ARITHMETIC RALLIS INNER PRODUCT FORMULAE FOR $(\mathrm{SO}_3, \widetilde{\mathrm{SL}_2})$

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Abstract

We prove arithmetic Rallis inner product formulae for $(\mathrm{SO}_3, \widetilde{\mathrm{SL}_2})$ over totally real fields. As a result, for GL_2 type abelian variety with trivial central character, the central derivative of the L-function is related to arithmetic inner product of the arithmetic theta lifting. The approach is different from arithmetic Siegel-Weil proposed by Kudla. Instead, the result follows from considering relations among various (arithmetic) Whittaker periods formulae for L-values, and comparison of local periods. We also establish explicit formulae for (arithmetic) Whittaker-Fourier periods. As a by product, we get (i) explicit (arithmetic) Rallis inner product (ii) generalization of Tunnell's work to any quadratic twist family of cuspidal automorphic irreducible representations of PGL_2 over a number field.

1. Introduction

Given a dual pair (G, H) over a number field, the Weil representation gives a construction of automorphic forms on $H \times G$, called theta series. Using these theta series as kernel functions, one could construct automorphic representations on one group from automorphic representations on another group, such a process is called theta lifting. Given a cuspidal irreducible automorphic representation of $H(\mathbb{A})$ such that its theta lifting is cuspidal and has no local obstructions to be nonvanishing. The Rallis inner product formulae, was first proposed by Rallis [31], connects inner product of lifted forms to relevant L-values. The key in the proof is the Siegel-Weil formulae, which connects inner product of lifted forms to an integral of the original forms with diagonal restriction of Siegel Eisenstein series. Via doubling methods of Piatetski-Shapiro and Rallis [11], the integral could be unfolded as product of local doubling zeta integral for pure tensor test vectors, and hence to related L-value.

In many cases when G and H have almost equal rank, for example, $(\mathrm{SO}_{2n+1}, \mathrm{Mp}_{2n})$, $(U(n), U(n))$, a necessary condition for theta lifting of (conjugate) self-dual representation to be nonvanishing is given by the global epsilon factor equals to $+1$ and the global obstruction of the nonvanishingness is given by nonvanishing of central L-value. The question is that if epsilon factor equals to -1, is there exist an arithmetic version of theta lifting whose global obstruction of nonvanishingness is given by central derivative of L-function and there exists an arithmetic version of Rallis inner product formulae?

Consider $(\mathrm{SO}_3, \widetilde{\mathrm{SL}_2})$, it was observed by Gross-Kohnen-Zagier [14] that certain family of Heegner points on modular curve form Fourier coefficients of a weight 3/2 modular form valued in the rational points of the Jacobian. Extending Gross-Kohnen-Zagier's work, Kudla constructed arithmetic theta kernels which are weight 3/2 modular forms valued in $\widehat{\mathrm{Ch}}^1$ of Shimura curves over $\mathbb{Q}$ with maximal level. In [18], it is conjectured that for the weight 3/2 eigen form φ corresponds to a weight 2 newform f with sign -1 and square-free level, arithmetic inner product of arithmetic theta lifting of φ is related to central derivative L-values of f. In the same paper, Kudla propose an arithmetic version of Siegel-Weil together with doubling methods to get his conjecture. Towards to such approach, there are many results, see Kudla-Rapoport-Yang [19] for the conjecture on Shimura curve with maximal level, Du-Yang [6] for a relevant arithmetic Siegel-Weil on modular curve $X_0(N)$ with N square-free. Ald also see Zhu [43] for local arithmetic Siegel-Weil on modular curve with general level. The local arithmetic Siegel-Weil in the case SO_3 is nonsplit is still open except the level is hyperspecial. In Kudla's approach, even formulation of arithmetic Siegel-Weil needs good model of Shimura curve at all places.

In this paper, we prove arithmetic inner product formulae for $(\mathrm{SO}_2, \widetilde{\mathrm{SL}_2})$ over totally real fields. We use modularity of Heegner points on generic fiber of Shimura curve [40] to formulate arithmetic theta lifting. Before introducing our method, let's first introduce main results. For simplicity, we consider everything over $\mathbb{Q}$ in the rest of the introduction.

