Mathematics

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Contents

1. Mathematics

1.1 Gradient, Divergence, Curl and Laplacian

1.1.1 Cartesian Coordinates

For Cartesian coordinates (x, y, z), line element:

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{1.1.1}$$

Metric:

$$g_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \tag{1.1.2}$$

Gradient:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \tag{1.1.3}$$

Divergence:

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{1.1.4}$$

Curl:

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_y}{\partial z} - \frac{\partial A_x}{\partial y}\right)\mathbf{k} \tag{1.1.5}$$

Laplacian:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{1.1.6}$$

1.1.2 Cylindrical Coordinates

For Cylindrical coordinates (ρ, θ, z), line element:

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\theta^2 + \mathrm{d}z^2 \tag{1.1.7}$$

Metric:

$$g_{ij} = \begin{pmatrix} 1 & & \\ & \rho^2 & \\ & & 1 \end{pmatrix} \tag{1.1.8}$$

Gradient:

$$\nabla f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\mathbf{e}_\phi + \frac{\partial f}{\partial z}\mathbf{e}_z \tag{1.1.9}$$

Divergence:

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \tag{1.1.10}$$

Curl:

$$\nabla \times \mathbf{A} = \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \mathbf{e}_\theta & \mathbf{e}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\theta & A_z \end{vmatrix} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\mathbf{e}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\mathbf{e}_\phi + \frac{1}{\rho}\left(\frac{\partial(\rho A_\theta)}{\partial \rho} - \frac{\partial A_\rho}{\partial \theta}\right)\mathbf{e}_z \tag{1.1.11}$$

Laplacian:

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \tag{1.1.12}$$

1.1.3 Spherical Coordinates

For Spherical coordinate (r, θ, ϕ), line element:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\Phi^2 \tag{1.1.13}$$

Metric:

$$g_{ij} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix} \tag{1.1.14}$$

Gradient:

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi \tag{1.1.15}$$

Divergence:

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \tag{1.1.16}$$

Curl:

$$\begin{aligned}\nabla \times \mathbf{A} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \\ &= \frac{1}{r\sin\theta}\left(\frac{\partial(A_\phi\sin\theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right)\mathbf{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial(rA_\phi)}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\mathbf{e}_\phi\end{aligned} \tag{1.1.17}$$

Laplacian:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \tag{1.1.18}$$

1.2 Vector Calculus Identities

1.2.1 Differentiation

Gradient

1. $\nabla(\psi + \phi) = \nabla\psi + \nabla\phi$
2. $\nabla(\psi\phi) = \phi\nabla\psi + \psi\nabla\phi$
3. $\nabla(\psi\mathbf{A}) = \nabla\psi \otimes \mathbf{A} + \psi\nabla\mathbf{A}$
4. $\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$

Divergence

1. $\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B}$
2. $\nabla\cdot(\psi\mathbf{A}) = \psi\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\psi$
3. $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = (\nabla\times\mathbf{A})\cdot\mathbf{B} - (\nabla\times\mathbf{B})\cdot\mathbf{A}$

Curl

1. $\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B}$
2. $\nabla\times(\psi\mathbf{A}) = \psi(\nabla\times\mathbf{A}) - (\mathbf{A}\times\nabla)\psi = \psi(\nabla\times\mathbf{A}) + (\nabla\psi)\times\mathbf{A}$
3. $\nabla\times(\psi\nabla\phi) = \nabla\psi\times\nabla\phi$
4. $\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$

Vector dot Del operator

1. $(\mathbf{A}\cdot\nabla)\mathbf{B} = \frac{1}{2}[\nabla(\mathbf{A}\cdot\mathbf{B}) - \nabla\times(\mathbf{A}\times\mathbf{B}) - \mathbf{B}\times(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla\times\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + \mathbf{A}(\nabla\cdot\mathbf{B})]$
2. $(\mathbf{A}\cdot\nabla)\mathbf{A} = \frac{1}{2}\nabla|\mathbf{A}|^2 - \mathbf{A}\times(\nabla\times\mathbf{A})$

Second derivatives

1. $\nabla\cdot(\nabla\times\mathbf{A}) = 0$
2. $\nabla\times(\nabla\phi) = 0$
3. $\nabla\cdot(\nabla\psi) = \nabla^2\psi$, (scalar Laplacian)
4. $\nabla(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A}) = \nabla^2\mathbf{A}$, (vector Laplacian)
5. $\nabla\cdot(\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi$
6. $\psi\nabla^2\phi - \phi\nabla^2\psi = \nabla\cdot(\psi\nabla\phi - \phi\nabla\psi)$
7. $\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2(\nabla\phi)\cdot(\nabla\psi) + (\nabla^2\phi)\psi$
8. $\nabla^2(\psi\mathbf{A}) = \mathbf{A}\nabla^2\psi + 2(\nabla\psi\cdot\nabla)\mathbf{A} + \psi\nabla^2\mathbf{A}$
9. $\nabla^2(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\nabla^2\mathbf{B} - \mathbf{B}\cdot\nabla^2\mathbf{A} + 2\nabla\cdot[(\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{B}\times(\nabla\times\mathbf{A})]$, (Green's vector identity)

Third derivatives

1. $\nabla^2(\nabla\psi) = \nabla(\nabla\cdot(\nabla\psi)) = \nabla(\nabla^2\psi)$
2. $\nabla^2(\nabla\cdot\mathbf{A}) = \nabla\cdot(\nabla(\nabla\cdot\mathbf{A})) = \nabla\cdot(\nabla^2\mathbf{A})$
3. $\nabla^2(\nabla\times\mathbf{A}) = -\nabla\times(\nabla\times(\nabla\times\mathbf{A})) = \nabla\times(\nabla^2\mathbf{A})$

1.2.2 Integration

Surface-volume integrals

1. $\oint_{\partial V} \mathbf{A}\cdot \mathrm{d}\mathbf{S} = \int_V (\nabla\cdot\mathbf{A})\,\mathrm{d}V,$ (divergence theorem)
2. $\oint_{\partial V} \psi\,\mathrm{d}\mathbf{S} = \int_V \nabla\psi\,\mathrm{d}V$
3. $\oint_{\partial V} \mathbf{A}\times\mathrm{d}\mathbf{S} = -\int_V \nabla\times\mathbf{A}\,\mathrm{d}V$
4. $\oint_{\partial V} \psi\nabla\phi\cdot\mathrm{d}\mathbf{S} = \int_V (\psi\nabla^2\phi + \nabla\phi\cdot\nabla\psi)\,\mathrm{d}V,$ (Green's first identity)
5. $\oint_{\partial V} (\psi\nabla\phi - \phi\nabla\psi)\cdot\mathrm{d}\mathbf{S} = \oint_{\partial V}\left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right)\mathrm{d}S = \int_V (\psi\nabla^2\phi - \phi\nabla^2\psi)\,\mathrm{d}V,$ (Green's second identity)
6. $\int_V \mathbf{A}\cdot\nabla\psi\,\mathrm{d}V = \oint_{\partial V}\psi\mathbf{A}\cdot\mathrm{d}\mathbf{S} - \int_V \psi\nabla\cdot\mathbf{A}\,\mathrm{d}V,$ (integration by parts)
7. $\int_V \psi\nabla\cdot\mathbf{A}\,\mathrm{d}V = \oint_{\partial V}\psi\mathbf{A}\cdot\mathrm{d}\mathbf{S} - \int_V \mathbf{A}\cdot\nabla\psi\,\mathrm{d}V,$ (integration by parts)

