cos^2\frac{\alpha}{2})$

17. (1) 令 $\begin{cases} x = s \\ y = -s+t \\ z = 12s-4t-5 \end{cases}$

$t \in [0,1],\ s \in [0,t]$.

$(A, B, C) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 12 \\ 0 & 1 & -4 \end{vmatrix} = (-8, 4, 1)$

$I = \int_0^1 dt \int_0^t [8(-s+t) + 4s]\, ds$

$= \int_0^1 dt \int_0^t (8t-4s)\, ds$

$= \int_0^1 6t^2\, dt$

$= 2$

(2) 由对称性，$I = 0$

15. 令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r^2 \end{cases}$

满足 $r^2 \le r\sin\theta$，即 $0 \le r \le \sin\theta$.

$P'_r = (\cos\theta, \sin\theta, 2r)$

$P'_\theta = (-r\sin\theta, r\cos\theta, 0)$

$E = 4r^2+1,\ F = 0,\ G = r^2$

$\sqrt{EG-F^2} = r\sqrt{4r^2+1}$

$$I = \int_0^\pi d\theta \int_0^{\sin\theta} \sqrt{1+4r^2} \cdot r\sqrt{4r^2+1}\, dr$$

$$= \int_0^\pi \left(\sin^4\theta + \frac{1}{2}\sin^2\theta\right) d\theta$$

$$= \frac{5}{8}\pi$$

16. 设 $\begin{cases} x = \sin\varphi\cos\theta \\ y = \sin\varphi\sin\theta \\ z = \cos\varphi \end{cases}$

$\varphi \in [0, \alpha],\ \theta \in [0, 2\pi]$

设质心坐标为 z_0

$$I_1 \triangleq \int_0^\alpha \int_0^{2\pi} \cos\varphi \sin\varphi\, d\theta\, d\varphi$$

$$= \pi \int_0^\alpha \sin 2\varphi\, d\varphi$$

$$= \frac{\pi}{2}(1-\cos 2\alpha)$$

$$I_2 \triangleq \int_0^\alpha \int_0^{2\pi} \sin\varphi\, d\theta\, d\varphi$$

$$= 2\pi \int_0^\alpha \sin\varphi\, d\varphi$$

$$= 2\pi(1-\cos\alpha)$$

$$z_0 = I_1/I_2 = \frac{1-\cos 2\alpha}{4(1-\cos\alpha)} = \cos^2\frac{\alpha}{2}$$

故质心坐标为 $(0, 0, \cos^2\frac{\alpha}{2})$

17. (1) 令 $\begin{cases} x = s \\ y = -s+t \\ z = 12s-4t-5 \end{cases}$

$t \in [0,1],\ s \in [0,t]$.

$$(A,B,C) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 12 \\ 0 & 1 & -4 \end{vmatrix} = (-8, 4, 1)$$

$$I = \int_0^1 dt \int_0^t [8(-s+t)+4s]\, ds$$

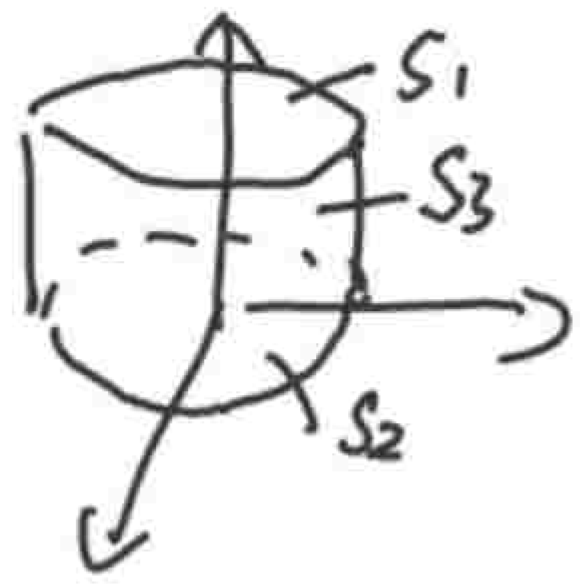
$$= \int_0^1 dt \int_0^t (8t-4s)\, ds$$

$$= \int_0^1 6t^2\, dt$$

$$= 2$$

(2) 由对称性，$I = 0$

(3)



$S_1: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 1 \end{cases}$

$\theta \in [0, 2\pi]$, $r \in [0, 1]$.

$$(A,B,C) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (0, 0, r)$$

因此 $I_1 = 0$，同理 $I_2 = 0$

$S_3: \begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = z \end{cases}$

$\theta \in [0, 2\pi]$, $z \in [0, 1]$

$$(A,B,C) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta, \sin\theta, 0)$$

$$I_3 = \int_0^1 \int_0^{2\pi} (\cos^2\theta \sin\theta + \sin^2\theta \cdot z)\, d\theta\, dz$$

$$= \frac{1}{2}\pi$$

$$I = I_1 + I_2 + I_3 = \frac{1}{2}\pi$$

18.(1) 由Gauss公式，

$$I = \iiint 3(x^2+y^2+z^2)\, dx\,dy\,dz$$

令 $\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$

$r \in [0,1]$, $\varphi \in [0, \pi]$, $\theta \in [0, 2\pi]$.

$$I = \int_0^{\pi}\int_0^{2\pi}\int_0^1 3r^4 \sin\varphi\, dr\, d\theta\, d\varphi$$

$$= \frac{12}{5}\pi$$

(2) $S: \begin{cases} x = \sin\varphi\cos\theta \\ y = \sin\varphi\sin\theta \\ z = \cos\varphi \end{cases}$ $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$.

$$(A,B,C) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= (\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi)$$

$$I = \int_0^{2\pi}\int_0^{\pi} \sin^3\varphi \cos^2\theta\, d\varphi\, d\theta$$

$$= \frac{4}{3}\int_0^{2\pi} \cos^2\theta\, d\theta$$

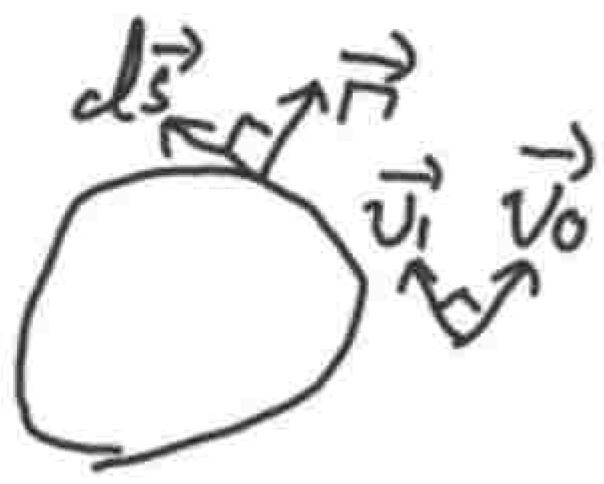
$$= \frac{4}{3}\pi$$

19. 不妨设 $|\vec{v_0}| = 1$，记 $\vec{v_0}$ 逆时针旋转 $\frac{\pi}{2}$ 后的向量为 $\vec{v_1}$.

$$\oint_\Gamma \cos(\vec{v_0}, \vec{n})\, ds = \oint_\Gamma \vec{v_1} \cdot d\vec{s}$$

$$= \iint_D 0\, dx\,dy$$

$$= 0$$



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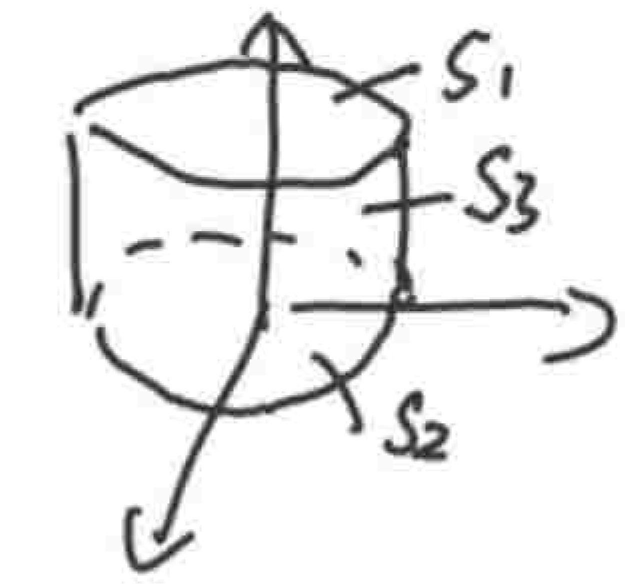


S_1: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 1 \end{cases}$

$\theta \in [0, 2\pi]$, $r \in [0, 1]$.

$$(A, B, C) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (0, 0, r)$$

因此 $I_1 = 0$，同理 $I_2 = 0$

S_3: $\begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = z \end{cases}$

$\theta \in [0, 2\pi]$, $z \in [0, 1]$

$$(A, B, C) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta, \sin\theta, 0)$$

$$I_3 = \int_0^1 \int_0^{2\pi} (\cos^2\theta \sin\theta + \sin^2\theta \cdot z)\, d\theta\, dz$$

$$= \frac{1}{2}\pi$$

$$I = I_1 + I_2 + I_3 = \frac{1}{2}\pi$$

18. (1) 由Gauss公式，

$$I = \iiint 3(x^2 + y^2 + z^2)\, dx\,dy\,dz$$

令 $\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$

$r \in [0, 1]$, $\varphi \in [0, \pi]$, $\theta \in [0, 2\pi]$.

$$I = \int_0^{\pi} \int_0^{2\pi} \int_0^1 3r^4 \sin\varphi\, dr\, d\theta\, d\varphi$$

$$= \frac{12}{5}\pi$$

(2) S: $\begin{cases} x = \sin\varphi\cos\theta \\ y = \sin\varphi\sin\theta \\ z = \cos\varphi \end{cases}$ $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$.

$$(A, B, C) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= (\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi)$$

$$I = \int_0^{2\pi} \int_0^{\pi} \sin^3\varphi \cos^2\theta\, d\varphi\, d\theta$$

$$= \frac{4}{3} \int_0^{2\pi} \cos^2\theta\, d\theta$$

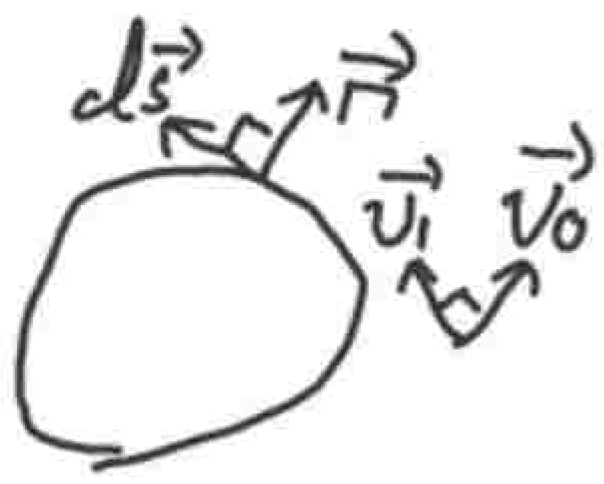
$$= \frac{4}{3}\pi$$

19. 不妨设 $|\vec{v_0}| = 1$，记 $\vec{v_0}$ 逆时针旋转 $\frac{\pi}{2}$ 后的向量为 $\vec{v_1}$.