Let $\mathbb{B}$ be an incoherent definite quaternion algebra over $\mathbb{A}$ and let $\mathbb{V} = \mathbb{B}^{\mathrm{tr}=0}$ be the quadratic space over $\mathbb{A}$ with quadratic form given by minus of the reduced norm. Let $\mathbb{H} = \mathbb{A}^\times\backslash\mathbb{B}^\times \simeq \mathrm{SO}(\mathbb{V})$ and $\mathbb{G} = \widetilde{\mathrm{SL}_2(\mathbb{A})}$. Fix a nontrivial additive character ψ of $\mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$.

Let π be an cuspidal irreducible automorphic representation of $\mathbb{H}$ such that π^{JL} corresponds to an elliptic curve A over $\mathbb{Q}$. The representation π has a model $\mathrm{Hom}_\xi(X, A)_\mathbb{Q}$, where X is the Shimura curve associated to $\mathbb{H}$ and Hom_ξ means use Hodge cycle ξ as base point.

Consider the case

$$\epsilon(\pi) = -1.$$

Fix decomposition $\pi = \otimes_v \pi_v$, and for each v let $\theta_{\psi_v}(\pi_v)$ be the local theta correspondence of π_v with respect to ψ_v, then

$$\Theta := \otimes_v \theta_{\psi_v}(\pi_v)$$

is an irreducible cuspidal automorphic representation of $\mathbb{G}$.

The work of Yuan-Zhang-Zhang establish a $\mathbb{H}\times\mathbb{G}$ equivalent map:

$$\mathcal{S}(\mathbb{V}) \to \mathrm{Ch}^1(X)_\mathbb{Q} \otimes_\mathbb{Q} \mathcal{A}(\mathbb{G}), \quad \phi \mapsto \vartheta_\phi.$$

Identify π and Θ with their model of morphisms and automorphic forms respectively, the arithmetic theta lifting from $\mathbb{H}$ to $\mathbb{G}$ is

$$\pi \otimes \mathcal{S}(\mathbb{V}) \to \Theta \otimes_\mathbb{Q} L, \quad (f,\phi) \mapsto \vartheta_\phi^f := f \circ \vartheta_\phi,$$

where $L \subset A(\mathbb{Q})_\mathbb{Q}$ is the space generated by image of Heegner points on Shimura curve associated to $\mathbb{H}$ via modular parameterizations. The space L has dimension ≤ 1. The image is denoted by $\vartheta_\psi(\pi)$. Then $\vartheta_\psi(\pi)$ is either zero or equals to $\Theta \otimes_\mathbb{Q} L$ with L has dimension one.

Fix decomposition $(\ ,\ ) = \otimes_v (\ ,\ )_v$ of Petersson norm on π.

Theorem 1.1. *Let $f_i = \otimes f_{i,v} \in \pi$, $\phi = \otimes \phi_{i,v} \in \mathcal{S}(\mathbb{V})$, $i = 1, 2$ be pure tensor vectors,*

$$(\vartheta_{\phi_1}^{f_1}, \vartheta_{\phi_2}^{f_2})_{NT} = \frac{L'(1/2,\pi)}{L(2,1_\mathbb{Q})} \prod_v Z^*(\phi_{1,v}, \phi_{2,v}, f_{1,v}, f_{2,v}),$$

where $Z^*(\phi_{1,v}, \phi_{2,v}, f_{1,v}, f_{2,v})$ is the normalized doubling zeta integral:

$$\frac{L(2,1_v)}{L(1/2,\pi_v)} \int_{H_v} (h_v\phi_{1,v}, \phi_{2,v})_v \overline{(h_v f_{1,v}, f_{2,v})_v} dh_v$$

same as the one appeared in the classical Rallis inner product formula.

Similar for the other direction of arithmetic theta lifting.