Curve-Surface integrals

1. $\oint_{\partial S} \mathbf{A}\cdot\mathrm{d}\boldsymbol{\ell} = \int_S (\nabla\times\mathbf{A})\cdot\mathrm{d}\mathbf{S},$ (Stoke's theorem)
2. $\oint_{\partial S} \psi\,\mathrm{d}\boldsymbol{\ell} = -\int_S \nabla\psi\times\mathrm{d}\mathbf{S}$
3. $\oint_{\partial S} \mathbf{A}\cdot\mathrm{d}\boldsymbol{\ell} = -\oint \mathbf{A}\cdot\mathrm{d}\boldsymbol{\ell}$

1.3 Limits and Series in Zhihu

1.3.1 Limits

1. Link: $\lim_{x\to\infty}\frac{1}{x^2}\int_0^x t|\sin t|\,\mathrm{d}t = \frac{1}{\pi}$
2. Link: $\lim_{x\to 0+}\frac{1}{x}\int_0^x \cos^n\frac{1}{t}\,\mathrm{d}t = \begin{cases} 0, & \text{for } n \text{ odd} \\ \frac{(n-1)!!}{(n)!!}, & \text{for } n \text{ even}\end{cases}$

3. Link: $\lim_{t\to 0^+}\int_{-2022}^{2022}\frac{t\cos x}{x^2+t^2}\,\mathrm{d}x=\pi$

4. Link: $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n^2+n+i}=\frac{1}{2}$

5. Link: $\lim_{x\to\infty}\left[\frac{x^n}{(x-1)(x-2)\cdots(x-n)}\right]^{2x}=e^{n(n+1)}$

6. Link: $\lim_{n\to\infty}\int_0^1\sqrt[n]{x^n+(1-x)^n}\,\mathrm{d}x=\frac{3}{4}$

$$\lim_{n\to\infty}n^2\left(\int_0^1\sqrt[n]{\sqrt{x^n+(1-x)^n}}\,\mathrm{d}x-\frac{3}{4}\right)=\frac{\pi^2}{48}$$

7. Link: $\lim_{n\to\infty}\left\{\left[\left(\int_0^1\frac{x^{n-1}}{1+x}\,\mathrm{d}x\right)n-\frac{1}{2}\right]\frac{n}{2}\right\}=\frac{1}{8}$

8. Link: $\lim_{n\to\infty}\int_{-\sqrt{n}}^{\infty}\exp\left(-\frac{x^2}{2}+\frac{x^3}{3\sqrt{n}}-\frac{x^4}{4n}+\cdots\right)\mathrm{d}x=\sqrt{2\pi}$

9. Link: $\lim_{n\to\infty}n\int_0^{\pi}\left(\sqrt[n]{\sin x}-1\right)\mathrm{d}x=-\pi\ln 2$

1.3.2 Series

1. Link: $\sum_{n=1}^{\infty}\arctan\frac{2}{n^2}=\frac{3\pi}{4}$

2. Link: $\sum_{k=1}^{n}\sqrt{k}=\zeta\left(-\frac{1}{2}\right)+\frac{2}{3}n^{\frac{3}{2}}+\frac{1}{2}n^{\frac{1}{2}}+\sum_{k=1}^{\infty}\frac{B_{2k}\Gamma\left(\frac{3}{2}\right)}{(2k)!\Gamma\left(\frac{5}{2}-2k\right)}n^{\frac{3}{2}-2k}$

$$\sum_{k=1}^{n}k^s=\zeta(-k)+\frac{1}{s+1}n^{s+1}+\frac{1}{2}n^s+\sum_{k=1}^{\infty}\frac{B_{2k}\Gamma(s+1)}{(2k)!\Gamma(s+1-k)}n^{s-k},\quad (s\neq -1)$$

3. Link: $\sum_{n=1}^{\infty}\frac{a^n}{(n+b)^n}=a\int_0^1 x^{b-ax}\,\mathrm{d}x$

$$\sum_{n=1}^{\infty}\frac{a^n}{(cn+b)^n}=\frac{a}{c}\int_0^1 x^{\frac{b-ax}{c}}\,\mathrm{d}x$$

4. Link: $\sum_{k=-\infty}^{\infty}\mathrm{e}^{i2\pi k\frac{t}{T}}=T\sum_{k=-\infty}^{\infty}\delta(t-kT)$

5. Link: $\prod_{m=1}^{n}\sin\frac{m\pi}{2n+1}=\frac{\sqrt{2n+1}}{2^n},\quad \prod_{m=1}^{n}\cos\frac{m\pi}{2n+1}=\frac{1}{2^n},\quad \prod_{m=1}^{n}\tan\frac{m\pi}{2n+1}=\sqrt{2n+1}$

6. Link: $\prod_{k=1}^{\infty}\frac{(k-a_1)(k-a_2)\cdots(k-a_m)}{(k-b_1)(k-b_2)\cdots(k-b_m)}=\frac{\Gamma(1-b_1)\Gamma(1-b_2)\cdots\Gamma(1-b_m)}{\Gamma(1-a_1)\Gamma(1-a_2)\cdots\Gamma(1-a_m)},\quad$ where $\sum_{i=1}^{m}a_i=\sum_{i=1}^{m}b_i$

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{k^2}\right)=\frac{\sinh\pi}{\pi}$$

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{k^3}\right)=\frac{1}{\pi}\cosh\frac{\sqrt{3}}{2}\pi$$

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{k^4}\right)=\frac{1}{2\pi^2}\left(\cosh\sqrt{2}\pi-\cos\sqrt{2}\pi\right)$$

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{k^6}\right)=\frac{\sinh\pi}{2\pi^3}\left(\cosh\pi-\cos\sqrt{3}\pi\right)$$

$$\cdots$$

$$\prod_{k=m}^{\infty}\left(1+\frac{z^n}{k^n}\right)=\Gamma^n(m)\prod_{k=0}^{n-1}\frac{1}{\Gamma(m-ze^{(2k+1)\pi i/n})},\quad m>0,n>1$$

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{k^2}\right)=\frac{1}{2}$$

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{k^3}\right)=\frac{1}{3\pi}\cosh\frac{\sqrt{3}}{2}\pi,$$ Link

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{k^4}\right)=\frac{\sinh\pi}{4\pi}$$

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{k^6}\right)=\frac{1}{12\pi^2}\left(1+\cosh\sqrt{3}\pi\right)$$

$$\cdots$$

$$\prod_{k=m}^{\infty}\left(1-\frac{z^n}{k^n}\right)=\Gamma^n(m)\prod_{k=0}^{n-1}\frac{1}{\Gamma(m-ze^{2k\pi i/n})},\quad m>0,n>1$$

1.4 Integrals in Zhihu

1.4.1 Indefinite Integrals

1. Link: $$\int\frac{\cos x}{a\sin x+b\cos x}\,\mathrm{d}x=\frac{b}{a^2+b^2}x+\frac{a}{a^2+b^2}\ln|a\sin x+b\cos x|+C$$

$$\int\frac{\sin x}{a\sin x+b\cos x}\,\mathrm{d}x=\frac{a}{a^2+b^2}x-\frac{b}{a^2+b^2}\ln|a\sin x+b\cos x|+C$$