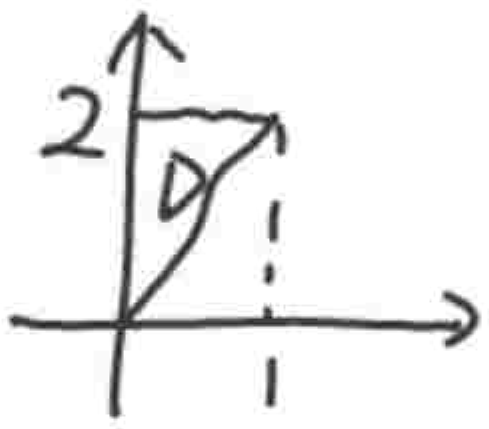
$$\oint_\Gamma \cos(\vec{v_0}, \vec{n})\, ds = \oint_\Gamma \vec{v_1} \cdot d\vec{s}$$

$$= \iint_D 0\, dx\,dy$$

$$= 0$$

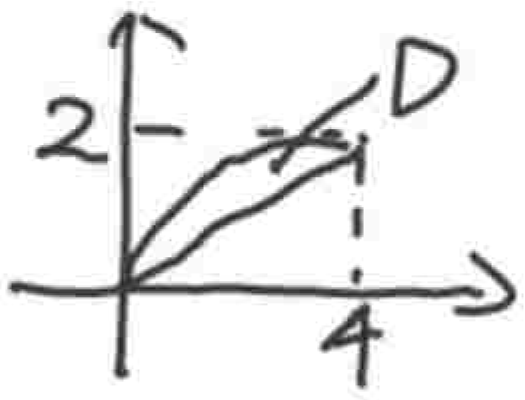


20. (1)



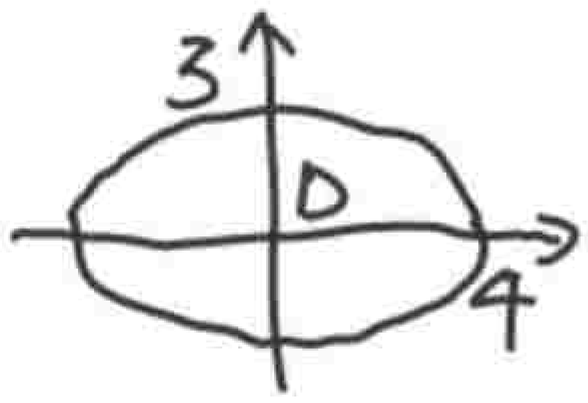
$$I=\iint_D(-4x^2)\,dxdy$$
$$=\int_0^2 dy\int_0^{\frac{y}{2}}(-4x^2)\,dx$$
$$=-\frac{1}{6}\int_0^2 y^3\,dy$$
$$=-\frac{2}{3}$$

(2)



$$I=\iint_D(-2x)\,dxdy$$
$$=\int_0^4 dx\int_{\frac{x}{2}}^{\sqrt{x}}(-2x)\,dy$$
$$=-2\int_0^4\left(x^{\frac{3}{2}}-\frac{1}{2}x^2\right)dx$$
$$=-\frac{64}{15}$$

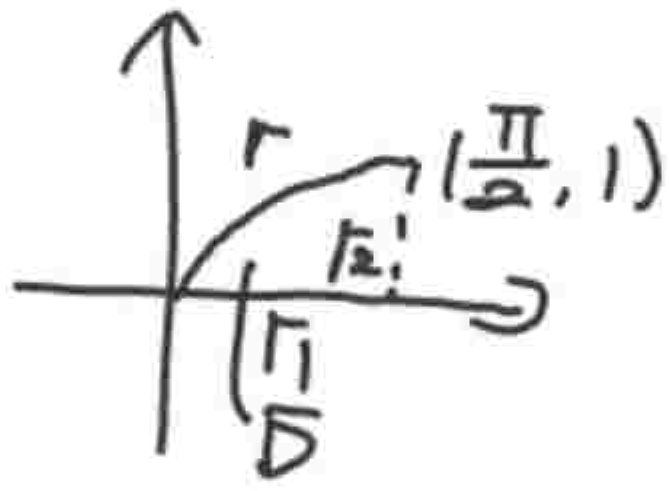
(3)



$$I=\iint_D(4x-4x)\,dxdy=0$$

(4) $$I=\iint_D(y^2+x^2)\,dxdy$$
$$=\int_2^4 dr\int_0^{2\pi}r^2\cdot r\,d\theta$$
$$=120\pi$$

(5)



$$\int_{\vec{\Gamma}_1}(2x^2y-y^2\cos x)\,dx+(1-2y\sin x+3x^2y^2)\,dy=0$$

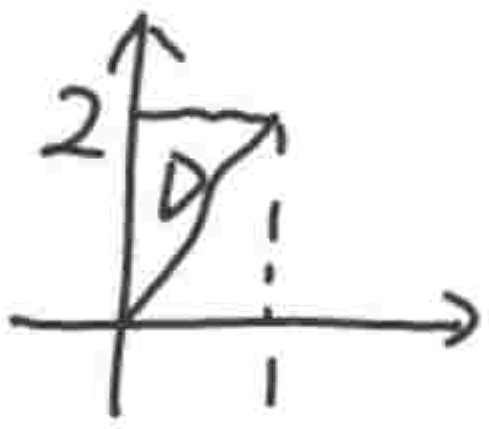
$$\int_{\vec{\Gamma}_2}(2x^2y-y^2\cos x)\,dx+(1-2y\sin x+3x^2y^2)\,dy$$
$$=\int_0^1\left(1-2y+\frac{3\pi^2}{4}y^2\right)dy$$
$$=\left(y-y^2+\frac{\pi^2}{4}y^3\right)\Big|_0^1$$
$$=\frac{\pi^2}{4}$$

由Green公式，

$$\int_{\vec{\Gamma}\cup\vec{\Gamma}_1\cup\vec{\Gamma}_2}\cdots=\iint_D(-2y\cos x+6xy^2-2x^2+2y\cos x)\,dxdy$$
$$=\int_0^1\int_{\frac{\pi}{2}y^2}^{\frac{\pi}{2}}(6xy^2-2x^2)\,dxdy$$
$$=\int_0^1\left[3y^2\left(\frac{\pi^2}{4}-\frac{\pi}{4}y^4\right)-\frac{2}{3}\left(\frac{\pi^3}{8}-\frac{\pi^3}{8}y^6\right)\right]dy$$
$$=\frac{\pi^2}{4}-\frac{3}{28}\pi^2-\frac{\pi^3}{12}+\frac{\pi^3}{84}$$
$$=\frac{1}{7}\pi^2-\frac{1}{14}\pi^3$$

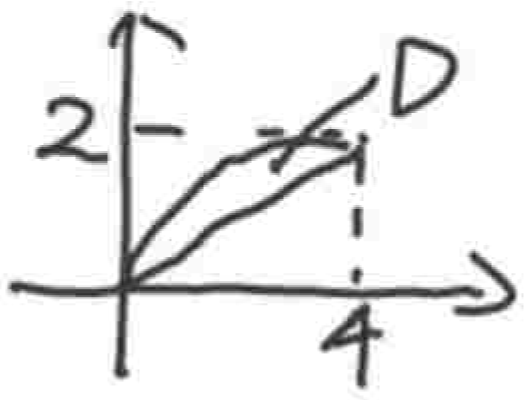
$$I=-\left(\frac{\pi^2}{7}-\frac{\pi^3}{14}\right)+\frac{\pi^2}{4}=\frac{1}{28}\pi^2(2\pi+3)$$

20. (1)



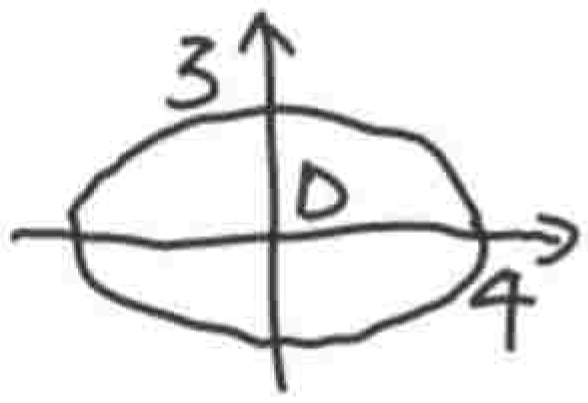
$$I=\iint_D(-4x^2)\,dxdy$$
$$=\int_0^2 dy\int_0^{\frac{y}{2}}(-4x^2)\,dx$$
$$=-\frac{1}{6}\int_0^2 y^3\,dy$$
$$=-\frac{2}{3}$$

(2)



$$I=\iint_D(-2x)\,dxdy$$
$$=\int_0^4 dx\int_{\frac{x}{2}}^{\sqrt{x}}(-2x)\,dy$$
$$=-2\int_0^4\left(x^{\frac{3}{2}}-\frac{1}{2}x^2\right)dx$$
$$=-\frac{64}{15}$$

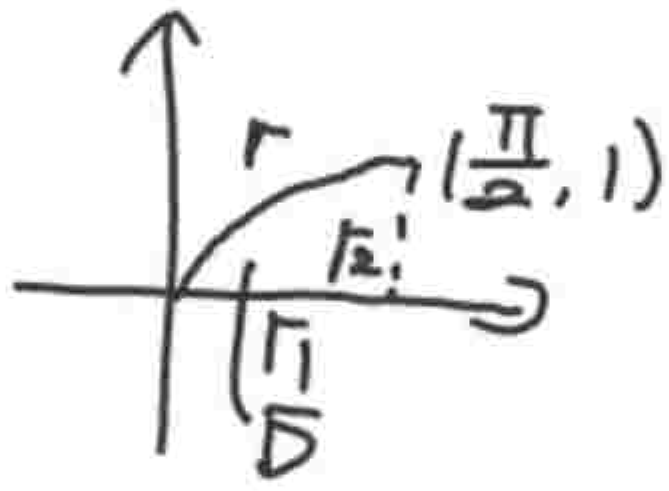
(3)



$$I=\iint_D(4x-4x)\,dxdy=0$$

(4) $$I=\iint_D(y^2+x^2)\,dxdy$$
$$=\int_2^4 dr\int_0^{2\pi}r^2\cdot r\,d\theta$$
$$=120\pi$$

(5)



$$\int_{\vec{\Gamma}_1}(2x^2y-y^2\cos x)\,dx+(1-2y\sin x+3x^2y^2)\,dy=0$$
$$\int_{\vec{\Gamma}_2}(2x^2y-y^2\cos x)\,dx+(1-2y\sin x+3x^2y^2)\,dy$$
$$=\int_0^1\left(1-2y+\frac{3\pi^2}{4}y^2\right)dy$$
$$=\left(y-y^2+\frac{\pi^2}{4}y^3\right)\Big|_0^1$$
$$=\frac{\pi^2}{4}$$

由Green公式，

$$\int_{\vec{\Gamma}\cup\vec{\Gamma}_1\cup\vec{\Gamma}_2}\cdots=\iint_{\bar{D}}(-2y\cos x+6xy^2-2x^2+2y\cos x)\,dxdy$$
$$=\int_0^1\int_{\frac{\pi}{2}y^2}^{\frac{\pi}{2}}(6xy^2-2x^2)\,dxdy$$
$$=\int_0^1\left[3y^2\left(\frac{\pi^2}{4}-\frac{\pi^2}{4}y^4\right)-\frac{2}{3}\left(\frac{\pi^3}{8}-\frac{\pi^3}{8}y^6\right)\right]dy$$
$$=\frac{\pi^2}{4}-\frac{3}{28}\pi^2-\frac{\pi^3}{12}+\frac{\pi^3}{84}$$
$$=\frac{1}{7}\pi^2-\frac{1}{14}\pi^3$$

$$I=-\left(\frac{\pi^2}{7}-\frac{\pi^3}{14}\right)+\frac{\pi^2}{4}=\frac{1}{28}\pi^2(2\pi+3)$$

(6) 

$$\int_{\vec{\Gamma}_1}\cdots = \int_0^{\pi}(e^x\sin y - x - y)\,dx$$
$$= \int_0^{\pi}(-x)\,dx$$
$$= -\frac{\pi^2}{2}$$

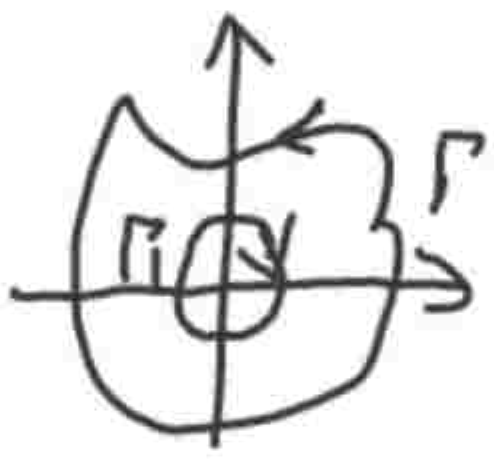
$$\int_{\vec{\Gamma}\cup\vec{\Gamma}_1}\cdots = \iint_D \left(e^x\cos y - 1 - (e^x\cos y - 1)\right)dx\,dy$$
$$= 0$$

故 $I = -\int_{\vec{\Gamma}}\cdots = -\frac{\pi^2}{2}$

21. 由Green公式，

$$I = \iint_D (a_2 - b_1)\,dx\,dy$$
$$= (a_2 - b_1)A$$

22. 取R充分小，使 $\Gamma_1 = \{(x,y)\mid x^2+y^2=R^2\}$ 落在Γ内部.



由Green公式，$\int_{\vec{\Gamma}\cup\vec{\Gamma}_1}\cdots = \iint_D 0\,dx\,dy = 0$.

$$\int_{\vec{\Gamma}_1}\cdots = \int_{2\pi}^{0}\frac{1}{R^2}\left[(aR\cos\theta - bR\sin\theta)(-R\sin\theta) + (bR\cos\theta + aR\sin\theta)R\cos\theta\right]d\theta$$
$$= \int_{2\pi}^{0} b\,d\theta$$
$$= -2b\pi.$$

故 $\int_{\vec{\Gamma}}\cdots = 2b\pi$.