The main ingredient is that we make full use of arithmetic periods for L-functions and study their relations. To get arithmetic Rallis inner product, the basic observation is that the central derivative of base change L-function of a cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ has a natural splitting into product of two twist L-functions. There are two Whittaker-Fourier periods formulae for arithmetic theta lifting (See Section 4.2). The first one is that the toric periods of $f \in \pi$ along Heegner points appear in Whittaker-Fourier coefficients of arithmetic theta lifting ϑ_ϕ^f, and the ratio of Whittaker-Fourier coefficients by the toric periods is product of local Whittaker functional constructed from Waldspurger explicit local theta lifting. As a consequence, the Néron -Tate height of arithmetic Whittaker-Fourier coefficients of $\vartheta_\psi(\pi)$ is connected to central derivative of base change L-function. The second formula connects Néron -Tate height of arithmetic Whittaker-Fourier coefficients to product of quadratic twist central L-value and arithmetic inner product, the ratio is given by product of local Whittaker periods constructed from matrix coefficients. The second arithmetic Whittaker-Fourier periods formulae is a variant of a formula for usual theta lifting. The Arithmetic Rallis inner product formulae follow from global comparison of two arithmetic Whittaker-Fourier periods formulae and composition of local periods 2.3. The same approach works for the other side of arithmetic theta lifting. For the other direction, one just interchange the role of arithmetic toric periods and arithmetic Whittaker-Fourier periods. The two sides of arithmetic Rallis inner product formulae are in fact equivalent.

The approach could also be applied to higher rank dual pair, like $(\mathrm{SO}_{2n+1}, \mathrm{Mp}_{2n})$, $(U(n), U(n))$, which connects AGGP to arithmetic Rallis inner product formulae. In the classical case, the problem was considered by Furusawa [8].

In the unitary shimura curve case, parallel to Yuan-Zhang-Zhang's approach to Gross-Zagier formula, Liu [20] proves a weak version of arithmetic Siegel-Weil only involves modularity of cycles on generic fiber which is enough to get arithmetic Rallis. Following Liu's approach, one may get the same result as ours. Yet our approach has benefit for getting explicit formulae and further arithmetic applications.

Our second result is the explicit formulae for (arithmetic) Whittaker-Fourier periods formulae and (arithmetic) Rallis inner product formulae.

The explicit version for first (arithmetic) Whittaker-Fourier periods formulae is based on Cai-Shu-Tian's work on explicit (Gross-Zagier) Waldspurger formulae and our choice of test Schwartz functions. As a consequence, the explicit Whittaker-Fourier periods formulae generalize the Tunnell-Gross type formulae on connection between quadratic twist L-values of elliptic curves to representation problem of ternary quadratic forms.

The explicit (arithmetic) Rallis inner product formulae follows from the comparison of two explicit (arithmetic) Whittaker periods formulae. More precisely, the normalized local doubling zeta integral could be interpreted by normalized local toric periods and local Whittaker periods which have 1 dimensional integral domain and whose explicit formulae is relative easy to understand.

In the next paper, we will consider further arithmetic application of these formulae to arithmetic of quadratic twist family of elliptic curves. Let's introduce arithmetic question for motivation of the explicit formulae.

Tunnell-Gross type formulae

Let's recall Tunnell, Gross's work on Shimura-Waldspurger correspondence, which related ternary quadratic forms to quadratic twist central L-values of elliptic curves.

For $a = 1, 2$, let $\{Q_{a,1}, Q_{a,2}\}$ be the genus class of ternary quadratic forms

$$\{x^2 + 2ay^2 + 32z^2, \quad 2ax^2 + 4y^2 + 9z^2 - 4yz\}.$$

Theorem 1.2 (Tunnell-Qin). *For any $n \equiv 1, 2, 3 \pmod 8$ positive square-free*

$$\frac{L(E^{(n)}, 1)}{\Omega/\sqrt{n}} = \frac{a}{16}\left(\sum_{Q_{a,1}(x,y,z)=n/a} 1 - \sum_{\substack{Q_{a,2}(x,y,z)=n/a \\ x\in\mathbb{Z}^3}} 1\right)^2$$

where $a = 1$ if $2 \nmid n$, $a = 2$ if $2|n$, $\Omega = \int_1^\infty \frac{dx}{\sqrt{x^3 - x}}$.

The work of Tunnell was based on Waldspurger's work which established connection between Fourier coefficients of half-integer modular form and quadratic twist central L-values of elliptic newform under Shimura-Waldspurger correspondence [36]. To apply Waldspurger's result, one needs to find weight 3/2 modular form that

- it is a Shimura-Waldspurger image of a given elliptic newform,
- the preassigned Fourier coefficient of weight 3/2 modular form is nonzero.