2. Link: $$\int\frac{\sin x}{1+\sin x\cos x}\,\mathrm{d}x=\frac{1}{\sqrt{3}}\operatorname{arctanh}\left(\frac{\sin x-\cos x}{\sqrt{3}}\right)-\arctan(\sin x+\cos x)+C$$

$$\int\frac{\cos x}{1+\sin x\cos x}\,\mathrm{d}x=\frac{1}{\sqrt{3}}\operatorname{arctanh}\left(\frac{\sin x-\cos x}{\sqrt{3}}\right)+\arctan(\sin x+\cos x)+C$$

3. Link: $$\int \frac{1}{(1+x^2)^2}\,\mathrm{d}x = \frac{\arctan x}{2} + \frac{x}{2(1+x^2)} + C$$
$$\int \frac{x^2}{(1+x^2)^2}\,\mathrm{d}x = \frac{\arctan x}{2} - \frac{x}{2(1+x^2)} + C$$

4. Link: $$\int \frac{1}{(\cos^2 x + k^2\sin^2 x)^2}\,\mathrm{d}x = \frac{k^2+1}{2k^3}\arctan(k\tan x) + \frac{k^2-1}{2k^2}\frac{\tan x}{k^2\tan^2 x+1} + C$$
$$\int \frac{\cos x}{(\cos^2 x + k^2\sin^2 x)^2}\,\mathrm{d}x = \frac{1}{2\sqrt{k^2-1}}\arctan\left(\sqrt{k^2-1}\sin x\right) + \frac{\sin x}{2\cos^2 x + 2k^2\sin^2 x} + C$$
$$\int \frac{\cos^2 x}{(\cos^2 x + k^2\sin^2 x)^2}\,\mathrm{d}x = \frac{1}{2k}\arctan(k\tan x) + \frac{1}{2}\frac{\tan x}{k^2\tan^2 x+1} + C$$

5. Link: $$\int \frac{2n!\sin x + x^n}{e^x + \sin x + \cos x + \sum_{k=0}^{n}\frac{x^k}{k!}}\,\mathrm{d}x = n!\,x - n!\ln\left|e^x + \sin x + \cos x + \sum_{k=0}^{n}\frac{x^k}{k!}\right| + C$$
$$\int \sqrt{\frac{\csc x - \cot x}{\csc x + \cot x}}\frac{\sec x}{\sqrt{1+2\sec x}}\,\mathrm{d}x = \pm\,\mathrm{arcsec}(1+\cos x) + C$$

1.4.2 Algebraic functions and Ratio Functions

1. Link: $$\int_0^a \sqrt{x^n(2a-x)^n}\,\mathrm{d}x = a^{n+1}\frac{\sqrt{\pi}}{2}\frac{\Gamma(n/2+1)}{\Gamma(n/2+3/2)}, \quad n\in\mathbb{N}^+, a\in\mathbb{R}^+$$

2. Link: Link: $$\int_a^b \frac{1}{\sqrt{(b-x)(x-a)}}\,\mathrm{d}x = \pi$$

3. Link: $$\int_a^b \sqrt{(x-a)(b-x)}\,\mathrm{d}x = \frac{\pi(b-a)^2}{8}$$
$$\int_a^b \sqrt{\frac{b-x}{x-a}}\,\mathrm{d}x = \frac{\pi}{2}(b-a)$$
$$\int_a^b \sqrt{\frac{x-a}{b-x}}\,\mathrm{d}x = \frac{\pi}{2}(b-a)$$
$$\int_a^b x\sqrt{\frac{b-x}{x-a}}\,\mathrm{d}x = \frac{\pi}{8}(b-a)(3a+b)$$
$$\int_a^b x\sqrt{\frac{x-a}{b-x}}\,\mathrm{d}x = \frac{\pi}{8}(b-a)(a+3b)$$
$$\int_a^b \frac{x}{\sqrt{(x-a)(b-x)}}\,\mathrm{d}x = \frac{\pi}{2}(a+b)$$

4. Link: $$\int_0^\infty \frac{1}{8+4x+x^2}\,\mathrm{d}x = \frac{\pi}{8}$$

5. Link: $$\int_0^\infty \frac{1}{(1+x+x^2)^2+1}\,\mathrm{d}x = \frac{3\pi}{20} - \frac{\ln 2}{5}$$

6. Link: $\displaystyle\int_0^1 \left(\frac{x^{p-1}}{1-x} - \frac{x^{q-1}}{1-x}\right) \mathrm{d}x = -\psi(p) + \psi(q), \quad \mathrm{Re}(p), \mathrm{Re}(q) > 0$

7. Link: $\displaystyle\int_0^\infty \frac{x^m}{1+x^n} \mathrm{d}x = \frac{\pi}{n} \csc\left(\frac{\pi(m+1)}{n}\right), \quad (\mathrm{Re}(m) > -1, \mathrm{Re}(m-n) < -1)$

8. Link: $\displaystyle\int_0^1 \frac{x}{(1+x^2)\sqrt{1-x^2}} \mathrm{d}x = \frac{1}{\sqrt{2}} \ln\left(1+\sqrt{2}\right)$

9. Link: $\displaystyle\int_0^1 x^x \,\mathrm{d}x = -\sum_{n=1}^\infty \frac{(-)^n}{n^n}$

$$\int_0^1 \frac{1}{x^x} \mathrm{d}x = \sum_{n=1}^\infty \frac{1}{n^n}$$

1.4.3 Trigonometric Functions and Inverse Trigonometric Functions

1. Link: $\displaystyle\int_0^\infty \sin x^2 \,\mathrm{d}x = \sqrt{\frac{\pi}{8}}$

2. Link: $\displaystyle\int_0^{\frac{\pi}{2}} \cos^5 x \sin x \sqrt{1-\sin^3 x} \,\mathrm{d}x = \frac{6\sqrt{\pi}}{7} \frac{\Gamma(2/3)}{\Gamma(1/6)} - \frac{4\sqrt{\pi}}{55} \frac{\Gamma(1/3)}{\Gamma(5/6)} + \frac{4}{45}$

3. Link: $\displaystyle\int_{-\pi}^{\pi} \frac{\sin^2 x}{5+4\cos x} \mathrm{d}x = \frac{\pi}{4}$

$$\int_{-\pi}^{\pi} \frac{\sin^2 x}{a^2 - 2ab\cos x + b^2} \mathrm{d}x = \frac{\pi}{a^2}, \quad (a \geq b > 0)$$

4. Link: $\displaystyle\int_0^{2\pi} \frac{1}{(1+\cos x+\sin x)^2+1} \mathrm{d}x = \frac{\sqrt{2+2\sqrt{2}}}{2}\pi$

5. Link: $\displaystyle\int_0^{2\pi} \frac{1}{2+(\sqrt{2}+\sin x+\cos x)^2} \mathrm{d}x = \sqrt{\frac{\sqrt{5}+1}{10}}\pi$