(6) 

$$\int_{\vec{\Gamma}_1}\cdots = \int_0^{\pi}(e^x\sin y - x - y)\,dx$$
$$= \int_0^{\pi}(-x)\,dx$$
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$$\int_{\vec{\Gamma}\cup\vec{\Gamma}_1}\cdots = \iint_D \left(e^x\cos y - 1 - (e^x\cos y - 1)\right)dx\,dy$$
$$= 0$$

故 $I = -\int_{\vec{\Gamma}}\cdots = -\frac{\pi^2}{2}$

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$$I = \iint_D (a_2 - b_1)\,dx\,dy$$
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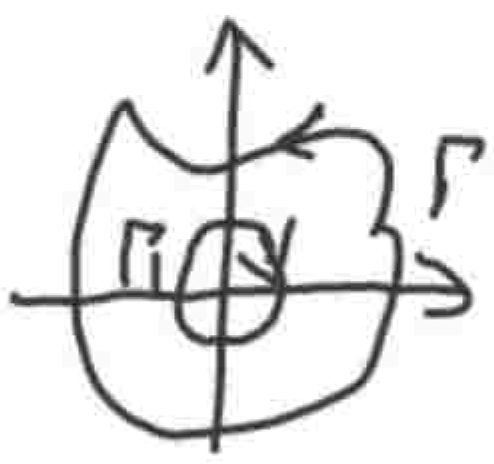

由Green公式, $\int_{\vec{\Gamma}\cup\vec{\Gamma}_1}\cdots = \iint_D 0\,dx\,dy = 0$.

$$\int_{\vec{\Gamma}_1}\cdots = \int_{2\pi}^{0}\frac{1}{R^2}\left[(aR\cos\theta - bR\sin\theta)(-R\sin\theta) + (bR\cos\theta + aR\sin\theta)R\cos\theta\right]d\theta$$
$$= \int_{2\pi}^{0} b\,d\theta$$
$$= -2b\pi.$$

故 $\int_{\vec{\Gamma}}\cdots = 2b\pi$.

23. $A=\iint_D dxdy$

$=\frac{1}{2}\int_{\vec{P}} -ydx+xdy$

$=\frac{1}{2}\int_{\vec{P}}[-r\sin\theta(-r\sin\theta)+r\cos\theta(r\cos\theta)]d\theta$

$=\frac{1}{2}\int_{\vec{P}} r^2 d\theta$

$A=\frac{1}{2}\int_0^{\frac{\pi}{2}} 9\sin^2 2\theta d\theta$

$=\frac{9}{8}\pi$

24. 由Green公式，

$I=\iint_D(-3x^2-3y^2)dxdy$

$=-3\int_0^{2\pi}d\theta\int_0^1 r^3 dr$

$=-\frac{3}{2}\pi$

25. 由Green公式，

$I=\iint_D \frac{y}{x}dxdy$

$=\int_1^3 dy\int_{e^y}^{e^{y^2}}\frac{y}{x}dx$

$=\int_1^3(y^3-y^2)dy$

$=\frac{34}{3}$

26. 由Green公式，

$I=\oint_{\partial D} adx+bdy$

$=\iint_D 0dxdy$

$=0$

27. 取$P=0, Q=x$

由Green公式，

$S=\iint_{\bar{D}} dxdy=\int_{\vec{P}} xdy$

$\int_{\vec{P}} xdy=\int_0^{2\pi} a\cos^3 t\, d a\sin^3 t$

$=3a^2\int_0^{2\pi}\sin^2 t\cos^4 t\, dt$

$=\frac{3}{8}a^2\pi$

28. (1) $I=\iiint_D 2(x+y+z)dxdydz$

$=2\int_0^1 dx\int_0^1 dy\int_0^1(x+y+z)dz$

$=2\int_0^1 dx\int_0^1(x+y+\frac{1}{2})dy$

$=2\int_0^1(x+1)dx$

$=3$

(2) $I=\iiint_D 3dxdydz$

令 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases}$

$z\in[0,4], r\in[0,\sqrt{4-z}], \theta\in[0,2\pi]$

$I=\int_0^4 dz\int_0^{\sqrt{4-z}}dr\int_0^{2\pi}3rd\theta$

$=6\pi\int_0^4 dz\int_0^{\sqrt{4-z}} rdr$

$=3\pi\int_0^4(4-z)dz$

$=24\pi$

(3) $I=\iiint_D 4dxdydz$

$=4(\frac{4}{3}\pi 2^3-\frac{4}{3}\pi 1^3)$

$=\frac{112}{3}\pi$

23. $A=\iint_D dxdy$

$=\frac{1}{2}\int_{\vec{P}} -ydx+xdy$

$=\frac{1}{2}\int_{\vec{P}}[-r\sin\theta(-r\sin\theta)+r\cos\theta(r\cos\theta)]d\theta$

$=\frac{1}{2}\int_{\vec{P}} r^2 d\theta$

$A=\frac{1}{2}\int_0^{\frac{\pi}{2}} 9\sin^2 2\theta d\theta$

$=\frac{9}{8}\pi$

24. 由Green公式，

$I=\iint_D(-3x^2-3y^2)dxdy$

$=-3\int_0^{2\pi}d\theta\int_0^1 r^3 dr$

$=-\frac{3}{2}\pi$

25. 由Green公式，

$I=\iint_D \frac{y}{x}dxdy$

$=\int_1^3 dy\int_{e^y}^{e^{y^2}}\frac{y}{x}dx$

$=\int_1^3(y^3-y^2)dy$

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$S=\iint_{\bar{D}} dxdy=\int_{\vec{P}} xdy$

$\int_{\vec{P}} xdy=\int_0^{2\pi} a\cos^3 t\, d a\sin^3 t$

$=3a^2\int_0^{2\pi}\sin^2 t\cos^4 t\,dt$

$=\frac{3}{8}a^2\pi$

28. (1) $I=\iiint_D 2(x+y+z)dxdydz$

$=2\int_0^1 dx\int_0^1 dy\int_0^1(x+y+z)dz$

$=2\int_0^1 dx\int_0^1(x+y+\frac{1}{2})dy$

$=2\int_0^1(x+1)dx$

$=3$

(2) $I=\iiint_D 3dxdydz$

令 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases}$

$z\in[0,4], r\in[0,\sqrt{4-z}], \theta\in[0,2\pi]$

$I=\int_0^4 dz\int_0^{\sqrt{4-z}}dr\int_0^{2\pi}3r d\theta$

$=6\pi\int_0^4 dz\int_0^{\sqrt{4-z}} r dr$

$=3\pi\int_0^4(4-z)dz$

$=24\pi$

(3) $I=\iiint_D 4dxdydz$

$=4(\frac{4}{3}\pi 2^3-\frac{4}{3}\pi 1^3)$

$=\frac{112}{3}\pi$

(4) 由Gauss公式,

$$I=\iiint_D 3z^2\,dxdydz$$

$$=\int_0^{\pi}\int_0^{2\pi}\int_0^1 3r^2\cos^2\varphi\cdot r^2\sin\varphi\,drd\theta d\varphi$$

$$=\frac{4}{5}\pi$$

(5) $I=\iiint_D(2-x)\,dxdydz$

$$=\int_0^3 dx\int_{-2}^{2}dy\int_0^{4-y^2}(2-x)\,dz$$

$$=\int_0^3 dx\int_{-2}^{2}(2-x)(4-y^2)\,dy$$

$$=\frac{32}{3}\int_0^3(2-x)\,dx=16$$

29. 不妨设 $\vec{r_0}=(\cos\alpha,\cos\beta,\cos\gamma)$, $|\vec{r_0}|=1$

$$I=\iint_S\vec{r_0}\cdot\vec{n}\,dS$$

$$=\iint_{\vec{S}}\cos\alpha\,dydz+\cos\beta\,dzdx+\cos\gamma\,dxdy$$

由Gauss公式之得 $I=0$

30. $\iint_{\vec{S}}\frac{\partial f}{\partial\vec{n}}dS=\iint_{\vec{S}}\nabla f\cdot\vec{n}\,dS$

$$=\iint_S\nabla f\cdot d\vec{S}$$

$$=\iiint_D\Delta f\,dV$$

31. (1) $I=\iint_{\vec{S}}2\,dydz-2\,dxdy$

令 $\begin{cases}x=\frac{1}{\sqrt{2}}r\cos\theta-1\\ y=r\sin\theta\\ z=\frac{1}{\sqrt{2}}r\cos\theta+1\end{cases}$

$r\in[0,\sqrt{6}]$, $\theta\in[0,2\pi]$

$$(A,B,C)=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{1}{\sqrt{2}}\cos\theta&\sin\theta&\frac{1}{\sqrt{2}}\cos\theta\\ -\frac{1}{\sqrt{2}}r\sin\theta&r\cos\theta&-\frac{1}{\sqrt{2}}r\sin\theta\end{vmatrix}$$

$$=\left(-\frac{1}{\sqrt{2}}r,0,\frac{1}{\sqrt{2}}r\right)$$

$$I=\int_0^{\sqrt{6}}dr\int_0^{2\pi}(-\sqrt{2}r-\sqrt{2}r)\,d\theta$$

$$=-4\sqrt{2}\pi\int_0^{\sqrt{6}}r\,dr$$

$$=-12\sqrt{2}\pi$$

(4) 由Gauss公式,

$$I=\iiint_D 3z^2\,dxdydz$$

$$=\int_0^{\pi}\int_0^{2\pi}\int_0^1 3r^2\cos^2\varphi\cdot r^2\sin\varphi\,dr\,d\theta\,d\varphi$$

$$=\frac{4}{5}\pi$$

(5) $$I=\iiint_D(2-x)\,dxdydz$$

$$=\int_0^3 dx\int_{-2}^{2}dy\int_0^{4-y^2}(2-x)\,dz$$

$$=\int_0^3 dx\int_{-2}^{2}(2-x)(4-y^2)\,dy$$

$$=\frac{32}{3}\int_0^3(2-x)\,dx=16$$

29. 不妨设 $\vec{r}_0=(\cos\alpha,\cos\beta,\cos\gamma)$, $|\vec{r}_0|=1$

$$I=\iint_S \vec{r}_0\cdot\vec{n}\,dS$$

$$=\iint_{\vec{S}}\cos\alpha\,dydz+\cos\beta\,dzdx+\cos\gamma\,dxdy$$

由Gauss公式之得 $I=0$

30. $$\iint_{\vec{S}}\frac{\partial f}{\partial\vec{n}}\,dS=\iint_{\vec{S}}\nabla f\cdot\vec{n}\,dS$$

$$=\iint_S\nabla f\cdot d\vec{S}$$

$$=\iiint_D\Delta f\,dV$$

31. (1) $$I=\iint_{\vec{S}}2\,dydz-2\,dxdy$$

令 $$\begin{cases}x=\frac{1}{\sqrt2}r\cos\theta-1\\ y=r\sin\theta\\ z=\frac{1}{\sqrt2}r\cos\theta+1\end{cases}$$

$r\in[0,\sqrt6]$, $\theta\in[0,2\pi]$

$$(A,B,C)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{1}{\sqrt2}\cos\theta & \sin\theta & \frac{1}{\sqrt2}\cos\theta\\ -\frac{1}{\sqrt2}r\sin\theta & r\cos\theta & -\frac{1}{\sqrt2}r\sin\theta\end{vmatrix}$$

$$=\left(-\frac{1}{\sqrt2}r,\ 0,\ \frac{1}{\sqrt2}r\right)$$

$$I=\int_0^{\sqrt6}dr\int_0^{2\pi}(-\sqrt2 r-\sqrt2 r)\,d\theta$$

$$=-4\sqrt2\pi\int_0^{\sqrt6}r\,dr$$

$$=-12\sqrt2\pi$$

31.(2)

$$I=\iint_{\vec{S}}(-6xy-y)dydz+(-1+3y^2)dzdx+3x^2dxdy$$

令 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=2-r \end{cases}$

$r\in[0,2]$, $\theta\in[0,2\pi]$

$$(A,B,C)=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & -1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}=(r\cos\theta,\ r\sin\theta,\ r)$$

$$I=\int_0^{2\pi}\int_0^{2}[(-6r^2\sin\theta\cos\theta-r\sin\theta)r\cos\theta$$
$$+(3r^2\sin^2\theta-1)r\sin\theta+3r^3\cos^2\theta]drd\theta$$

$$=\int_0^{2\pi}[-24\sin\theta\cos^2\theta-\frac{8}{3}\sin\theta\cos\theta$$
$$+12\sin^3\theta-2\sin\theta+12\cos^2\theta]d\theta$$

$$=12\int_0^{2\pi}\cos^2\theta d\theta$$

$$=12\pi$$

(3) $I=\iint_{\vec{S}}6dxdy=-6\pi$

31.(2)

$$I = \iint_{\vec{S}} (-6xy - y)\,dy\,dz + (-1 + 3y^2)\,dz\,dx + 3x^2\,dx\,dy$$

令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 2 - r \end{cases}$

$r \in [0, 2]$, $\theta \in [0, 2\pi]$

$$(A, B, C) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & -1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (r\cos\theta, r\sin\theta, r)$$

$$I = \int_0^{2\pi}\int_0^2 [(-6r^2\sin\theta\cos\theta - r\sin\theta)r\cos\theta + (3r^2\sin^2\theta - 1)r\sin\theta + 3r^3\cos^2\theta]\,dr\,d\theta$$

$$= \int_0^{2\pi} [-24\sin\theta\cos^2\theta - \frac{8}{3}\sin\theta\cos\theta + 12\sin^3\theta - 2\sin\theta + 12\cos^2\theta]\,d\theta$$

$$= 12\int_0^{2\pi} \cos^2\theta\,d\theta$$

$$= 12\pi$$

(3) $I = \iint_{\vec{S}} 6\,dx\,dy = -6\pi$

(4) $I=\int_{\vec{\Gamma}}(2Rx-x^2)dx+(2Rx-y^2)dy+(2Rx-z^2)dz$

$$=\iint_{\vec{S}} -2R\,dz\,dx+2R\,dx\,dy$$

令 $\begin{cases} x=l\cos\theta+r \\ y=l\sin\theta \\ z=\sqrt{2(R+r)(l\cos\theta+r)} \end{cases}$