For congruent elliptic curve $y^2 = x^3 - x$, Tunnell constructed its Shimura-Waldspurger image by tensoring weight 1/2 modular forms and weight 1 modular forms, whose structure well understood by work of Serre-Stark [33] and Deligne-Serre [4]. Recently, Qin [26] reinterpreted forms in Tunnell's work as constructed from theta series associated to ternary quadratic forms in the same genus class.

In the case of prime conductor, Gross [13] gave a method to construct Shimura-Waldspurger lifting via ternary quadratic forms whose Fourier coefficients are certain toric periods. There are also generalizations of Gross's work to the whole quadratic twist family in the prime conductor case [21] and to square-free conductor case by S. Bocherer and R. Schulze-Pillot [2] by their investigation of the Yoshida lift.

As an application of the explicit formula for Whittaker-Fourier period in the sign +1 case, we get general Tunnell-Gross type formula.

Let $\mathcal{A}$ be a quadratic twist family of elliptic curves over $\mathbb{Q}$. We call p a bad place of $\mathcal{A}$ if any $A \in \mathcal{A}$ has bad reduction at p. Let Σ be a set of finite places containing bad places of $\mathcal{A}$ and 2∞. Given $A \in \mathcal{A}$ and $\mathfrak{X}$ a Σ equivalent class with equivalent relation defined by a fiber of the map $\mathbb{Q}^\times \to \prod_{v \in \Sigma} \mathbb{Q}_v^\times / \mathbb{Q}^{\times 2}$.

Theorem 1.3. *Given $A \in \mathcal{A}$ with $L(A,1) \neq 0$, $\mathfrak{X}$ and Σ equivalent class with $\epsilon(A \otimes \mathfrak{X}) = 1$, exists an explicit weight 3/2 modular form $\sum_n a_n q^n$ and a constant C such that*

$$|a_{|n|}|^2 = \begin{cases} C \cdot L^{alg}(A^{(n)}, 1), & n \in \mathfrak{X} \text{ fundamental discriminant} \\ 0, n \notin \mathfrak{X}. \end{cases}$$

We give several explanations and remarks.

- The whole quadratic twist family $\mathcal{A}$ could be covered by finitely many $(A_i, \mathfrak{X}_i)$.
- When $\mathfrak{X} \subset \mathbb{Q}_{<0}$, the Fourier coefficients of the form in the above theorem have a simple form like Tunnell's work which connects to arithmetic of ternary quadratic forms, indefinite ternary quadratic form also involved. In particular, there is an effective algorithm to determine $a_{|n|}$ in $O(n^{3/2})$ steps. The formula exactly generalizes Tunnell and Gross's work with full generality. If there exist d_1, d_2 with distinct signs such that both $L(A^{(n_1)}, 1)$ and $L(A^{(n_2)}, 1)$ are non-zero, then exists a covering $(A_i \otimes \mathfrak{X}_i)_i$ of $\mathcal{A}$ such that their Fourier coefficients are always related to definite ternary quadratic forms. In particular, it holds if $\exists A \in \mathcal{A}$ has non-square conductor or has CM. For a counter example that the condition is not satisfied, for example the quadratic twist family of the elliptic curve $y^2 = x^3 - 91x + 182$, see [5].

Examples

(1) In the general situation, there may be local obstructions given by Atkin-Lehner operators, and is necessary to consider oriented lattice points to replace to counting of the whole lattice points (See Proposition 2.17.) Let $A = X_0(14)$ be the elliptic curve of conductor 14 whose $L(A,1) \neq 0$:

$$A: \quad y^2 + xy + y = x^3 + 4x - 6.$$

For the class $\mathfrak{X}$ containing negative fundamental discriminant $n \equiv -3 \bmod 56$, there will be local obstruction at $p = 2, 7$:

let $Q = (x + 14y + 4z)^2 + (x - 14y - 2z)^2 + x^2$. For each $n \in \mathfrak{X}$,

$$\frac{L(A^{(n)}, 1)}{\Omega(A^{(-1)})/\sqrt{|n|}} = 2 \left(\sum_{\substack{Q(x,y,z)=|n|, \\ 3x+2z\equiv 3 \pmod 4 \\ 3x+2z\equiv 3 \pmod 7}} 1 - \sum_{\substack{Q(x,y,z)=|n|, \\ 3x+2z\equiv 3 \pmod 4 \\ 3x+2z\equiv -3 \pmod 7}} 1 \right)^2.$$