6. Link: $\displaystyle\int_0^{2\pi} \frac{\sin x}{(1+\cos x+\sin x)^2+1} \mathrm{d}x = -\frac{\sqrt{\sqrt{2}-1}}{2}\pi$

$$\int_0^{2\pi} \frac{\cos x}{(1+\cos x+\sin x)^2+1} \mathrm{d}x = -\frac{\sqrt{\sqrt{2}-1}}{2}\pi$$

7. Link: $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{\sin^2 x} \mathrm{d}x = \frac{n\pi}{2}, \quad (n \in \mathbb{N}^+)$

8. Link: $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt[3]{\tan x}}{(\sin x+\cos x)^2} \mathrm{d}x = \frac{2\pi}{3\sqrt{3}}$

9. Link: $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+k^2\tan^2 x} \mathrm{d}x = \frac{\pi}{2(k+1)}, \quad (k > 0)$

10. Link: $\displaystyle\int_0^\infty x^{p-1}\sin x\,\mathrm{d}x = \Gamma(p)\sin\frac{p\pi}{2}, \quad 0 < |\,\mathrm{Re}(p)| < 1$

$$\int_0^\infty x^{p-1}\cos x\,\mathrm{d}x = \Gamma(p)\cos\frac{p\pi}{2}, \quad 0 < \mathrm{Re}(p) < 1$$

11. Link: $\displaystyle\int_0^\pi \frac{x}{1+\sin^2 x}\,\mathrm{d}x = \int_0^\pi \frac{x}{1+\cos^2 x}\,\mathrm{d}x = \frac{\pi^2}{2\sqrt{2}}$

12. Link: $\displaystyle\int_0^\pi \frac{x\cos x}{1+\sin^2 x}\,\mathrm{d}x = \ln^2(\sqrt{2}+1) - \frac{\pi^2}{4}$

13. Link: Link: $\displaystyle\int_0^\infty \frac{\cos bx}{a^2+x^2}\,\mathrm{d}x = \frac{\pi}{2a}e^{-ab}$

14. Link: $\displaystyle\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}$

15. Link: $\displaystyle\int_0^\infty \frac{\sin x}{x}\frac{\sin\left(\frac{x}{3}\right)}{\frac{x}{3}}\,\mathrm{d}x = \frac{\pi}{2}$

$$\int_0^\infty \frac{\sin ax}{\frac{x}{a}}\frac{\sin bx}{\frac{x}{b}}\,\mathrm{d}x = \frac{ab\pi}{4}(|a+b|-|a-b|), \quad (a,b\in\mathbb{R})$$

16. Link: $\displaystyle\int_0^\infty \left(\frac{1}{1+x^n} - \cos x\right)\frac{\mathrm{d}x}{x} = \gamma, \quad \mathrm{Re}(n) > 0$

17. Link: $\displaystyle\int_0^\infty \frac{\cos x - \cos x^2}{x}\,\mathrm{d}x = -\frac{\gamma}{2}$

$$\int_0^\infty \frac{\cos x^a - \cos x^b}{x}\,\mathrm{d}x = \frac{b-a}{ab}\gamma$$

18. Link: $\displaystyle\int_0^{\frac{\pi}{2}} \arctan(\sin x)\,\mathrm{d}x = \frac{\pi^2}{8} - \frac{1}{2}\ln^2(\sqrt{2}+1)$

19. Link: $\displaystyle\int_{-1}^1 \frac{\arccos x}{1+x^2}\,\mathrm{d}x = \frac{\pi^2}{4}$

20. Link: $\displaystyle\int_0^1 \frac{x\arctan x}{1+x}\,\mathrm{d}x = \frac{\pi}{4} - \frac{\pi}{8}\ln 2 - \frac{\ln 2}{2}$

21. Link: $\displaystyle\int_0^1 \frac{\arctan x}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{\pi^2}{8} - \frac{1}{2}\ln^2(\sqrt{2}+1)$

22. Link: $\displaystyle\int_0^1 \frac{\arctan x}{x\sqrt{1-x^2}}\,\mathrm{d}x = \frac{\pi}{2}\ln\left(1+\sqrt{2}\right)$

23. Link: $\displaystyle\int_0^\infty \frac{\arctan 2x}{1+x^2}\,\mathrm{d}x = \frac{\pi^2}{8} - \frac{\ln 2\ln 3}{2} - \frac{1}{2}\mathrm{Li}_2(-2)$

$$\int_0^\infty \frac{\arctan kx}{1+x^2}\,\mathrm{d}x = \frac{\pi^2}{12} - \frac{\ln k\ln(k+1)}{2} - \frac{1}{2}(\mathrm{Li}_2(1-k)+\mathrm{Li}_2(-k)), \quad (k>1)$$

24. Link: $\displaystyle\int_0^1 \frac{\arcsin x}{(1+x^2)^2}\,\mathrm{d}x = \frac{\pi}{8} - \frac{1}{2\sqrt{2}}\ln\left(1+\sqrt{2}\right) + \frac{1}{4}\ln^2(1+\sqrt{2})$

1.4.4 Exponential Functions, Logarithmic Functions and Hyperbolic Functions

1. Link: $\displaystyle\int_0^\infty \frac{\mathrm{e}^{-ax}-\mathrm{e}^{-bx}}{x}\,\mathrm{d}x = \ln\frac{b}{a}, \quad a,b>0$

2. Link: $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}}e^{-\frac{(x-1)^2}{x}}\,\mathrm{d}x = \sqrt{\pi}$

3. Link: $\displaystyle\int_0^\infty \left(\frac{1}{1+x^n} - e^{-x}\right)\frac{\mathrm{d}x}{x} = \gamma, \quad \mathrm{Re}(n)>0$

4. Link: $\displaystyle\int_{\mathbb{R}^n} |\mathbf{x}|^2 e^{-|\mathbf{x}|^2}\,\mathrm{d}^n x = \frac{n}{2}\pi^{\frac{n}{2}}$

5. Link: $\displaystyle\int_0^\infty \frac{\mathrm{e}^{-x^a}-\mathrm{e}^{-x^b}}{x}\,\mathrm{d}x = \frac{a-b}{ab}\gamma$

6. Link: $\displaystyle\int_{-\infty}^\infty e^{i(x-a^2x^3)t}\,\mathrm{d}x = \frac{2\pi}{(3a^2t)^{\frac{1}{3}}}\mathrm{Ai}\left[-\left(\frac{t^2}{3a^2}\right)^{\frac{1}{3}}\right]$

7. Link: $\displaystyle\int_0^\infty \frac{x}{(\mathrm{e}^x-1)^{2/3}}\,\mathrm{d}x = -\frac{\pi^2}{3} + \sqrt{3}\pi\ln 3$

$$\int_0^\infty \frac{x}{(\mathrm{e}^x-1)^{1/3}}\,\mathrm{d}x = \frac{\pi^2}{3} + \sqrt{3}\pi\ln 3$$

$$\int_0^\infty \frac{x}{(\mathrm{e}^x-1)^{1/4}}\,\mathrm{d}x = \frac{\pi^2}{\sqrt{2}} + 3\sqrt{2}\pi\ln 2$$

$$\int_0^\infty \frac{x}{(\mathrm{e}^x-1)^{1/5}}\,\mathrm{d}x = \frac{\pi^2}{\sqrt{5}}\phi^2 + \pi\sqrt{\sqrt{5}\phi}(\sqrt{5}\ln\sqrt{5}+\ln\phi), \quad \text{where } \phi = \frac{\sqrt{5}+1}{2}$$