$l\in[0,r]$, $\theta\in[0,2\pi]$

$$(A,B,C)=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & \dfrac{\sqrt{R+r}\cos\theta}{\sqrt{2(l\cos\theta+r)}} \\ -l\sin\theta & l\cos\theta & \dfrac{-\sqrt{R+r}\,l\sin\theta}{\sqrt{2(l\cos\theta+r)}} \end{vmatrix}$$

$$=\left(\frac{-\sqrt{R+r}\,l}{\sqrt{2(l\cos\theta+r)}},\ 0,\ l\right)$$

$$I=-\int_0^{2\pi}d\theta\int_0^{r}2Rl\,dl$$

$$=-2\pi Rr^2$$

(4) $I=\int_{\vec{\Gamma}}(2Rx-x^2)dx+(2Rx-y^2)dy+(2Rx-z^2)dz$

$$=\iint_{\vec{S}} -2R\,dz\,dx+2R\,dx\,dy$$

令 $\begin{cases} x=l\cos\theta+r \\ y=l\sin\theta \\ z=\sqrt{2(R-r)(l\cos\theta+r)} \end{cases}$

$l\in[0,r]$, $\theta\in[0,2\pi]$

$$(A,B,C)=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & \dfrac{\sqrt{R-r}\cos\theta}{\sqrt{2(l\cos\theta+r)}} \\ -l\sin\theta & l\cos\theta & \dfrac{-\sqrt{R-r}\,l\sin\theta}{\sqrt{2(l\cos\theta+r)}} \end{vmatrix}$$

$$=\left(\frac{-\sqrt{R-r}\,l}{\sqrt{2(l\cos\theta+r)}},\ 0,\ l\right)$$

$$I=-\int_0^{2\pi}d\theta\int_0^{r}2Rl\,dl$$

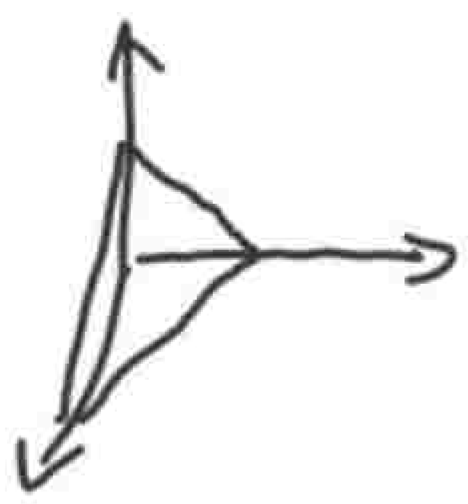
$$=-2\pi Rr^2$$

(5) $I=\iint_{\vec{S}} 3dydz+dzdx+3dxdy$

(a) $I=\iint_{\vec{S}} 3dxdy=3\pi$

注：该题未指明Γ的方向

(b) 令 $\begin{cases} x=u \\ y=v-u \\ z=1-v \end{cases}$



$$(A,B,C)=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix}=(1,1,1)$$

$$I=\iint_S 7dudv$$

$$=7\int_0^1 dv\int_0^v du$$

$$=\frac{7}{2}$$

32. 由Stokes公式，

$I=\oint_{\vec{\Gamma}} Pdx+Qdy+Rdz$,

其中Γ为S的边界.

而闭曲面无边界，因此自然有$I=0$.

33. (1) $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$，故为保守场.

取折线 $\begin{cases} x=1+2t, \\ y=-2+6t, \end{cases}$

$t\in[0,1]$.

$$I=\int_0^1 \frac{1}{(1+2t)^2}[2(-2+6t)-6(1+2t)]\,dt$$

$$=-10\int_0^1 \frac{1}{(1+2t)^2}dt$$

$$=-\frac{10}{3}$$

(2) $rot(\vec{v})=\vec{0}$，故为保守场.

取折线 $\begin{cases} x=\pi t, \\ y=1-t, \\ z=t, \end{cases}$

$t\in[0,1]$.

$$I=\int_0^1 (\pi\sin\pi t-(1-t)^2+e^t)\,dt$$

$$=e+\frac{2}{3}$$

(3) $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$，故为保守场.

取折线 $\begin{cases} x=1+t, \\ y=1+2t, \end{cases}$

$t\in[0,1]$.

$$I=\int_0^1 \left[4(1+t)^3(1+2t)^3+\frac{1}{1+t}+6(1+t)^4(1+2t)^2-\frac{2}{1+2t}\right]dt$$

$$=431+\ln\frac{2}{3}$$

(4) $rot(\vec{v})=\vec{0}$，故为保守场.

取折线 $\begin{cases} x=1+t, \\ y=1-2t, \\ z=1+2t, \end{cases}$

$t\in[0,1]$.

$$I=\int_0^1 [(1-2t)(1+2t)-2(1+t)(1+2t)+2(1+t)(1-2t)]\,dt$$

$$=\int_0^1(-12t^2-8t+1)\,dt$$

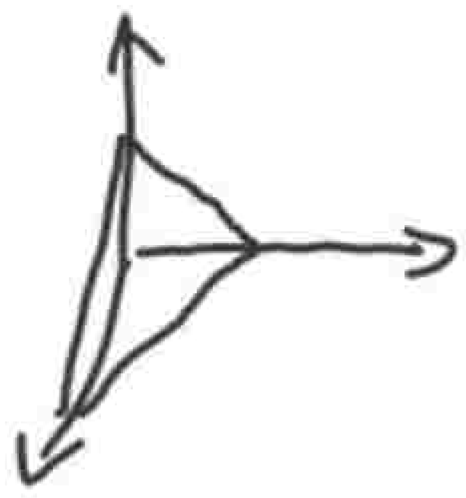
$$=-7$$

(5) $I=\iint_{S} 3dydz+dzdx+3dxdy$

(a) $I=\iint_{S} 3dxdy=3\pi$

注：该题未指明Γ的方向

(b) 令 $\begin{cases} x=u \\ y=v-u \\ z=1-v \end{cases}$



$$(A,B,C)=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix}=(1,1,1)$$

$$I=\iint_S 7dudv$$
$$=7\int_0^1 dv\int_0^v du$$
$$=\frac{7}{2}$$

32. 由Stokes公式，

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33. (1) $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$，故为保守场.

取折线 $\begin{cases} x=1+2t, \\ y=-2+6t, \end{cases}$

$t\in[0,1]$.

$$I=\int_0^1 \frac{1}{(1+2t)^2}[2(-2+6t)-6(1+2t)]\,dt$$
$$=-10\int_0^1 \frac{1}{(1+2t)^2}dt$$
$$=-\frac{10}{3}$$

(2) $rot(\vec{v})=\vec{0}$，故为保守场.

取折线 $\begin{cases} x=\pi t, \\ y=1-t, \\ z=t, \end{cases}$

$t\in[0,1]$.

$$I=\int_0^1(\pi\sin\pi t-(1-t)^2+e^t)\,dt$$
$$=e+\frac{2}{3}$$

(3) $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$，故为保守场.

取折线 $\begin{cases} x=1+t, \\ y=1+2t, \end{cases}$

$t\in[0,1]$.

$$I=\int_0^1\left[4(1+t)^3(1+2t)^3+\frac{1}{1+t}+6(1+t)^4(1+2t)^2-\frac{2}{1+2t}\right]dt$$
$$=431+\ln\frac{2}{3}$$

(4) $rot(\vec{v})=\vec{0}$，故为保守场.

取折线 $\begin{cases} x=1+t, \\ y=1-2t, \\ z=1+2t, \end{cases}$

$t\in[0,1]$.

$$I=\int_0^1[(1-2t)(1+2t)-2(1+t)(1+2t)+2(1+t)(1-2t)]\,dt$$
$$=\int_0^1(-12t^2-8t+1)\,dt$$
$$=-7$$

(5) $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$，故为保守场.

取折线 $\begin{cases} x=t, \\ y=t, \end{cases}$

$t\in[0,1]$.

$$I=\int_0^1 (t^3\cos t+2t^2\sin t-t^2e^t+t^2\sin t-2te^t)dt$$

$$=\sin 1-e$$

(6) $rot(\vec{v})=\vec{0}$，故为保守场.

取折线 $\begin{cases} x=1+t, \\ y=\pi t, \\ z=\frac{\pi}{2}+\pi t, \end{cases}$

$t\in[0,1]$.

$$I=\int_0^1 [\cos\pi t\sin(\tfrac{\pi}{2}+\pi t)-\pi(1+t)\sin\pi t\sin(\tfrac{\pi}{2}+\pi t)$$
$$+\pi(1+t)\cos\pi t\cos(\tfrac{\pi}{2}+\pi t)]dt$$

$$=\int_0^1 [\cos^2\pi t-2(\pi+\pi t)\sin\pi t\cos\pi t]dt$$

$$=\frac{1}{\pi}\int_0^\pi [\cos^2 x-(\pi+x)\sin 2x]dx \quad (x=\pi t)$$

$$=1$$

34. (1) 观察易得 $u=xye^{xy}+y\sin x+C$

(2) 观察易得 $u=x\sin yz+y\sin zx+z\sin xy+C$

(5) $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$，故为保守场.

取折线 $\begin{cases} x=t, \\ y=t, \end{cases}$

$t\in[0,1]$.

$$I=\int_0^1 (t^3\cos t+2t^2\sin t-t^2e^t+t^2\sin t-2te^t)\,dt$$

$$=\sin 1-e$$

(6) $\mathrm{rot}(\vec{v})=\vec{0}$，故为保守场.

取折线 $\begin{cases} x=1+t, \\ y=\pi t, \\ z=\frac{\pi}{2}+\pi t, \end{cases}$

$t\in[0,1]$.

$$I=\int_0^1 \Big[\cos\pi t\sin(\tfrac{\pi}{2}+\pi t)-\pi(1+t)\sin\pi t\sin(\tfrac{\pi}{2}+\pi t)$$
$$+\pi(1+t)\cos\pi t\cos(\tfrac{\pi}{2}+\pi t)\Big]dt$$

$$=\int_0^1 [\cos^2\pi t-2(\pi+\pi t)\sin\pi t\cos\pi t]\,dt$$

$$=\frac{1}{\pi}\int_0^\pi [\cos^2 x-(\pi+x)\sin 2x]\,dx \quad (x=\pi t)$$

$$=1$$

34. (1) 观察易得 $u=xye^{xy}+y\sin x+C$

(2) 观察易得 $u=x\sin yz+y\sin zx+z\sin xy+C$

35. (1) $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$，故 I 在 $\mathbb{R}^2\setminus\{\vec{0}\}$

的单连通区域内与路径无关.

取 $\Gamma_1=\{(x,y)\mid x^2+y^2=1\}$.

$$\int_{\vec{\Gamma}_1}\frac{x\,dy-y\,dx}{x^2+y^2}=\int_0^{2\pi}[\cos\theta\cos\theta-\sin\theta(-\sin\theta)]\,d\theta$$

$$=2\pi$$

$$\neq 0.$$

故 I 在 $\mathbb{R}^2$ 内与路径有关.

(2) 不妨设 $\Gamma\neq\Gamma_0$，设 $\Gamma\cup\Gamma_0\triangleq\Gamma_1$ 围成的区域为 D.

若 $\vec{0}\notin D$，$\int_{\vec{\Gamma}}\cdots-\int_{\vec{\Gamma}_0}\cdots=\pm\int_{\vec{\Gamma}_1}\cdots=0$.

若 $\vec{0}\in D$，取 R 充分小，使 $\Gamma_2=\{(x,y)\mid x^2+y^2=R^2\}\subseteq D$.

记 $\Gamma_1\cup\Gamma_2$ 围成的区域为 D_1.

由 Green 公式，$\int_{\vec{\Gamma}_1\cup\vec{\Gamma}_2}\cdots=\iint_{D_1}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=0$.

故 $\int_{\vec{\Gamma}}\cdots-\int_{\vec{\Gamma}_0}\cdots=\pm\int_{\vec{\Gamma}_1}\cdots=\mp\int_{\vec{\Gamma}_2}\cdots=\pm 2\pi$.

42. (1) $\nabla f=(e^x(x+1)\cos y,\ -xe^x\sin y)$

(2) $\nabla f=(yz\cos xyz,\ xz\cos xyz,\ xy\cos xyz)$

43. (1) $\mathrm{div}\vec{F}=2e^x\cos y+1$

$\mathrm{rot}\vec{F}=(0,0,2e^x\sin y)$

(2) $\mathrm{div}\vec{F}=0$

$\mathrm{rot}\vec{F}=\vec{0}$

35. (1) $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$，故 I 在 $\mathbb{R}^2\setminus\{\vec{0}\}$ 的单连通区域内与路径无关.

取 $\Gamma_1=\{(x,y)\mid x^2+y^2=1\}$.

$$\int_{\vec{\Gamma}_1}\frac{x\,dy-y\,dx}{x^2+y^2}=\int_0^{2\pi}[\cos\theta\cos\theta-\sin\theta(-\sin\theta)]\,d\theta$$

$$=2\pi$$

$$\neq 0.$$

故 I 在 $\mathbb{R}^2$ 内与路径有关.