(2) Different $(A_i, \mathfrak{X}_i)$ may interpret the same subfamily of quadratic twist L-values e.g. For congruent number elliptic curves, choose $A: y^2 = x^3 - x$ and $A^{(2)}$ as base curve: For $n > 0$ square-free

$$\sum_{x^2+2y^2+8z^2=n} (-1)^z = \pm \sum_{x^2+8y^2+16z^2=n} (-1)^{y+z}, \quad 0 < n \equiv 1 \pmod 8,$$

$$\sum_{x^2+2y^2+8z^2=n} (-1)^z = \pm 2 \sum_{x^2+2(x+4y)^2+16z^2=n} (-1)^z \quad 0 < n \equiv 3 \pmod 8,$$

$$\sum_{x^2+4y^2+8z^2=n/2} (-1)^z = \pm \sum_{x^2+16y^2+16z^2=n/2} (-1)^z \quad 0 < n \equiv 2 \pmod{16},$$

$$\sum_{x^2+4y^2+8z^2=n/2} (-1)^z = \pm 2 \sum_{x^2+4(x+4y)^2+16z^2=n/2} (-1)^z \quad 0 < n \equiv 10 \pmod{16}.$$

(3) The above Tunnell-Gross Type theorem is just the specialization of the Theorem 6.3 in the sign $+1$ case with $D_1 = 1$ and $\mathfrak{X} = \mathfrak{X}_2$ and Σ containing bad places of A_0, in general, we have a Whittaker-Fourier period formula associated to two families. Consider the quadratic twist family of congruent elliptic curve $E : y^2 = x^3 - x$. Let $A_0 = E$, $\Sigma = \{2, \infty\}$ $(\mathfrak{X}_1, \mathfrak{X}_2) = ([2], [-1])$ or $([2], [-1])$.

For each $p \nmid 2$, consider the weight function on $L_p = \mathbb{Z}_p[i, j, k]$ defined by the following: Fix $x_0 \in L_p \backslash pL_p$ such that $p | q(x_0)$. Let

$$w_{n,p}(x) = \begin{cases} 0, & p \nmid q(x) \\ \eta_{n,p}(-\langle x, x_0 \rangle), & p \nmid \langle x, x_0 \rangle \\ \eta_{n,p}(u), & p | \langle x, x_0 \rangle \end{cases},$$

here $\langle x, y \rangle = q(x + y) - q(x) - q(y)$, $u \in \mathbb{Z}_p^\times$ is such that $x \equiv ux_0 \pmod p$ and $\eta_{n,p}$ is the quadratic character of $\mathbb{Q}_p^\times$ associated to $\mathbb{Q}_p(\sqrt{n})/\mathbb{Q}_p$. For different choice of x_0, w differs by a constant in $\{\pm 1\}$.

Proposition 1.4.
(a) *For each positive square-free* $n_1 \equiv 2 \pmod{16}$ *and* $n_2 \equiv 1 \pmod 8$,

$$\frac{L(E^{(n_1)}, 1)}{\Omega/\sqrt{n_1}} \frac{L(E^{(n_2)}, 1)}{\Omega/\sqrt{n_2}} = \frac{1}{32} \Bigg(\sum_{(a-4b)^2+(a+4b)^2+64c^2=n_1n_2} \prod_{p|\ n_1/2} w_{n_1,p}(a - 4b, a + 4b, 4c) + \sum_{(2a+4b+c)^2+(2a-4b+c)^2+16c^2=n_1n_2} \prod_{p|n_1/2} w_{n_1,p}(2a + 4b + c, 2a - 4b + c, 4c) \Bigg)^2$$

In particular, if $n_2 = 1$, *for each positive square-free* $n \equiv 2 \pmod 8$,

$$\frac{L(E^{(n)}, 1)}{\Omega/\sqrt{n}} = \frac{1}{32} \Bigg(\sum_{(a-4b)^2+(a+4b)^2+64c^2=n} \prod_{p|\ n_1/2} w_{n_1,p}(a - 4b, a + 4b, 4c) + \sum_{(2a+4b+c)^2+(2a-4b+c)^2+16c^2=n_1n_2} \prod_{p|n_1/2} w_{n_1,p}(2a + 4b + c, 2a - 4b + c, 4c) \Bigg)^2$$