$\dots$

$$\int_0^\infty \frac{x}{(\mathrm{e}^x-1)^a}\,\mathrm{d}x = -\frac{\pi}{\sin\pi a}(\gamma+\psi(a))$$

8. Link: $\displaystyle\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{p^2}\,e^{i\mathbf{p}\cdot\mathbf{r}} = \frac{1}{4\pi r}$

$$\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{p_i}{p^2}\,e^{i\mathbf{p}\cdot\mathbf{r}} = \frac{ir_i}{4\pi r^3}$$

$$\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{p_ip_j}{p^2}\,e^{i\mathbf{p}\cdot\mathbf{r}} = \frac{1}{3}\delta_{ij}\delta(\mathbf{r}) + \frac{1}{4\pi r^3}(\delta_{ij}-3\hat{x}_i\hat{x}_j)$$

$$\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{p_ip_j}{p^4}\,e^{i\mathbf{p}\cdot\mathbf{r}} = \frac{1}{8\pi r}(\delta_{ij}-\hat{x}_i\hat{x}_j)$$

$$\int \frac{\mathrm{d}^3p}{(2\pi)^3}\ln p\,e^{i\mathbf{p}\cdot\mathbf{r}} = -\frac{1}{4\pi r^3}$$

$$\int \frac{\mathrm{d}^3p}{(2\pi)^3}\ln p\,p_ip_j\,e^{i\mathbf{p}\cdot\mathbf{r}} = \frac{1}{4\pi r^4}(15\hat{x}_i\hat{x}_j-3\delta_{ij})$$

9. Link: $\displaystyle\int_0^1 \frac{\ln(1+x)}{x}\,\mathrm{d}x = \frac{\pi^2}{12}$

10. Link: $\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}\,\mathrm{d}x = \frac{\pi\ln 2}{8}$

$$\int_0^1 \frac{\ln\left(1+x^2\right)}{1+x}\,\mathrm{d}x = \frac{3}{4}(\ln 2)^2 - \frac{\pi^2}{48}$$

11. Link: $\displaystyle\int_{-\infty}^{\infty} \frac{\ln\left(1+x^2\right)}{(1+x)^2+1}\,\mathrm{d}x = \pi\ln 5$

12. Link: $\displaystyle\int_{-\infty}^{\infty} \frac{\ln\left(1+x^2\right)}{(1+x+x^2)^2+1}\,\mathrm{d}x = -\frac{2\pi^2}{5} + \frac{\pi}{5}\ln 20 + \frac{4\pi}{5}\arctan 2$

13. Link: $\displaystyle\int_0^{\infty} \frac{\ln x}{x^2+a^2}\,\mathrm{d}x = \frac{\pi}{2a}\ln a, \quad (a>0)$

14. Link: $\displaystyle\int_0^{\infty} \frac{\ln x}{(1+x+x^2)^2+1}\,\mathrm{d}x = -\frac{3\pi^2}{80} + \frac{1}{40}\pi\ln 2 - \frac{\ln^2 2}{20}$

15. Link: $\displaystyle\int_0^1 \lfloor nx \rfloor \frac{\ln x + \ln(1-x)}{\sqrt{x(1-x)}}\,\mathrm{d}x = -2(n-1)\pi\ln 2$

16. Link: $\displaystyle\int_0^1 \frac{\ln\left(1-2x\cos a + x^2\right)}{x}\,\mathrm{d}x = \pi a - \frac{a^2}{2} - \frac{\pi^2}{3}$

17. Link: $\displaystyle\int_0^1 \frac{\ln^{n-1} x}{1-x}\,\mathrm{d}x = (-)^{n-1}\Gamma(n)\zeta(n), \quad n>1$

$$\int_0^1 \frac{\ln x}{1-x}\,\mathrm{d}x = -\frac{\pi^2}{6}$$

$$\int_0^1 \frac{\ln^2 x}{1-x}\,\mathrm{d}x = 2\zeta(3)$$

$$\int_0^1 \frac{\ln^3 x}{1-x}\,\mathrm{d}x = -\frac{\pi^4}{15}$$

$$\int_0^1 \frac{\ln^4 x}{1-x}\,\mathrm{d}x = 24\zeta(5)$$

$$\int_0^1 \frac{\ln^5 x}{1-x}\,\mathrm{d}x = -\frac{8\pi^6}{63}$$

18. Link: $\displaystyle\int_0^1 \frac{x-1}{\ln x}\,\mathrm{d}x = \ln 2$

$$\int_0^{\infty} \frac{x^a - x^b}{\ln x}\,\mathrm{d}x = \ln\frac{1+a}{1+b}, \quad a,b>-1$$

19. Link: $\int_0^\infty e^{-x}\ln x\,\mathrm{d}x = \Gamma'(1) = -\gamma$

$$\int_0^\infty e^{-x}\ln^2 x\,\mathrm{d}x = \Gamma''(1) = \frac{\pi^2}{6} + \gamma^2$$

20. Link: $\frac{1}{2\pi}\int_0^{2\pi}\ln|f(re^{i\theta})|\,\mathrm{d}\theta = \ln|f(0)| - \sum_{k=1}^{n}\ln\left(\frac{|a_k|}{r}\right),$ Jensen's formula

$$\frac{1}{2\pi}\int_0^{2\pi}\ln|e^{i\theta}-a|\,\mathrm{d}\theta = \begin{cases}\ln|a|, & \text{if } |a|>1\\ 0, & \text{otherwise}\end{cases}$$

21. Link: $\int_0^\infty \frac{x^k}{\cosh x+1}\,\mathrm{d}x = 2\Gamma(k+1)\eta(k)$

$$\int_0^\infty \frac{x^k}{\cosh x-1}\,\mathrm{d}x = 2\Gamma(k+1)\zeta(k)$$

$$\int_0^\infty \frac{x^k}{(\cosh x+1)^2}\,\mathrm{d}x = \frac{2}{3}\Gamma(k+1)[\eta(k)-\eta(k-2)]$$

$$\int_0^\infty \frac{x^k}{(\cosh x-1)^2}\,\mathrm{d}x = \frac{2}{3}\Gamma(k+1)[\zeta(k-2)-\zeta(k)]$$

22. Link: $\int_{-\infty}^\infty \frac{\cosh ax}{\cosh bx}\,\mathrm{d}x = \frac{\pi}{b}\sec\frac{\pi a}{2b}, \quad |\mathrm{Re}(a)|<|\mathrm{Re}(b)|$

$$\int_{-\infty}^\infty \frac{\sinh ax}{\sinh bx}\,\mathrm{d}x = \frac{\pi}{b}\tan\frac{\pi a}{2b}, \quad |\mathrm{Re}(a)|<|\mathrm{Re}(b)|$$

1.4.5 Combinations of Elementary Functions

1. Link: $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos x}{1+e^{-x}}\,\mathrm{d}x = \frac{\sqrt{2}}{2}$

2. Link: $\int_0^\infty \frac{|\cos x|}{\mathrm{e}^x}\,\mathrm{d}x = \frac{1}{2} + \frac{1}{2}\operatorname{csch}\frac{\pi}{2}$