(2) 不妨设 $\Gamma\neq\Gamma_0$，设 $\Gamma\cup\Gamma_0\triangleq\Gamma_1$ 围成的区域为 D.

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若 $\vec{0}\in D$，取 R 充分小，使 $\Gamma_2=\{(x,y)\mid x^2+y^2=R^2\}\subseteq D$.

记 $\Gamma_1\cup\Gamma_2$ 围成的区域为 D_1.

由Green公式，$\int_{\vec{\Gamma}_1\cup\vec{\Gamma}_2}\cdots=\iint_{D_1}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=0$.

故 $\int_{\vec{\Gamma}}\cdots-\int_{\vec{\Gamma}_0}\cdots=\pm\int_{\vec{\Gamma}_1}\cdots=\mp\int_{\vec{\Gamma}_2}\cdots=\pm 2\pi$.

42. (1) $\nabla f=(e^x(x+1)\cos y,\ -xe^x\sin y)$

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$\mathrm{rot}\vec{F}=(0,0,2e^x\sin y)$

(2) $\mathrm{div}\vec{F}=0$

$\mathrm{rot}\vec{F}=\vec{0}$

44. (1) $\nabla f = (f'_x, f'_y, f'_z)$

$div(\nabla f) = \nabla \cdot \nabla f = f''_{xx} + f''_{yy} + f''_{zz}$

(2) 记 $\vec{F} = (P, Q, R)$

$div\vec{F} = \nabla \cdot \vec{F} = P'_x + Q'_y + R'_z \triangleq M$

$\nabla(div\vec{F}) = (M'_x, M'_y, M'_z)$

(3) $grad f = (f'_x, f'_y, f'_z)$

$rot(grad f) = \nabla \times grad f$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f'_x & f'_y & f'_z \end{vmatrix}$$

$= (0, 0, 0)$

(4) 记 $\vec{F} = (P, Q, R)$

$div(f\vec{F}) = \nabla \cdot (f\vec{F})$

$= f'_x P + f P'_x + f'_y Q + f Q'_y$

$+ f'_z R + f R'_z$

$= grad f \cdot \vec{F} + f div(\vec{F})$

45. $rot\vec{F} = \vec{0}$

$div(rot\vec{F}) = div(\vec{0}) = 0$

46. 由44.(3)知 $rot(\nabla f) = \vec{0}$

47. (1) $rot(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(r)x & f(r)y & f(r)z \end{vmatrix}$

$\triangleq (P, Q, R).$

$P = \frac{\partial}{\partial x}(f(r)y) - \frac{\partial}{\partial y}(f(r)x)$

$= y f'(r)\frac{x}{r} - x f'(r)\frac{y}{r}$

$= 0.$

同理 $Q = R = 0$，故 $rot(\vec{F}) = \vec{0}$.

(2) $div\vec{F} = f'(r)\frac{x^2+y^2+z^2}{r} + 3f(r)$

$= r f'(r) + 3f(r)$

$= \frac{1}{r^2}(r^3 f(r))' = 0.$

故 $f(r) = cr^{-3}$.

48. $LHS = \frac{\partial^2 fg}{\partial x^2} + \frac{\partial^2 fg}{\partial y^2} + \frac{\partial^2 fg}{\partial z^2}$

$= f\frac{\partial^2 g}{\partial x^2} + g\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial x}$

$+ \cdots$

$= f\nabla^2(g) + g\nabla^2(f) + 2\nabla f \nabla g$

44. (1) $\nabla f=(f'_x, f'_y, f'_z)$

$div(\nabla f)=\nabla\cdot\nabla f=f''_{xx}+f''_{yy}+f''_{zz}$

(2) 记 $\vec{F}=(P,Q,R)$

$div\vec{F}=\nabla\cdot\vec{F}=P'_x+Q'_y+R'_z\triangleq M$

$\nabla(div\vec{F})=(M'_x, M'_y, M'_z)$

(3) $gradf=(f'_x, f'_y, f'_z)$

$rot(gradf)=\nabla\times gradf$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f'_x & f'_y & f'_z\end{vmatrix}$$

$=(0,0,0)$

(4) 记 $\vec{F}=(P,Q,R)$

$div(f\vec{F})=\nabla\cdot(f\vec{F})$

$=f'_xP+fP'_x+f'_yQ+fQ'_y$

$+f'_zR+fR'_z$

$=gradf\cdot\vec{F}+f\,div(\vec{F})$

45. $rot\vec{F}=\vec{0}$

$div(rot\vec{F})=div(\vec{0})=0$

46. 由44.(3)知 $rot(\nabla f)=\vec{0}$

47. (1) $rot(\vec{F})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f(r)x & f(r)y & f(r)z\end{vmatrix}$

$\triangleq(P,Q,R)$.

$P=\frac{\partial}{\partial x}(f(r)y)-\frac{\partial}{\partial y}(f(r)x)$

$=yf'(r)\frac{x}{r}-xf'(r)\frac{y}{r}$

$=0$.

同理 $Q=R=0$，故 $rot(\vec{F})=\vec{0}$.

(2) $div\vec{F}=f'(r)\frac{x^2+y^2+z^2}{r}+3f(r)$

$=rf'(r)+3f(r)$

$=\frac{1}{r^2}(r^3f(r))'=0$.

故 $f(r)=cr^{-3}$.

48. $LHS=\frac{\partial^2 fg}{\partial x^2}+\frac{\partial^2 fg}{\partial y^2}+\frac{\partial^2 fg}{\partial z^2}$

$=f\frac{\partial^2 g}{\partial x^2}+g\frac{\partial^2 f}{\partial x^2}+2\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial x}$

$+\cdots$

$=f\nabla^2(g)+g\nabla^2(f)+2\nabla f\nabla g$

习题十七

1. 取 $f(x,y)=R(y)$ 即可，其中 $R(\cdot)$ 为Riemann函数

2. 存在性显然.

$$|I(x+\Delta x,u+\Delta u)-I(x,u)| \le |I(x+\Delta x,u+\Delta u)-I(x+\Delta x,u)|$$
$$+|I(x+\Delta x,u)-I(x,u)|$$
$$=\left|\int_u^{u+\Delta u} f(x+\Delta x,y)\,dy\right|$$
$$+\left|\int_c^u (f(x+\Delta x,y)-f(x,y))\,dy\right|$$
$$\le\left|\int_u^{u+\Delta u} |f(x+\Delta x,y)|\,dy\right|$$
$$+\int_c^d |f(x+\Delta x,y)-f(x,y)|\,dy. \quad (*)$$

$\exists M,\ \forall (x,y)\in D,\ |f(x,y)|\le M.$

$\forall \varepsilon>0,\ \exists \delta,\ \forall |P_1-P_2|<\delta,\ |f(P_1)-f(P_2)|<\varepsilon.$

当 $|\Delta x|<\delta,\ |\Delta u|<\varepsilon$ 时，$(*)\le (M+d-c)\varepsilon.$

故 $I(x,u)$ 在 D 上连续.

3. 存在性显然.

$\forall \varepsilon>0,\ \exists \delta,\ \forall |P_1-P_2|<\delta,\ |f(P_1)-f(P_2)|<\varepsilon.$

当 $|\Delta\vec{x}|<\delta$ 时，$|I(\vec{x}+\Delta\vec{x})-I(\vec{x})| \le \int\cdots\int_{\overline{N(0,1)}} |f(\vec{x}+\Delta\vec{x},\vec{y})-f(\vec{x},\vec{y})|\,dV$

$$<\int\cdots\int_{\overline{N(0,1)}} \varepsilon\,dV$$
$$=\varepsilon.$$

故 $I(\vec{x})$ 在 $\overline{N(0,1)}$ 上连续.

习题十七

1. 取 $f(x,y)=R(y)$ 即可，其中 $R(\cdot)$ 为Riemann函数

2. 存在性显然.

$$\begin{aligned}|I(x+\Delta x,u+\Delta u)-I(x,u)| &\le |I(x+\Delta x,u+\Delta u)-I(x+\Delta x,u)| \\ &\quad +|I(x+\Delta x,u)-I(x,u)| \\ &= \left|\int_u^{u+\Delta u} f(x+\Delta x,y)\,dy\right| \\ &\quad +\left|\int_c^u (f(x+\Delta x,y)-f(x,y))\,dy\right| \\ &\le \left|\int_u^{u+\Delta u} |f(x+\Delta x,y)|\,dy\right| \\ &\quad +\int_c^d |f(x+\Delta x,y)-f(x,y)|\,dy. \qquad (*)\end{aligned}$$

$\exists M,\ \forall (x,y)\in D,\ |f(x,y)|\le M.$

$\forall \varepsilon>0,\ \exists \delta,\ \forall |P_1-P_2|<\delta,\ |f(P_1)-f(P_2)|<\varepsilon.$

当 $|\Delta x|<\delta,\ |\Delta u|<\varepsilon$ 时，$(*)\le (M+d-c)\varepsilon.$

故 $I(x,u)$ 在 D 上连续.

3. 存在性显然.

$\forall \varepsilon>0,\ \exists\delta,\ \forall |P_1-P_2|<\delta,\ |f(P_1)-f(P_2)|<\varepsilon.$

当 $|\Delta\vec{x}|<\delta$ 时，

$$\begin{aligned}|I(\vec{x}+\Delta\vec{x})-I(\vec{x})| &\le \int\cdots\int_{\overline{N(0,1)}} |f(\vec{x}+\Delta\vec{x},\vec{y})-f(\vec{x},\vec{y})|\,dV \\ &< \int\cdots\int_{\overline{N(0,1)}} \varepsilon\,dV \\ &= \varepsilon.\end{aligned}$$

故 $I(\vec{x})$ 在 $\overline{N(0,1)}$ 上连续.

4.(1) 原 $=\int_0^1 \frac{1}{\sqrt{1+y^2}}\,dy$

令 $y=\tan t,\ t\in[0,\frac{\pi}{4}]$

上 $=\int_0^{\frac{\pi}{4}} \frac{1}{\cos t}\,dt$

$=\frac{1}{2}\ln\frac{1+\sin t}{1-\sin t}\Big|_0^{\frac{\pi}{4}}$

$=\ln(\sqrt{2}+1)$

(2) 原 $=\int_0^1 \frac{dy}{1+y^2}=\frac{\pi}{4}$

5. $I=\int_0^{+\infty} dx\int_a^b e^{-xy}\,dy$

$=\int_a^b dy\int_0^{+\infty} e^{-yx}\,dx$

$=\int_a^b \frac{1}{y}\,dy$

$=\ln\frac{b}{a}$

6. $I=\int_0^{+\infty} dx\int_a^b e^{-x^2y}\,dy$

$=\int_a^b dy\int_0^{+\infty} e^{-yx^2}\,dx$

$=\frac{\sqrt{\pi}}{2}\int_a^b \frac{1}{\sqrt{y}}\,dy$ (Gauss积分)

$=\sqrt{\pi}(\sqrt{b}-\sqrt{a})$

7.(1) $F'(x)=\int_{a+x}^{b+x} \cos xy\,dy+\frac{\sin x(b+x)}{b+x}-\frac{\sin x(a+x)}{a+x}$

$=\frac{(b+2x)\sin x(b+x)}{x(b+x)}-\frac{(a+2x)\sin x(a+x)}{x(a+x)}$

(2) 记 $I(x,t)=\int_t^{\sin x} f(t,s)\,ds$

$I'_x=f(t,\sin x)\cos x$

$F'(x)=\int_x^{x^2} f(t,\sin x)\cos x\,dt+2x\int_{x^2}^{\sin x} f(x^2,s)\,ds-\int_x^{\sin x} f(x,s)\,ds$

(3) $F'(x)=\int_0^1 \frac{1}{x^2+y^2}\left(\sqrt{x^2+y^2}-x\cdot\frac{2x}{2\sqrt{x^2+y^2}}\right)dy$

$=\int_0^1 \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}\,dy$

$=\sinh^{-1}\left(\frac{1}{x}\right)-\frac{1}{\sqrt{1+x^2}}$

(4) $F'(x)=\int_0^x \left(-ye^{-xy}\cos xy-ye^{-xy}\sin xy\right)dy$

$+e^{-x^2}\cos x^2$

$=e^{-x^2}\cos x^2-\frac{1}{2x^2}\left[1+e^{-x^2}\sin x^2-(2x^2+1)e^{-x^2}\cos x^2\right]$

$=-\frac{1}{2x^2}\left[1+e^{-x^2}\sin x^2-(4x^2+1)e^{-x^2}\cos x^2\right]$

4. (1) 原 $= \int_0^1 \frac{1}{\sqrt{1+y^2}} dy$

令 $y = \tan t$, $t \in [0, \frac{\pi}{4}]$

$I = \int_0^{\frac{\pi}{4}} \frac{1}{\cos t} dt$

$= \frac{1}{2} \ln \frac{1+\sin t}{1-\sin t} \Big|_0^{\frac{\pi}{4}}$

$= \ln(\sqrt{2}+1)$

(2) 原 $= \int_0^1 \frac{dy}{1+y^2} = \frac{\pi}{4}$

5. $I = \int_0^{+\infty} dx \int_a^b e^{-xy} dy$

$= \int_a^b dy \int_0^{+\infty} e^{-yx} dx$

$= \int_a^b \frac{1}{y} dy$

$= \ln \frac{b}{a}$

6. $I = \int_0^{+\infty} dx \int_a^b e^{-x^2 y} dy$

$= \int_a^b dy \int_0^{+\infty} e^{-yx^2} dx$

$= \frac{\sqrt{\pi}}{2} \int_a^b \frac{1}{\sqrt{y}} dy$ (Gauss积分)