(b) *For each positive square-free* $n_1 \equiv 1 \pmod 8$ *and* $n_2 \equiv 2 \pmod{16}$ *with* $(n_1, n_2) = 1$,

$$\frac{L(E^{(n_1)}, 1)}{\Omega/\sqrt{n_1}} \frac{L(E^{(n_2)}, 1)}{\Omega/\sqrt{n_2}} = \frac{1}{32} \Bigg(\sum_{(a+4b)^2+(a-4b)^2+64c^2=n_1n_2} \prod_{p|\ n_1} w_{n_1,p}(a + 4b, a - 4b, 4c) - \sum_{(2a+4b+c)^2+(2a-4b+c)^2+16c^2=n_1n_2} \prod_{p|n_1} w_{n_1,p}(2a + 4b + c, 2a - 4b + c, 4c) \Bigg)^2$$

In particular, if $n_1 = 1$, *for each positive square-free* $n \equiv 2 \pmod{16}$,

$$\frac{L(E^{(n)}, 1)}{\Omega/\sqrt{n}} = \frac{1}{8} \Bigg(\sum_{x^2+16y^2+8z^2=\frac{n}{2}} (-1)^z \Bigg)^2$$

p-divisibility of Ш in sub quadratic twist family In the following, we introduce our explicit formulae on (arithmetic) Whittaker-Fourier periods formulae and (arithmetic) Rallis inner product formulae and the arithmetic application on p-divisibility of Ш. Consider distribution of Tate-Shafarevich group Ш(A) as A varies in a quadratic family of $\mathcal{A}$ elliptic curves over $\mathbb{Q}$. It is suggested that it has good behaviour as A varies in a Σ equivalent class. Here Σ is a finite set of places containing $\{p \mid p \text{ is bad for } \forall A \in \mathcal{A}\} \cup \{2, \infty\}$ and $A, A' \in \mathcal{A}$ are called Σ equivalent if and only if $A/\mathbb{Q}_v \simeq A'/\mathbb{Q}_v$, $\forall v \in \Sigma$. Elliptic curves in an equivalent class have the same sign. The work of Pan-Tian suggest that Ш$(A)[p^\infty]$ has good distribution behaviour as A varies in $\mathfrak{X}$ and the invariant

$$\mu_p(\mathfrak{X}) = \inf_{\substack{A \in \mathfrak{X} \\ \mathrm{ord}_{s=1} L(s,A) \le 1}} \mathrm{ord}_p \#\text{Ш}^{\mathrm{an}}(A)$$

could be strictly positive. Recall Kolyvagin conjectured that exists one $\mathfrak{X}$ with $\mu_p(\mathfrak{X})=0$ [17].

The Goldfeld conjecture predicts that density one of $A\in\mathfrak{X}$ has $\mathrm{ord}_{s=1}L(s,A)=\frac{1}{2}(1-\epsilon(\mathfrak{X}))$, where $\epsilon(\mathfrak{X})$ is the sign of $\mathfrak{X}$. The question is

Question 1.5. *How does $\mu_p(\mathfrak{X}_1)-\mu_p(\mathfrak{X}_2)$ varies as $\mathfrak{X}_i$ varies?*

Given $A\in\mathcal{A}$, $\mathfrak{X}_1,\mathfrak{X}_2\subset\mathcal{A}$ with $\epsilon(\mathfrak{X}_2)=+1$. We may also identify an equivalent class $\mathfrak{X}$ with subset of $\mathbb{Q}^\times$ consists of n such that $A^{(n)}\in\mathfrak{X}$. Assume $\mathfrak{X}_1\mathfrak{X}_2$ whenever $\epsilon(\mathfrak{X}_1)=-1$. Let $\mathbb{B}$ be the quaternion algebra over $\mathbb{A}$ unramified outside Σ such that

$$\epsilon(\mathbb{B}_v)=\eta_v(-1)\epsilon(\mathfrak{X}\otimes\mathbb{Q}_v)\epsilon(\mathfrak{X}_2\otimes\mathbb{Q}_v),\quad \forall v\in\Sigma,$$

where η_v is the quadratic twist character associated to $\mathbb{Q}_v(\sqrt{n_1n_2})$, $n_i\in\mathfrak{X}_i$.