3. Link: $\int_0^\pi e^{\cos\theta}\cos(\sin\theta)\,\mathrm{d}\theta = \pi$

4. Link: $\int_0^\infty \frac{\mathrm{e}^{\cos x}\sin(\sin x)}{\mathrm{d}x} = \frac{\pi}{2}(\mathrm{e}-1)$

5. Link: $\int_0^{2\pi} e^{\sin\theta}\cos^2\theta\,\mathrm{d}\theta = 2\pi I_1(1)$

6. Link: $\int_0^{2\pi} e^{\sin n\phi}\cos(\phi-\cos n\phi)\,\mathrm{d}\phi = 0, \quad n\in\mathbb{Z}$

$$\int_0^{2\pi} e^{\sin n\phi}\sin(\phi-\cos n\phi)\,\mathrm{d}\phi = \begin{cases}0, & \text{if } n\in\mathbb{Z}\backslash\{-1\}\\ -2\pi, & \text{if } n=-1\end{cases}$$

7. Link: $$\int_0^{2\pi} \mathrm{e}^{r\cos\phi}\cos(r\sin\phi - n\phi)\,\mathrm{d}\phi = \frac{2\pi r^n}{n!}, \quad r>0, n\in\mathbb{N}$$

$$\int_0^{2\pi} \mathrm{e}^{r\cos\phi}\sin(r\sin\phi - n\phi)\,\mathrm{d}\phi = 0, \quad r>0, n\in\mathbb{N}$$

8. Link: $$\int_0^{\infty} \frac{\cos x - e^{-x}}{x}\,\mathrm{d}x = 0$$

9. Link: $$\int_0^{\infty} e^{-ax}\sin bx\, x^{s-1}\,\mathrm{d}x = \frac{\Gamma(s)}{(a^2+b^2)^{\frac{s}{2}}}\sin\Big(s\arctan\frac{a}{b}\Big), \quad \mathrm{Re}(s)>-1, \mathrm{Re}(a)>|\,\mathrm{Im}(b)|$$

$$\int_0^{\infty} e^{-ax}\sin bx\ln x\,\mathrm{d}x = \frac{1}{a^2+b^2}\left(a\arctan\frac{b}{a} - b\gamma - \frac{b}{2}\ln\big(a^2+b^2\big)\right)$$

$$\int_0^{\infty} e^{-ax}\sin bx\, x\ln x\,\mathrm{d}x = \frac{x^2-b^2}{(x^2+b^2)^2}\arctan\left(\frac{b}{a}\right) - \frac{ab}{(a^2+b^2)^2}\left[-2+2\gamma+\ln\big(a^2+b^2\big)\right]$$

10. Link: $$\int_0^{\infty} e^{-ax}\cos bx\, x^{s-1}\,\mathrm{d}x = \frac{\Gamma(s)}{(a^2+b^2)^{\frac{s}{2}}}\cos\Big(s\arctan\frac{b}{a}\Big), \quad \mathrm{Re}(\mu)>0, \mathrm{Re}(a)>|\,\mathrm{Im}(b)|$$

11. Link: $$\int_0^{\infty} \frac{\sin\sqrt{x}}{x+\lambda}\mathrm{e}^{-xt}\,\mathrm{d}x = \frac{\pi}{2}\mathrm{e}^{\lambda t}\left[2\mathrm{e}^{-\sqrt{\lambda}} - \mathrm{e}^{\sqrt{\lambda}}\mathrm{erfc}\left(\frac{1}{2\sqrt{t}}+\sqrt{\lambda t}\right) - \mathrm{e}^{-\sqrt{\lambda}}\mathrm{erfc}\left(\frac{1}{2\sqrt{t}}-\sqrt{\lambda t}\right)\right]$$

$\lambda>0, t>0$

12. Link: $$\int_0^{\pi} \sin x\ln\left(\frac{k+\sin x}{k-\sin x}\right)\mathrm{d}x = 2\pi(k-\sqrt{k^2-1}), \quad (k>1)$$

13. Link: $$\int_0^{\infty} \frac{\cos\ln x}{(1+x)^2}\,\mathrm{d}x = \frac{\pi}{\sinh\pi}$$

14. Link: $$\int_0^{\frac{\pi}{2}} \ln\sin x\,\mathrm{d}x = -\frac{\pi}{2}\ln 2$$

$$\int_0^{\frac{\pi}{2}} \ln^2\sin x\,\mathrm{d}x = \frac{\pi\ln^2 2}{2} + \frac{\pi^3}{24}$$

$$\int_0^{\frac{\pi}{2}} \ln\sin x\ln\cos x\,\mathrm{d}x = \frac{\pi}{2}\ln^2 2 - \frac{\pi^3}{48}$$

15. Link: $$\int_0^{\frac{\pi}{4}} \ln\cos x\,\mathrm{d}x = -\frac{\pi}{4}\ln 2 + \frac{1}{2}\mathbf{G}$$

$$\int_0^{\frac{\pi}{4}} \ln\sin x\,\mathrm{d}x = -\frac{\pi}{4}\ln 2 - \frac{1}{2}\mathbf{G}$$

$$\int_0^{\frac{\pi}{4}} \ln\cot x\,\mathrm{d}x = \mathbf{G}$$

16. Link: $$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,\mathrm{d}x = \frac{\pi\ln 2}{8}$$

$$\int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan x)}{\tan x}\,\mathrm{d}x = \frac{7\pi^2}{96} - \frac{\ln^2 2}{8}$$

$$\int_0^{\frac{\pi}{2}} \frac{\ln(1+\tan x)}{\tan x}\,\mathrm{d}x = \frac{5\pi^2}{48}$$

17. Link: $$\int_0^{\pi} \ln\left(1-2a\cos x+a^2\right)\mathrm{d}x = \begin{cases} 2\pi\ln|a|, & \text{if } |a|>1 \\ 0, & \text{otherwise} \end{cases}$$

18. Link: $$\int_0^{\pi} \cos^2 x\ln\left(1+2\mathrm{e}\cos x+\mathrm{e}^2\right)\mathrm{d}x = \pi - \frac{\pi}{4\mathrm{e}^2}$$

$$\int_0^{\pi} \cos^2 x\ln\left(1+2a\cos x+a^2\right)\mathrm{d}x = \begin{cases} \pi\ln a - \frac{\pi}{4a^2}, & |a|>1 \\ -\frac{\pi}{4a^2}, & |a|<1 \end{cases}$$

19. Link: $$\int_0^{\infty} \frac{\arctan x\ln\left(1+x^2\right)}{x(1+x^2)}\,\mathrm{d}x = \frac{\pi}{2}\ln^2 2$$

20. Link: $$\int_0^{1} \arctan x\ln(1+x)\,\mathrm{d}x = \frac{\ln 2}{2} - \frac{(\ln 2)^2}{8} - \frac{\pi}{4} - \frac{\pi^2}{96} + \frac{3\pi\ln 2}{8}$$