$= \sqrt{\pi}(\sqrt{b}-\sqrt{a})$

7. (1) $F'(x) = \int_{a+x}^{b+x} \cos xy \, dy + \frac{\sin x(b+x)}{b+x} - \frac{\sin x(a+x)}{a+x}$

$= \frac{(b+2x)\sin x(b+x)}{x(b+x)} - \frac{(a+2x)\sin x(a+x)}{x(a+x)}$

(2) 记 $I(x,t) = \int_t^{\sin x} f(t,s) ds$

$I'_x = f(t, \sin x)\cos x$

$F'(x) = \int_x^{x^2} f(t, \sin x)\cos x \, dt + 2x \int_{x^2}^{\sin x} f(x^2, s) ds - \int_x^{\sin x} f(x,s) ds$

(3) $F'(x) = \int_0^1 \frac{1}{x^2+y^2} \left(\sqrt{x^2+y^2} - x \cdot \frac{2x}{2\sqrt{x^2+y^2}} \right) dy$

$= \int_0^1 \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}} dy$

$= \sinh^{-1}\left(\frac{1}{x}\right) - \frac{1}{\sqrt{1+x^2}}$

(4) $F'(x) = \int_0^x (-ye^{-xy}\cos xy - ye^{-xy}\sin xy) dy + e^{-x^2}\cos x^2$

$= e^{-x^2}\cos x^2 - \frac{1}{2x^2}\left[1 + e^{-x^2}\sin x^2 - (2x^2+1)e^{-x^2}\cos x^2\right]$

$= -\frac{1}{2x^2}\left[1 + e^{-x^2}\sin x^2 - (4x^2+1)e^{-x^2}\cos x^2\right]$

8. $F(x)$的存在性显然.

$$\frac{1}{\Delta x}[F(x+\Delta x)-F(x)]=\iint_D \frac{1}{\Delta x}[f(x+\Delta x,y,z)-f(x,y,z)]dydz$$

$$=\iint_D f_x'(x+\theta\Delta x,y,z)dydz \text{（L-中值）},$$

其中$\theta=\theta(\Delta x,y,z)\in[0,1]$.

欲证$F'(x)=\lim\limits_{\Delta x\to 0}\frac{1}{\Delta x}[F(x+\Delta x)-F(x)]$

$$=\iint_D f_x'(x,y,z)dydz,$$

只需证$\lim\limits_{\Delta x\to 0}\iint_D|f_x'(x+\theta\Delta x,y,z)-f_x'(x,y,z)|dydz=0$.

由于$\forall\varepsilon>0,\exists\delta,\forall|P_1-P_2|<\delta,|f_x'(P_1)-f_x'(P_2)|<\varepsilon$,

故$|\Delta x|<\delta$时$\iint_D|f_x'(x+\theta\Delta x,y,z)-f_x'(x,y,z)|dydz$

$$<\iint_D\varepsilon dydz$$

$$=A\varepsilon,$$

其中A为D的面积.

综上，$\exists F'(x)=\iint_D f_x'(x,y,z)dydz$.

8. $F(x)$的存在性显然.

$$\frac{1}{\Delta x}[F(x+\Delta x)-F(x)] = \iint_D \frac{1}{\Delta x}[f(x+\Delta x,y,z)-f(x,y,z)]\,dydz$$

$$= \iint_D f'_x(x+\theta\Delta x,y,z)\,dydz \text{ (L-中值)},$$

其中$\theta=\theta(\Delta x,y,z)\in[0,1]$.

欲证 $F'(x)=\lim\limits_{\Delta x\to 0}\frac{1}{\Delta x}[F(x+\Delta x)-F(x)]$

$$=\iint_D f'_x(x,y,z)\,dydz,$$

只需证 $\lim\limits_{\Delta x\to 0}\iint_D |f'_x(x+\theta\Delta x,y,z)-f'_x(x,y,z)|\,dydz=0.$

由于$\forall\varepsilon>0,\exists\delta,\forall|P_1-P_2|<\delta,\ |f'_x(P_1)-f'_x(P_2)|<\varepsilon$,

故$|\Delta x|<\delta$时 $\iint_D |f'_x(x+\theta\Delta x,y,z)-f'_x(x,y,z)|\,dydz$

$$<\iint_D \varepsilon\,dydz$$

$$=A\varepsilon,$$

其中A为D的面积.

综上,$\exists F'(x)=\iint_D f'_x(x,y,z)\,dydz$.

9. $u'_x=\frac{1}{2}[f'(x+at)+f'(x-at)]+\frac{1}{2a}[g(x+at)-g(x-at)]$

$u'_t=\frac{a}{2}[f'(x+at)-f'(x-at)]+\frac{1}{2}[g(x+at)+g(x-at)]$

$u''_{xx}=\frac{1}{2}[f''(x+at)+f''(x-at)]+\frac{1}{2a}[g'(x+at)-g'(x-at)]$

$u''_{tt}=\frac{a^2}{2}[f''(x+at)+f''(x-at)]+\frac{a}{2}[g'(x+at)-g'(x-at)]$

$u''_{tx}=u''_{xt}=\frac{a}{2}[f''(x+at)-f''(x-at)]+\frac{1}{2}[g'(x+at)+g'(x-at)]$

显然$u''_{xx},u''_{tt},u''_{xt},u''_{tx}$连续.

(1) 显然

(2) $u(x,0)=\frac{1}{2}[f(x)+f(x)]+\frac{1}{2a}\int_x^x g(y)dy$

$=f(x)$

(3) $u'_t(x,0)=\frac{a}{2}[f'(x)-f'(x)]+\frac{1}{2}[g(x)+g(x)]$

$=g(x)$

10. 设$f(t)$的Fourier级数的部分和为$S_n(t)$.
由Fourier级数的平方逼近最佳性,即知
该最小值为$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(t)-S_n(t)]^2dt$.

11. $\varphi'_x=\frac{1}{\alpha}\int_0^x f(t)\alpha\cos\alpha(x-t)dt+\frac{1}{\alpha}f(x)\sin\alpha(x-x)$

$=\int_0^x f(t)\cos\alpha(x-t)dt$

$\varphi''_x=\int_0^x f(t)(-\alpha)\sin\alpha(x-t)dt+f(x)\cos\alpha(x-x)$

$=-\alpha\int_0^x f(t)\sin\alpha(x-t)dt+f(x)$

$=-\alpha^2\varphi(x)+f(x)$

$\varphi(0)=\frac{1}{\alpha}\int_0^0 f(t)\sin(-\alpha t)dt=0$

$\varphi'(0)=\int_0^0 f(t)\cos(-\alpha t)dt=0$

9. $u'_x=\frac{1}{2}[f'(x+at)+f'(x-at)]+\frac{1}{2a}[g(x+at)-g(x-at)]$

$u'_t=\frac{a}{2}[f'(x+at)-f'(x-at)]+\frac{1}{2}[g(x+at)+g(x-at)]$

$u''_{xx}=\frac{1}{2}[f''(x+at)+f''(x-at)]+\frac{1}{2a}[g'(x+at)-g'(x-at)]$

$u''_{tt}=\frac{a^2}{2}[f''(x+at)+f''(x-at)]+\frac{a}{2}[g'(x+at)-g'(x-at)]$

$u''_{tx}=u''_{xt}=\frac{a}{2}[f''(x+at)-f''(x-at)]+\frac{1}{2}[g'(x+at)+g'(x-at)]$

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12. $J_0'(x) = \frac{1}{\pi}\int_0^\pi -\sin(x\sin\theta)\sin\theta\, d\theta$

$= \frac{1}{\pi}\int_0^\pi \sin(x\sin\theta)\, d\cos\theta$

$= \frac{1}{\pi}\sin(x\sin\theta)\cos\theta\Big|_0^\pi - \frac{1}{\pi}\int_0^\pi \cos^2\theta\cos(x\sin\theta)\, x\, d\theta$

$= -\frac{1}{\pi}\int_0^\pi \cos^2\theta\cos(x\sin\theta)\, x\, d\theta$

$J_0''(x) = \frac{1}{\pi}\int_0^\pi -\cos(x\sin\theta)\sin^2\theta\, d\theta$

$LHS = \frac{1}{\pi}\int_0^\pi \left[x^2\cos(x\sin\theta) - x^2\cos(x\sin\theta)\cos^2\theta \right.$
$\left. - x^2\cos(x\sin\theta)\sin^2\theta\right] d\theta$

$= 0$

13. $f_y' = \int_{\frac{1}{2}}^1 \cos(x+yt)\, dt$

$= \frac{2}{y}\sin\frac{y}{4}\cos(x+\frac{3}{4}y)$

$\neq 0.$

由隐函数存在定理之得结论.

14. $I'(r) = \int_0^{2\pi} \frac{2r-2\cos\theta}{1-2r\cos\theta+r^2}\, d\theta$

$= \frac{1}{r}\int_0^{2\pi}\left(1 - \frac{1-r^2}{1-2r\cos\theta+r^2}\right) d\theta$

$= 0$

$I(r) = I(0) + \int_0^r I'(r)\, dr$

$= 0$

12. $J_0'(x)=\frac{1}{\pi}\int_0^{\pi}-\sin(x\sin\theta)\sin\theta\,d\theta$

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$=-\frac{1}{\pi}\int_0^{\pi}\cos^2\theta\cos(x\sin\theta)\,x\,d\theta$

$J_0''(x)=\frac{1}{\pi}\int_0^{\pi}-\cos(x\sin\theta)\sin^2\theta\,d\theta$

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$-x^2\cos(x\sin\theta)\sin^2\theta]\,d\theta$

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$=\frac{1}{r}\int_0^{2\pi}(1-\frac{1-r^2}{1-2r\cos\theta+r^2})\,d\theta$

$=0$

$I(r)=I(0)+\int_0^{r}I'(r)\,dr$

$=0$

15. (1) 取 $f(x,y)=\frac{1}{e^{xy}\sqrt{y}}$,

$g(x,y)=\sin y$.

由Dirichlet判别法立得.

(2) $\frac{x^2}{1+x^2y^2}\le\frac{m^2}{1+m^2y^2}$.

由M判别法立得.

(3) 注意到 $\left|\frac{\sin(x^2y)\ln(1+y)}{x^2+y^2}\right|\le\frac{\ln(1+y)}{a^2+y^2}$,

由M判别法立知一致收敛.

(4) 注意到 $\left|\frac{\sin\sqrt{xy}}{y^4+x}\right|\le\frac{1}{4\sqrt{y}}$,

由M判别法立知一致收敛.

16. (1) (a) $\frac{1}{(xy+\frac{x}{y})^2}<\frac{1}{a^2}\cdot\frac{1}{(y+\frac{1}{y})^2}$

由M判别法即知一致收敛.

(b) $\forall A>0$, 取 $n\ge A$, $x_A=\frac{1}{n}$.

$$\int_n^{2n}\frac{dy}{(xy+\frac{x}{y})^2}\ge\int_n^{2n}\frac{dy}{(2+\frac{1}{n^2})^2}$$

$$\ge\int_n^{2n}\frac{1}{9}dy$$

$$=\frac{n}{9}$$

$$\ge\frac{1}{9}$$

因此不一致收敛.

(2) (a) $y=0$ 不是瑕点.

$$\left|\frac{\sqrt{xy}}{x^2+y^2}\right|\le\frac{\sqrt{by}}{a^2+y^2}.$$

故由M判别法知一致收敛.

(b) 取 $x=\frac{1}{n}$.

$$\int_{\frac{1}{n}}^{\frac{2}{n}}\frac{\sqrt{xy}}{x^2+y^2}dy=\int_{\frac{1}{n}}^{\frac{2}{n}}\frac{n\sqrt{ny}}{n^2y^2+1}dy$$

$$\ge\int_{\frac{1}{n}}^{\frac{2}{n}}\frac{n}{5}dy$$

$$=\frac{1}{5}.$$

由Cauchy准则即知不一致收敛.

(3) (a) $\forall x$, 当 y 充分大时有 $0<y-a\le y-x$,

从而 $e^{-(x-y)^2}\le e^{-(a-y)^2}$,

由M判别法即知一致收敛.

(b) 取 $x=n$.

$$\int_n^{n+1}e^{-(x-y)^2}dy=\int_n^{n+1}e^{-(y-n)^2}dy$$

$$=\int_0^1 e^{-y^2}dy$$

$$\ge\frac{1}{e}.$$

由Cauchy准则即知不一致收敛.

15. (1) 取 $f(x,y)=\frac{1}{e^{xy}\sqrt{y}}$,

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由Dirichlet判别法立得.

(2) $\frac{x^2}{1+x^2y^2}\le\frac{m^2}{1+m^2y^2}$.