For $n\in\mathbb{Q}^\times$, let η_n be the quadratic character associated to $\mathbb{Q}(\sqrt{n})/\mathbb{Q}$. Let π be the cuspidal automorphic irreducible representation of $\mathbb{H}=\mathbb{A}^\times\backslash\mathbb{B}^\times$ associated to A, Let $\Theta\subset\mathcal{A}_{0,3/2}(\mathbb{A})$ be the irreducible representation only depends on $\mathfrak{X}_1$ such that $\Theta_v\simeq\theta_{\psi_{n^{-1},v}}(\pi_v\otimes\eta_{n,v})$ for all v and $n\in\mathfrak{X}_1$. Let Σ be minimal for simplicity. The following result is a direct consequence of our explicit formulae (See Theorem 5.5 and Theorem 5.7).

Theorem 1.6. *There is a distinguished one dimensional Hecke eigen space in π (Cai-Shu-Tian), say generated by f, and uniform choice of integral ϕ_{D_1} for $D_1\in\mathfrak{X}_1$ fundamental discriminants such that the theta lifting $\theta_{D_1}:=\sum_n a_n^{D_1}q^n$ of $(f_{D_1}:=f\otimes\eta_{D_1},\phi_{D_1})$ in Θ (with respect to $\psi_{D_1^{-1}}$) satisfies that:*

(a) *Support of Whittaker-Fourier coefficients of θ_{D_1} is on $|\mathfrak{X}_2|$ for each fundamental discriminant $D_1\in\mathfrak{X}_1$. Further more, take f rational, for any fundamental discriminant $D_1\in\mathfrak{X}_1$:*

$$\mathbb{Q}\ni\begin{cases}|a_{|D_2|}^{D_1}/\Omega_f^{\mathrm{sign}(D_1)}|^2\\ |a_{|D_2|}^{D_1}|^2\\ |a_{|D_2|}^{D_1}|_{NT}^2/R_{A^{(D_1)}}\end{cases}=\alpha_0\begin{cases}\frac{L(A^{(D_1)},1)}{\Omega_A^{\mathrm{sign}(D_1)}/\sqrt{|D_1|}}\frac{L(A^{(D_2)},1)}{\Omega_A^{\mathrm{sign}(D_2)}/\sqrt{|D_2|}}, & \epsilon(A\otimes\mathfrak{X}_1)=+1,\ \mathfrak{X}_1\mathfrak{X}_2>0\\ \frac{L(A^{(D_1)},1)}{\Omega_A^{\mathrm{sign}(D_1)}/\sqrt{|D_1|}}\frac{L(A^{(D_2)},1)}{\Omega_A^{\mathrm{sign}(D_2)}/\sqrt{|D_2|}}, & \epsilon(A\otimes\mathfrak{X}_1)=+1,\ \mathfrak{X}_1\mathfrak{X}_2<0,\\ \frac{L'(A^{(D_1)},1)}{R_{A^{(D_1)}}\cdot\Omega_A^{\mathrm{sign}(D_1)}/\sqrt{|D_1|}}\frac{L(A^{(D_2)},1)}{\Omega_A^{\mathrm{sign}(D_2)}/\sqrt{|D_2|}} & \epsilon(A\otimes\mathfrak{X}_1)=-1,\ \mathfrak{X}_1\mathfrak{X}_2<0\end{cases}$$

$$\alpha_0=C'\cdot\begin{cases}\frac{(f,f)}{(\phi_A,\phi_A)}\frac{\Omega_A^{\mathrm{sign}(D_1)}\Omega_A^{\mathrm{sign}(D_2)}}{2\pi i\Omega_f^{\mathrm{sign}(D_1)}2\pi i\Omega_f^{\mathrm{sign}(D_2)}}\\ \frac{(f,f)\Omega_A^+\Omega_A^-}{\pi^3(\phi_A,\phi_A)}\\ \frac{(f,f)\Omega_A^+\Omega_A^-}{\pi^3(\phi_A,\phi_A)}\end{cases}$$ *with $C'\in\mathbb{Q}^\times$ an explicit constant (essentially) does not depends on D_i and $p|C'$ only if*

$$p|\prod_{q\in\Sigma\ or\ A/\mathbb{Q}_q\ bad}q(q^2-1)L_2(A^{(D_1)},1)L_2(A^{(D_2)},1).$$

Here $\Omega_A^\pm$ are Néron periods of A, Ω_f^ϵ are Shimura periods associated to f, $R_{A^{(D_1)}}$ is the regulator of $A^{(D_1)}$, ϕ_A is the weight 2 newform associated to A.