21. Link: $$\int_{-\pi}^{\pi} \frac{x\sin x\arctan e^x}{1+\cos^2 x}\,\mathrm{d}x = \frac{\pi^3}{8}$$

22. Link: $$\int_0^{\infty} \frac{\sin\alpha x}{\sinh\beta x}\,\mathrm{d}x = \frac{\pi}{2\beta}\tanh\frac{\alpha\pi}{2\beta}, \quad (\mathrm{Re}(\beta)>0, \quad a>0)$$

$$\int_0^{\infty} \frac{\cos\alpha x}{\cosh\beta x}\,\mathrm{d}x = \frac{\pi}{2\beta}\operatorname{sech}\left(\frac{\alpha\pi}{2\beta}\right), \quad (\mathrm{Re}(\beta)>0, \quad \text{all real } \alpha)$$

23. Link: $$\int_{-\infty}^{\infty} \frac{\cos\gamma x}{x}\tanh\alpha x = 2\ln\coth\frac{\pi\gamma}{4\alpha}, \quad \gamma\in\mathbb{R}\backslash\{0\}$$

$$\int_{-\infty}^{\infty} \frac{\cos\gamma x}{x}\frac{\sinh\alpha x}{\cosh\beta x}\,\mathrm{d}x = \ln\left(\frac{\cosh\frac{\pi\gamma}{2\beta}+\sin\frac{\pi\alpha}{2\beta}}{\cosh\frac{\pi\gamma}{2\beta}-\sin\frac{\pi\alpha}{2\beta}}\right), \quad |\mathrm{Re}(\alpha)|+|\mathrm{Im}(\gamma)|\le|\mathrm{Re}(\beta)|$$

1.4.6 Multiple Integrals

1. Link: $$\int_0^{\frac{\pi}{2}}\mathrm{d}\phi\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\cos(2z\sin\phi\sin\theta) = \left(\int_0^{\frac{\pi}{2}}\cos(z\sin\xi)\,\mathrm{d}\xi\right)^2$$

2. Link: $$\iint_{x^2+y^2\le 1}\sin x^2\sin y^2\,\mathrm{d}x\,\mathrm{d}y = \frac{\pi}{2}J_0(1) + \frac{\pi^2}{4}[J_1(1)\mathbf{H}_0(1) - J_0(1)\mathbf{H}_1(1)] - \frac{\pi}{2}\sin 1$$

3. Link: $$\iiint_{B(0,1)}\cos(x+y+z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{4\pi}{3\sqrt{3}}(\sin\sqrt{3}-\sqrt{3}\cos\sqrt{3})$$

$$\iiint_{B(0,1)}\cos(k_x x+k_y y+k_z z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{4\pi}{k^3}(\sin k - k\cos k)$$

4. Link: $$\int_0^{\infty} p\,\mathrm{d}p\int_0^{\pi}\mathrm{d}\theta\,\frac{\cos n\theta}{(p^2+k^2-2pk\cos\theta)^{1/2}} = \frac{n}{n^2-1}\pi k$$

1.5 Special Functions

1. Link: $J_\nu(z)=\dfrac{1}{2\pi i}\displaystyle\int_{-\infty}^{(0+)} \mathrm{e}^{\frac{z}{2}(t-t^{-1})}t^{-\nu-1}\,\mathrm{d}t\,,\quad |\arg z|<\dfrac{\pi}{2}, |\arg t|<\pi$

$$\sum_{n=-\infty}^{\infty} J_n(z)t^n=\exp\left[\frac{z}{2}\left(t-t^{-1}\right)\right]$$

2. Link: $\displaystyle\int_0^\infty \frac{J_\nu^2(x)}{x}\,\mathrm{d}x=\frac{1}{2\nu},\quad \mathrm{Re}(\nu)>0$

$$\int_0^\infty \frac{J_\mu(ax)J_\nu(ax)}{x}\,\mathrm{d}x=\frac{2}{\pi}\frac{\sin[\pi(\mu-\nu)/2]}{\mu^2-\nu^2},\quad \mathrm{Re}(\mu+\nu)>0$$

3. Link: $\displaystyle\int_0^\infty J_\nu(k_1r)J_\nu(k_2r)r\,\mathrm{d}r=\frac{\delta(k_1-k_2)}{k_1},\quad (\nu\geq -1,\quad k_1,k_2>0)$

4. Link: $\displaystyle\int_0^\infty J_n(x)e^{-px}\,\mathrm{d}x=\mathcal{L}\{J_n(x)\}=\frac{(\sqrt{p^2+1}-p)^n}{\sqrt{p^2+1}}$

$$\int_0^\infty t^nJ_n(x)e^{-px}\,\mathrm{d}x=\mathcal{L}\{t^nJ_n(x)\}=\frac{2^n}{\sqrt{\pi}}\frac{\Gamma\left(n+\frac{1}{2}\right)}{(p^2+1)^{n+\frac{1}{2}}}$$

$$\int_0^\infty \frac{J_n(x)}{x}e^{-px}\,\mathrm{d}x=\mathcal{L}\left\{\frac{J_n(x)}{x}\right\}=\frac{(\sqrt{p^2+1}-p)^n}{n}$$

$$\int_0^\infty J_0(2\sqrt{x})e^{-px}\,\mathrm{d}x=\mathcal{L}\{J_0(2\sqrt{x})\}=\frac{1}{p}\,e^{-\frac{1}{p}}$$

5. Link: $\mathrm{Ai}(x)=\dfrac{1}{\pi}\displaystyle\int_0^\infty\cos\left(\frac{1}{3}t^3+xt\right)\mathrm{d}t\,,\quad \mathrm{Bi}(x)=\dfrac{1}{\pi}\displaystyle\int_0^\infty\left[\exp\left(-\frac{t^3}{3}+xt\right)+\sin\left(\frac{t^3}{3}+xt\right)\right]\mathrm{d}t$

$$\mathrm{Ai}(x)=\begin{cases}\dfrac{\sqrt{x}}{3}\left[I_{-\frac{1}{3}}\left(\dfrac{2}{3}x^{3/2}\right)-I_{\frac{1}{3}}\left(\dfrac{2}{3}x^{3/2}\right)\right]=\dfrac{1}{\pi}\sqrt{\dfrac{x}{3}}K_{\frac{1}{3}}\left(\dfrac{2}{3}x^{3/2}\right), & (x>0)\\[2ex] \dfrac{\sqrt{|x|}}{3}\left[J_{-\frac{1}{3}}\left(\dfrac{2}{3}|x|^{3/2}\right)+J_{\frac{1}{3}}\left(\dfrac{2}{3}|x|^{3/2}\right)\right], & (x<0)\end{cases}$$

$$\mathrm{Bi}(x)=\begin{cases}\sqrt{\dfrac{x}{3}}\left[I_{-\frac{1}{3}}\left(\dfrac{2}{3}x^{3/2}\right)+I_{\frac{1}{3}}\left(\dfrac{2}{3}x^{3/2}\right)\right], & (x>0)\\[2ex] \sqrt{\dfrac{|x|}{3}}\left[J_{-\frac{1}{3}}\left(\dfrac{2}{3}|x|^{3/2}\right)-J_{\frac{1}{3}}\left(\dfrac{2}{3}|x|^{3/2}\right)\right], & (x<0)\end{cases}$$

6. Link: $H_n(x)=(-1)^ne^{x^2}\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-x^2}\right)=n!\displaystyle\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^m}{m!(n-2m)!}(2x)^{n-2m}$