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(3) 注意到 $\left|\frac{\sin(x^2y)\ln(1+y)}{x^2+y^2}\right|\le\frac{\ln(1+y)}{a^2+y^2}$,

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$$=\int_0^1 e^{-y^2}dy$$

$$\ge\frac{1}{e}.$$

由Cauchy准则即知不一致收敛.

17. 由Cauchy准则，$\forall \varepsilon>0$，$\exists A$，$\forall A_1, A_2>A$，

$\forall x\in(a,b)$，$\left|\int_{A_1}^{A_2} f(x,y)dy\right|<\frac{\varepsilon}{2}$.

由于$f(x,y)$在$[a,b]\times[0,+\infty)$连续，

故$\int_{A_1}^{A_2} f(x,y)dy$在$x\in[a,b]$连续，

从而$\left|\int_{A_1}^{A_2} f(a,y)dy\right| = \lim_{x\to a}\left|\int_{A_1}^{A_2} f(x,y)dy\right|<\varepsilon$，

同理$\left|\int_{A_1}^{A_2} f(b,y)dy\right|<\varepsilon$.

综上，$\forall x\in[a,b]$，$\left|\int_{A_1}^{A_2} f(x,y)dy\right|<\varepsilon$，

即$\int_0^{+\infty} f(x,y)dy$在$[a,b]$一致收敛.

注意到$\int_0^{+\infty} e^{-ax}\sin x\,dx$在$x=0$时发散，

而由以上讨论，若其在$(0,+\infty)$一致收敛，

则必在$[0,+\infty)$一致收敛，矛盾，故不一致收敛.

17. 由Cauchy准则，$\forall \varepsilon>0$，$\exists A$，$\forall A_1, A_2>A$，

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则必在$[0,+\infty)$一致收敛，矛盾，故不一致收敛.

18. $\forall x\in[m,+\infty)$, $m>0$.

$\frac{1}{1+y^2}\downarrow$, $\frac{1}{1+y^2}\rightrightarrows 0$, $y\to+\infty$.

$$\left|\int_1^A \cos xy\,dy\right| = \frac{1}{|x|}\left|\int_x^{Ax}\cos xy\,dxy\right|$$
$$\le \frac{2}{|x|}$$
$$\le \frac{2}{m},\ \forall A\ge 1.$$

于是由Dirichlet判别法知$f(x)$在$(0,+\infty)$内闭一致收敛，从而$f(x)\in C(0,+\infty)$.

$$\lim_{x\to 0} f(x) = f(0) = \int_1^{+\infty}\frac{1}{1+y^2}dy = \frac{\pi}{4}.$$

$\forall\varepsilon>0$，由一致收敛，$\exists A$，

$$\left|\int_A^{+\infty}\frac{\cos xy}{1+y^2}dy\right|<\varepsilon,\ \forall x.$$

又由Riemann-Lebesgue引理，

$$\lim_{x\to+\infty}\int_1^A\frac{\cos xy}{1+y^2}dy=0.$$

故$\lim\limits_{x\to+\infty}f(x)=0$.

19. 由M判别法，显然$F(\alpha)$在$\alpha\in(-\infty,+\infty)$一致收敛，故

$$\int_0^{+\infty}\frac{d}{d\alpha}\left(e^{-x^2-\frac{\alpha^2}{x^2}}\right)dx = \int_0^{+\infty}-\frac{2\alpha}{x^2}e^{-x^2-\frac{\alpha^2}{x^2}}dx$$
$$= -\int_0^{+\infty}2e^{-t^2-\frac{\alpha^2}{t^2}}dt\ \left(t=\frac{\alpha}{x}\right)$$
$$= -2F(\alpha)$$

在$(-\infty,+\infty)$一致收敛，从而$F'(\alpha)=-2F(\alpha)$.

易解得$F(\alpha)=F(0)e^{-2\alpha}$，$\alpha\in\mathbb{R}$.

$$F(0)=\int_0^{+\infty}e^{-x^2}dx=\frac{\sqrt{\pi}}{2}.$$

于是$\int_0^{+\infty}e^{-x^2-\frac{1}{x^2}}dx=F(1)=\frac{\sqrt{\pi}}{2e^2}$.

18. $\forall x\in[m,+\infty)$, $m>0$.

$\frac{1}{1+y^2}\downarrow$, $\frac{1}{1+y^2}\rightrightarrows 0$, $y\to+\infty$.

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$$\lim_{x\to 0} f(x) = f(0) = \int_1^{+\infty}\frac{1}{1+y^2}dy = \frac{\pi}{4}.$$

$\forall\varepsilon>0$，由一致收敛，$\exists A$，

$$\left|\int_A^{+\infty}\frac{\cos xy}{1+y^2}dy\right|<\varepsilon,\ \forall x.$$

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易解得$F(\alpha)=F(0)e^{-2\alpha}$，$\alpha\in\mathbb{R}$.

$$F(0)=\int_0^{+\infty}e^{-x^2}dx=\frac{\sqrt{\pi}}{2}.$$

于是$\displaystyle\int_0^{+\infty}e^{-x^2-\frac{1}{x^2}}dx=F(1)=\frac{\sqrt{\pi}}{2e^2}$.

20. 记 $I(x)=\int_0^{+\infty}\frac{\sin^2 xy}{y^2}dy$.

$y=0$ 不是瑕点，易知 $I(x)$ 在 $x\in\mathbb{R}$ 收敛.

$$\int_0^{+\infty}\frac{d}{dx}\left(\frac{\sin^2 xy}{y^2}\right)dy=\int_0^{+\infty}\frac{\sin 2xy}{y}dy$$

$$=\int_0^{+\infty}\frac{\sin 2xy}{2xy}d(2xy)$$

$$=\frac{\pi}{2}\operatorname{sgn}(x),\ x\in\mathbb{R}.$$

显然上式在 $x\in\mathbb{R}$ 一致收敛，故 $I'(x)=\frac{\pi}{2}\operatorname{sgn}(x),\ x\neq 0$.

可计算得 $I(0)=0$，$I(\pm 1)=\frac{\pi}{2}$（22题），于是 $I(x)=\frac{\pi}{2}|x|,\ x\in\mathbb{R}$.

21. $I=\int_0^{+\infty}\int_0^1\frac{\sin xy}{x}dydx$

由Dirichlet判别法，易知

$\int_0^{+\infty}\frac{\sin xy}{x}dx$ 对 $y\in[0,1]$ 一致收敛.

故 $I=\int_0^1\int_0^{+\infty}\frac{\sin yx}{x}dxdy$

$$=\int_0^1\int_0^{+\infty}\frac{\sin yx}{yx}d(yx)dy$$

$$=\int_0^1\frac{\pi}{2}dy$$

$$=\frac{\pi}{2}.$$

20. 记 $I(x)=\int_0^{+\infty}\frac{\sin^2 xy}{y^2}dy$.

$y=0$ 不是瑕点，易知 $I(x)$ 在 $x\in\mathbb{R}$ 收敛.

$$\int_0^{+\infty}\frac{d}{dx}\left(\frac{\sin^2 xy}{y^2}\right)dy=\int_0^{+\infty}\frac{\sin 2xy}{y}dy$$

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$$=\int_0^1\int_0^{+\infty}\frac{\sin yx}{yx}d(yx)dy$$

$$=\int_0^1\frac{\pi}{2}dy$$

$$=\frac{\pi}{2}.$$

22. 记 $I(x)=\int_0^{+\infty} e^{-y}\cos xy\,dy$.

显然 $I(x)$ 在 $x\in\mathbb{R}$ 收敛.

设 $f(x)=\int_0^{+\infty}\frac{d}{dx}(e^{-y}\cos xy)\,dy=\int_0^{+\infty}-ye^{-y}\sin xy\,dy$

$=\int_0^{+\infty} y\sin xy\,de^{-y}$

$=-\int_0^{+\infty} e^{-y}(\sin xy+xy\cos xy)\,dy$

$\triangleq f_1(x)+f_2(x)$.

$f_2(x)=-x\int_0^{+\infty} ye^{-y}\cos xy\,dy$

$=x\int_0^{+\infty} y\cos xy\,de^{-y}$

$=-x\int_0^{+\infty} e^{-y}(\cos xy-xy\sin xy)\,dy$

$\triangleq f_3(x)-x^2 f(x)$.

$f_1(x)=-\int_0^{+\infty} e^{-y}\sin xy\,dy$

$=\frac{e^{-y}}{x^2+1}(\sin xy+x\cos xy)\Big|_0^{+\infty}$

$=-\frac{x}{x^2+1}$.

$f_3(x)=-x\int_0^{+\infty} e^{-y}\cos xy\,dy$

$=\frac{xe^{-y}}{x^2+1}(\cos xy-x\sin xy)\Big|_0^{+\infty}$

$=-\frac{x}{x^2+1}$.

故 $f(x)=-\frac{2x}{(x^2+1)^2}$.

显然 $f(x)$ 在 $x\in\mathbb{R}$ 一致收敛, 故 $I'(x)=f(x)$, $x\in\mathbb{R}$.

$I(x)=\int_0^x I'(t)\,dt+I(0)=\frac{1}{x^2+1}$, $x\in\mathbb{R}$.

22. 记 $I(x)=\int_0^{+\infty} e^{-y}\cos xy\,dy$.

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$=-\frac{x}{x^2+1}$.

$f_3(x)=-x\int_0^{+\infty}e^{-y}\cos xy\,dy$

$=\frac{xe^{-y}}{x^2+1}(\cos xy-x\sin xy)\Big|_0^{+\infty}$

$=-\frac{x}{x^2+1}$.

故 $f(x)=-\frac{2x}{(x^2+1)^2}$.

显然 $f(x)$ 在 $x\in\mathbb{R}$ 一致收敛，故 $I'(x)=f(x)$, $x\in\mathbb{R}$.

$I(x)=\int_0^x I'(t)dt+I(0)=\frac{1}{x^2+1}$, $x\in\mathbb{R}$.

23. $I=-\int_0^{+\infty}\sin^2x\,d\frac{1}{x}$

$$=-\frac{\sin^2 x}{x}\Big|_0^{+\infty}+\int_0^{+\infty}\frac{\sin 2x}{x}dx$$

$$=0+\int_0^{+\infty}\frac{\sin 2x}{2x}d(2x)$$

$$=\frac{\pi}{2}$$

24. $\int_0^{+\infty}\sin e^{xy}dy$

$$=\int_1^{+\infty}\frac{\sin e^{xy}}{xe^{xy}}d(e^{xy})$$

$$=\int_1^{+\infty}\frac{\sin t}{xt}dt\quad(t=e^{xy})$$

$$=\frac{1}{x}\left(\frac{\pi}{2}-\int_0^1\frac{\sin t}{t}dt\right)$$

$$\to 0\quad(x\to+\infty)$$

25. 记 $\varphi(x)=\int_1^{+\infty}\frac{d}{dx}\left(\frac{\sin y}{y^x}\right)dy$

$$=\int_1^{+\infty}\frac{-\ln y\sin y}{y^x}dy.$$

由Dirichlet判别法，易知 $\varphi(x)$在$(0,+\infty)$内闭一致收敛.
从而$F'(x)=\varphi(x)\in C(0,+\infty)$.

26. $I=\int_0^{+\infty}dx\int_a^b e^{-\alpha x}\cos xy\,dy$.

由Dirichlet判别法，易知$\int_0^{+\infty}e^{-\alpha x}\cos xy\,dx$
对$y\in[a,b]$一致收敛，故

$$I=\int_a^b dy\int_0^{+\infty}e^{-\alpha x}\cos yx\,dx$$

$$=\int_a^b\frac{\alpha}{\alpha^2+y^2}dy$$

$$=\arctan\left(\frac{x}{\alpha}\right)\Big|_a^b$$

$$=\arctan\frac{b}{\alpha}-\arctan\frac{a}{\alpha}.$$

27. 这是习题八.23.(1).

$$I=\frac{\pi}{2}\ln\frac{b}{a}.$$

28. 注意到$I(t)=\int_0^{+\infty}x^{t-1}\cdot xf(x)dx=\int_0^{+\infty}x^{t+1}\cdot\frac{f(x)}{x}dx$,
由Abel判别法易知$I(t)$在$t\in(-1,1)$收敛.

设$f(t)=\int_0^{+\infty}\frac{d}{dt}(x^tf(x))dx$

$$=\int_0^{+\infty}x^t\ln x f(x)dx$$

$$=\int_0^{+\infty}x^{t-1}\ln x\cdot xf(x)dx$$

$$=\int_0^{+\infty}x^{t+1}\ln x\cdot\frac{f(x)}{x}dx.$$

由Abel判别法，易知$f(t)$在$t\in(-1,1)$内闭一致收敛，故$I'(t)=f(t)\in C(-1,1)$.