7. Link: $P_n(\cos\theta)=\displaystyle\sum_{k=0}^{n}\frac{(2k-1)!!}{(2k)!!}\frac{(2n-2k-1)!!}{(2n-2k)!!}\cos[(n-2k)\theta]$

8. Link: $\displaystyle\int_{-1}^{1}P_\ell(x)e^{ikrx}\,\mathrm{d}x=2i^\ell j_\ell(kr)$

9. Link: $\int_0^\pi P_\ell(\cos\theta)\cos n\theta\,\mathrm{d}\theta = \pi\frac{(\ell-n-1)!!}{(\ell-n)!!}\frac{(\ell+n-1)!!}{(\ell+n)!!}$

10. Link: $\int_0^\infty x^\mu \mathrm{e}^{-x} L_n^\mu(x) L_{n'}^\mu(x)\,\mathrm{d}x = \frac{\Gamma(\mu+n+1)}{n!}\delta_{nn'}$

$$\sum_{n=0}^\infty \frac{x^n}{\Gamma(n+\mu+1)} L_n^\mu(z) = J_\mu(2\sqrt{xz})\mathrm{e}^x (xz)^{-\mu/2}$$

$$\int_0^\infty x^{\mu/2}\mathrm{e}^{-x} L_n^\mu(x) J_\mu(2\sqrt{xz})\,\mathrm{d}x = \frac{1}{n!} z^{n+\mu/2}\mathrm{e}^{-z}$$

11. Link: $e^{i\mathbf{k}\cdot\mathbf{r}} = 4\pi\sum_{\ell=0}^\infty\sum_{m=-\ell}^{\ell} i^\ell j_\ell(kr) Y_{\ell m}^*(\hat{\mathbf{k}}) Y_{\ell m}(\hat{\mathbf{r}})$

$$e^{ikr\cos\theta} = \sum_{\ell=0}^\infty i^\ell (2\ell+1) j_\ell(kr) P_\ell(\cos\theta)$$

1.6 Other Questions in Zhihu

1.6.1 Inequality

1. Link: $\sum_{1\le i,j\le n} \frac{a_i a_j}{1+|i-j|} \ge 0, \quad \forall a_i \in \mathbb{R}$

2. Link: $(x+y+z)\left(\frac{1}{\sqrt{y^2+z^2+yz}}+\frac{1}{\sqrt{z^2+x^2+zx}}+\frac{1}{\sqrt{x^2+y^2+xy}}\right) \ge 4+\frac{2}{\sqrt{3}}, \quad x,y,z\ge 0$

3. Link: $(a+b)(b+c)(c+a) \ge 4(a+b+c-1), \quad abc=1, a,b,c\in\mathbb{R}^+$

4. Link: $\frac{1}{2} < \int_0^1 \frac{1}{\sqrt{4-x^2+x^3}}\,\mathrm{d}x < \frac{\pi}{6}$

5. Link: $\int_0^1 f(g(x))\,\mathrm{d}x \le \int_0^1 f(x)\,\mathrm{d}x + \int_0^1 g(x)\,\mathrm{d}x\,,$ where $f(x), g(x) \in [0,1] \to C[0,1]$ and $f(x)\uparrow$

6. Link: $\int_0^1 x f^3(x)\,\mathrm{d}x \Big/ \int_0^1 x f^2(x)\,\mathrm{d}x \ge \int_0^1 f^3(x)\,\mathrm{d}x \Big/ \int_0^1 f^2(x)\,\mathrm{d}x\,,$ where $f(x)\in C[0,1]$ and $\uparrow$

$$\int_a^b p(x)\,\mathrm{d}x \int_a^b p(x)f(x)g(x)\,\mathrm{d}x \ge \int_a^b p(x)f(x)\,\mathrm{d}x \int_a^b p(x)g(x)\,\mathrm{d}x$$

where $p(x), f(x), g(x) \in C[a,b]$, $p(x) > 0$ and $f(x), g(x) \uparrow (\downarrow)$

1.6.2 Integral Transformation

1. Link: $\mathscr{F}\left\{\frac{x}{x^2+\lambda}\right\}(k) = i\sqrt{\frac{\pi}{2}}\mathrm{e}^{-\sqrt{\lambda}|k|}\operatorname{sign} k, \quad \lambda > 0$

$$\mathscr{F}\left\{\sin x\,\mathrm{e}^{-x^2\lambda}\right\}(k) = \frac{i}{2\sqrt{2\lambda}}\left[\exp\left(-\frac{(k-1)^2}{4\lambda}\right) - \exp\left(-\frac{(k+1)^2}{4\lambda}\right)\right], \quad \lambda > 0$$

2. Link: $\mathcal{L}^{-1}\left\{\frac{p}{p^2+\pi^2}e^{-\frac{p}{2}}+\frac{\pi}{p^2+\pi^2}e^{-p}\right\}(t)=\left[\theta\left(t-\frac{1}{2}\right)-\theta(t-1)\right]\sin\pi t$

3. Link: $\mathcal{L}^{-1}\left\{\frac{1}{p^2}\right\}(t)=t$

4. Link: $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{p}}\right\}(t)=\frac{1}{\sqrt{\pi t}}$

5. Link: $\mathcal{L}^{-1}\left\{\frac{\mathrm{e}^{-\sqrt{p}}}{p-\lambda}\right\}=\frac{\mathrm{e}^{\lambda t}}{2}\left[\mathrm{e}^{\sqrt{\lambda}}\mathrm{erfc}\left(\frac{1}{2\sqrt{t}}+\sqrt{\lambda t}\right)+\mathrm{e}^{-\sqrt{\lambda}}\mathrm{erfc}\left(\frac{1}{2\sqrt{t}}-\sqrt{\lambda t}\right)\right],\quad \lambda>0$

1.6.3 Others

1. Link: $\int_0^\pi xf(\sin x)\,\mathrm{d}x=\frac{\pi}{2}\int_0^\pi f(\sin x)\,\mathrm{d}x=\pi\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x$

2. Link: $\mathrm{PV}\int_{-\infty}^{\infty} f\left(x-\frac{1}{x}\right)\mathrm{d}x=\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x\,,$ Cauchy–Schlömilch transformation

3. Link: $\int_0^1\frac{\mathrm{d}t}{\sqrt{1-t^4}}=\sqrt{2}\int_0^1\frac{\mathrm{d}t}{\sqrt{1+t^4}}$

$$\int_0^u\frac{\mathrm{d}t}{\sqrt{1-t^4}}=\sqrt{2}\int_0^v\frac{\mathrm{d}t}{\sqrt{1+t^4}},\quad \text{where } u=\frac{\sqrt{2}v}{\sqrt{1+v^4}}$$

$$\int_0^u\frac{\mathrm{d}t}{\sqrt{1-t^4}}=2\int_0^v\frac{\mathrm{d}t}{\sqrt{1-t^4}},\quad \text{where} u=\frac{2v\sqrt{1-v^4}}{1+v^4}$$

$$\int_0^u\frac{\mathrm{d}t}{\sqrt{1-t^4}}+\int_0^v\frac{\mathrm{d}t}{\sqrt{1-t^4}}=\int_0^{T(u,v)}\frac{\mathrm{d}t}{\sqrt{1-t^4}},\quad \text{where } T(u,v)=\frac{u\sqrt{1-v^4}+v\sqrt{1-u^4}}{1+u^2v^2}$$