23. $I=-\int_0^{+\infty}\sin^2 x\, d\frac{1}{x}$

$=-\frac{\sin^2 x}{x}\Big|_0^{+\infty}+\int_0^{+\infty}\frac{\sin 2x}{x}dx$

$=0+\int_0^{+\infty}\frac{\sin 2x}{2x}d(2x)$

$=\frac{\pi}{2}$

24. $\int_0^{+\infty}\sin e^{xy}dy$

$=\int_1^{+\infty}\frac{\sin e^{xy}}{xe^{xy}}d(e^{xy})$

$=\int_1^{+\infty}\frac{\sin t}{xt}dt\ (t=e^{xy})$

$=\frac{1}{x}\left(\frac{\pi}{2}-\int_0^1\frac{\sin t}{t}dt\right)$

$\to 0\ (x\to+\infty)$

25. 记 $\varphi(x)=\int_1^{+\infty}\frac{d}{dx}\left(\frac{\sin y}{y^x}\right)dy$

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由Dirichlet判别法，易知

$\varphi(x)$在$(0,+\infty)$内闭一致收敛.

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$=\int_0^{+\infty}x^t\ln x\, f(x)dx$

$=\int_0^{+\infty}x^{t-1}\ln x\cdot xf(x)dx$

$=\int_0^{+\infty}x^{t+1}\ln x\cdot\frac{f(x)}{x}dx.$

由Abel判别法，易知$f(t)$在$t\in(-1,1)$内闭一致收敛，故$I'(t)=f(t)\in C(-1,1).$

29. $LHS=\int_1^{+\infty}-\frac{1}{(1+x)^2}dx=-\frac{1}{2}$.

$RHS=\int_1^{+\infty}\frac{1}{(1+y)^2}dy=\frac{1}{2}$.

$LHS\neq RHS$.

30. (1) 令 $x=\sqrt{\tan\theta}$, $\theta\in[0,\frac{\pi}{2})$

$$I=\int_0^{\frac{\pi}{2}}\frac{1}{1+\tan^2\theta}\cdot\frac{1}{2\sqrt{\tan\theta}}\cdot\frac{1}{\cos^2\theta}d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos^{\frac{1}{2}}\theta\sin^{-\frac{1}{2}}\theta\, d\theta$$

$$=\frac{1}{4}B(\frac{3}{4},\frac{1}{4})$$

$$=\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{4\Gamma(1)}$$

$$=\frac{\pi}{4\sin\frac{\pi}{4}}$$

$$=\frac{\sqrt{2}}{4}\pi$$

(2) $I=\int_0^{\frac{\pi}{2}}\cos^{-\frac{1}{2}}x\sin^{\frac{1}{2}}x\,dx$

$$=\frac{1}{2}B(\frac{1}{4},\frac{3}{4})$$

$$=\frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{2\Gamma(1)}$$

$$=\frac{\sqrt{2}}{2}\pi$$

(3) 令 $t=-\ln x$, $t\in(+\infty,0)$

$$I=\int_0^{+\infty}(-1)^n t^n e^{-t}dt$$

$$=(-1)^n\Gamma(n+1)$$

$$=(-1)^n n!$$

(4) $I=\int_0^{\frac{\pi}{2}}\cos^{-p}x\sin^{p}x\,dx$

$$=\frac{1}{2}B(\frac{1-p}{2},\frac{1+p}{2})$$

$$=\frac{\Gamma(\frac{1-p}{2})\Gamma(\frac{1+p}{2})}{2\Gamma(1)}$$

$$=\frac{\pi}{2\sin\frac{1-p}{2}\pi}$$

$$=\frac{\pi}{2\cos\frac{p}{2}\pi}$$

(5) 令 $t=2x$, $t\in(0,+\infty)$

$$I=\int_0^{+\infty}\frac{\sqrt{2}}{2}t^{-\frac{1}{2}}e^{-t}dt$$

$$=\frac{\sqrt{2}}{2}\Gamma(\frac{1}{2})$$

$$=\sqrt{\frac{\pi}{2}}$$

(6) $I=\frac{1}{2}B(3,\frac{5}{2})$

$$=\frac{\Gamma(3)\Gamma(\frac{5}{2})}{2\Gamma(\frac{11}{2})}$$

$$=\frac{2!}{2\times\frac{5}{2}\times\frac{7}{2}\times\frac{9}{2}}$$

$$=\frac{8}{315}$$

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$$=\frac{8}{315}$$

(7) 令 $x=\sin\theta,\ \theta\in[0,\frac{\pi}{2}]$

$$I=\int_0^{\frac{\pi}{2}}\cos^{\frac{1}{3}}\theta\, d\theta$$

$$=\frac{1}{2}B\left(\frac{2}{3},\frac{1}{2}\right)$$

$$=\frac{\Gamma(\frac{2}{3})\Gamma(\frac{1}{2})}{2\Gamma(\frac{7}{6})}$$

$$=\frac{\sqrt{\pi}\,\Gamma(\frac{2}{3})}{2\Gamma(\frac{7}{6})}$$

(8) 令 $t=x\ln 2,\ t\in(0,+\infty)$

$$I=\frac{1}{\ln^2 2}\int_0^{+\infty}te^{-t}\,dt$$

$$=\frac{1}{\ln^2 2}\Gamma(2)$$

$$=\frac{1}{\ln^2 2}$$

31.(1) 令 $x=\frac{t}{1+t},\ t\in(0,+\infty)$

$$B(\alpha,\beta)=\int_0^{+\infty}\left(\frac{t}{1+t}\right)^{\alpha-1}\left(\frac{1}{1+t}\right)^{\beta-1}\frac{1}{(1+t)^2}\,dt$$

$$=\int_0^{+\infty}\frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}\,dt$$

$$=\int_0^{+\infty}\frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}}\,dx$$

$$B(\alpha,\beta)=B(\beta,\alpha)=\int_0^{+\infty}\frac{x^{\beta-1}}{(1+x)^{\alpha+\beta}}\,dx$$

(2) $$B(\alpha,\beta)=\int_0^{+\infty}\frac{x^{\beta-1}}{(1+x)^{\alpha+\beta}}\,dx$$

$$=\int_0^{1}\frac{x^{\beta-1}}{(1+x)^{\alpha+\beta}}\,dx+\int_1^{+\infty}\frac{x^{\beta-1}}{(1+x)^{\alpha+\beta}}\,dx$$

$$=\int_0^{1}\frac{x^{\beta-1}}{(1+x)^{\alpha+\beta}}\,dx+\int_0^{1}\frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}\,dt\quad\left(t=\frac{1}{x}\right)$$

$$=\int_0^{1}\frac{x^{\alpha-1}+x^{\beta-1}}{(1+x)^{\alpha+\beta}}\,dx$$

(7) 令 $x=\sin\theta$, $\theta\in[0,\frac{\pi}{2}]$

$$I=\int_0^{\frac{\pi}{2}}\cos^{\frac{1}{3}}\theta\,d\theta$$

$$=\frac{1}{2}B\left(\frac{2}{3},\frac{1}{2}\right)$$

$$=\frac{\Gamma(\frac{2}{3})\Gamma(\frac{1}{2})}{2\Gamma(\frac{7}{6})}$$

$$=\frac{\sqrt{\pi}\,\Gamma(\frac{2}{3})}{2\Gamma(\frac{7}{6})}$$

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$$=\int_0^{+\infty}\frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}\,dt$$

$$=\int_0^{+\infty}\frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}}\,dx$$

$$B(\alpha,\beta)=B(\beta,\alpha)=\int_0^{+\infty}\frac{x^{\beta-1}}{(1+x)^{\alpha+\beta}}\,dx$$

(2) $$B(\alpha,\beta)=\int_0^{+\infty}\frac{x^{\beta-1}}{(1+x)^{\alpha+\beta}}\,dx$$

$$=\int_0^{1}\frac{x^{\beta-1}}{(1+x)^{\alpha+\beta}}\,dx+\int_1^{+\infty}\frac{x^{\beta-1}}{(1+x)^{\alpha+\beta}}\,dx$$

$$=\int_0^{1}\frac{x^{\beta-1}}{(1+x)^{\alpha+\beta}}\,dx+\int_0^{1}\frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}\,dt\quad\left(t=\frac{1}{x}\right)$$

$$=\int_0^{1}\frac{x^{\alpha-1}+x^{\beta-1}}{(1+x)^{\alpha+\beta}}\,dx$$

32. 令 $\begin{cases} x = r\cos^3\theta, \\ y = r\sin^3\theta, \end{cases}$ $r\in[0,1]$, $\theta\in[0,\frac{\pi}{2}]$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos^3\theta & \sin^3\theta \\ -3r\cos^2\theta\sin\theta & 3r\sin^2\theta\cos\theta \end{vmatrix}$$

$$= 3r\sin^2\theta\cos^2\theta$$

$$= \frac{3}{4}r\sin^2 2\theta$$

$$S = \int_0^1 dr\int_0^{2\pi}\left|\frac{3}{4}r\sin^2 2\theta\right| d\theta$$

$$= \int_0^1 dr\int_0^{\frac{\pi}{4}} 8\left(\frac{3}{4}r\sin^2 2\theta\right) d\theta$$

$$= \int_0^1 dr\left(r\sin^2 2\theta\right)\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{3}{4}\pi\int_0^1 r\,dr$$

$$= \frac{3}{8}\pi$$

注：这提供了习题七.42.(1)在多元微积分视角下的解法.

33. 令 $\begin{cases} x = r\sin^4\varphi\cos^4\theta \\ y = r\sin^4\varphi\sin^4\theta \\ z = r\cos^4\varphi \end{cases}$

$r\in[0,a]$, $\theta\in[0,\frac{\pi}{2}]$, $\varphi\in[0,\frac{\pi}{2}]$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = 2r^2\sin^3 2\theta\sin^7\varphi\cos^3\varphi$$

$$V = \int_0^a dr\int_0^{\frac{\pi}{2}} d\theta\int_0^{\frac{\pi}{2}} 2r^2\sin^3 2\theta\sin^7\varphi\cos^3\varphi\, d\varphi$$

$$= \int_0^a dr\int_0^{\frac{\pi}{2}} d\theta\left(r^2\sin^3 2\theta\, B(2,4)\right)$$

$$= \int_0^a dr\int_0^{\frac{\pi}{2}} \frac{r^2}{20}\sin^3 2\theta\, d\theta$$

$$= \frac{1}{30}\int_0^a r^2 dr = \frac{1}{90}a^3$$

34. 令 $t = x^\alpha$, $t\in[0,+\infty)$

$$\int_0^{+\infty} e^{-x^\alpha}dx = \int_0^{+\infty}\frac{1}{\alpha}t^{\frac{1}{\alpha}-1}e^{-t}dt$$

$$= \frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)$$

$$= \Gamma\left(\frac{1}{\alpha}+1\right)$$

$$\to \Gamma(1)\quad(\alpha\to+\infty)$$

$$= 1$$

35. 令 $t = x^\alpha$, $t\in[0,+\infty)$

$$\int_0^{+\infty}\frac{1}{1+x^\alpha}dx = \frac{1}{\alpha}\int_0^{+\infty}\frac{1}{1+t}t^{\frac{1}{\alpha}-1}dt$$

$$= \frac{1}{\alpha}B\left(\frac{1}{\alpha}, 1-\frac{1}{\alpha}\right)$$

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32. 令 $\begin{cases} x = r\cos^3\theta, \\ y = r\sin^3\theta, \end{cases}$ $r\in[0,1], \theta\in[0,\frac{\pi}{2}]$

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$$= 3r\sin^2\theta\cos^2\theta$$

$$= \frac{3}{4}r\sin^2 2\theta$$

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$$= \int_0^1 dr\int_0^{\frac{\pi}{4}} 8\left(\frac{3}{4}r\sin^2 2\theta\right) d\theta$$

$$= \int_0^1 dr\left(r\sin^3 2\theta\right)\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{3}{4}\pi\int_0^1 r dr$$

$$= \frac{3}{8}\pi$$

注：这提供了习题七.42.(1)在多元微积分视角下的解法.

33. 令 $\begin{cases} x = r\sin^4\varphi\cos^4\theta \\ y = r\sin^4\varphi\sin^4\theta \\ z = r\cos^4\varphi \end{cases}$

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$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = 2r^2\sin^3 2\theta\sin^7\varphi\cos^3\varphi$$

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$$= \int_0^a dr\int_0^{\frac{\pi}{2}} \frac{r^2}{20}\sin^3 2\theta\, d\theta$$

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34. 令 $t = x^\lambda$, $t\in[0,+\infty)$

$$\int_0^{+\infty} e^{-x^\lambda}dx = \int_0^{+\infty}\frac{1}{\lambda}t^{\frac{1}{\lambda}-1}e^{-t}dt$$

$$= \frac{1}{\lambda}\Gamma\left(\frac{1}{\lambda}\right)$$

$$= \Gamma\left(\frac{1}{\lambda}+1\right)$$

$$\to \Gamma(1) \quad (\lambda\to+\infty)$$

$$= 1$$

35. 令 $t = x^\lambda$, $t\in[0,+\infty)$

$$\int_0^{+\infty}\frac{1}{1+x^\lambda}dx = \frac{1}{\lambda}\int_0^{+\infty}\frac{1}{1+t}t^{\frac{1}{\lambda}-1}dt$$

$$= \frac{1}{\lambda}B\left(\frac{1}{\lambda}, 1-\frac{1}{\lambda}\right)$$

$$= \frac{\pi}{\lambda\sin\frac{\pi}{\lambda}}$$

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