(b). $\theta_{\phi_{D_1}}^{\theta_{\phi_{D_1}}^{f_{D_1}}}\in\mathbb{C}\cdot f_{D_1}$ *and for any fundamental discriminant $D_1\in\mathfrak{X}_1$:*

$$\frac{(\theta_{\phi_{D_1}}^{f_{D_1}},\theta_{\phi_{D_1}}^{f_{D_1}})}{(f,f)}\cdot\begin{cases}\frac{\pi^3}{\Omega_A^{\mathrm{sign}(D_1)}}\\ \frac{\pi}{\Omega_A^{\mathrm{sign}(D_1)}}\\ \frac{\pi}{R_{A^{(D_1)}}\Omega_A^{\mathrm{sign}(D_1)}}\end{cases}=C\begin{cases}\frac{L(A^{(D_1)},1)}{\Omega_A^{\mathrm{sign}(D_1)}/\sqrt{|D_1|}}, & \epsilon(A\otimes\mathfrak{X}_1)=+1,\ \mathfrak{X}_1\mathfrak{X}_2>0\\ \frac{L(A^{(D_1)},1)}{\Omega_A^{\mathrm{sign}(D_1)}/\sqrt{|D_1|}}, & \epsilon(A\otimes\mathfrak{X}_1)=+1,\ \mathfrak{X}_1\mathfrak{X}_2>0,\\ \frac{L'(A^{(D_1)},1)}{R_{A^{(D_1)}}\Omega_A^{\mathrm{sign}(D_1)}/\sqrt{|D_1|}} & \epsilon(A\otimes\mathfrak{X}_1)=-1,\ \mathfrak{X}_1\mathfrak{X}_2<0\end{cases}$$

where $C\in\mathbb{Q}^\times$ an explicit constant (essentially) does not depends on D_i and $p|C$ only if

$$p|\prod_{q\in\Sigma\ or\ A/\mathbb{Q}_q\ bad}q(q^2-1)\cdot L_2(A^{(D_1)},1)L_2(A^{(D_2)},1)$$

To answer of the question 1.5 will follow from the above explicit formulae and p-integrality of two sides of (arithmetic) theta lifting. In fact, once have p-integrality of (arithmetic) theta lifting, the first formulae give relation between

$$\mu_p(\mathfrak{X}_1)+\mu_p(\mathfrak{X}_2)$$

and

$$\inf_{D_1\in\mathfrak{X}_1}\mathrm{ord}_p\theta_{D_1}.$$

And the second formulae gives lower bound of

$$\mu(\mathfrak{X}_1)$$

in terms of

$$\inf_{D_1\in\mathfrak{X}_1}\mathrm{ord}_p\theta_{D_1}.$$

Hence we have upper bound for

$$\mu_p(\mathfrak{X}_1)-\mu_p(\mathfrak{X}_2).$$

The p-integrality of (arithmetic) theta lifting will be considered in subsequent paper. See [25] for partial results on p-integrality of theta lifting in the square-free conductor case.

2. Local theory

There are two explicit realizations of local theta lifting for $(\mathrm{SO}_3,\widetilde{\mathrm{SL}_2})$. One involves doubling zeta integrals, the other involves toric models of representations of SO_3 and Whittaker models of representations of $\widetilde{\mathrm{SL}_2}$. The two realizations give two constructions of local inner product of local theta lifting. In this section, we first consider comparison of these two inner product formulae, which follows from relations among doubling zeta integrals, toric periods and Whittaker periods. The point is that the toric periods and Whittaker periods are relative easy to study than the doubling zeta integral, since their integral region have lower dimension. All these periods are related to special L-values. For arithmetic application, in the second part, we consider test vectors for these periods and their family behaviours under quadratic twists.

Notations. Let F be a local field of characteristic zero and let ψ be a non-trivial character of F. Choose Haar measure dx on F to be self-dual with respect to ψ. Denoted $d^\times x$ by the Haar measure on $F^\times$ defined by $d^\times x = L(1,1_F)\frac{dx}{|x|}$, where $|\cdot|$ is the normalized valuation on $F^\times$.

We will identify an algebraic group over F or its metaplectic covering with its F points. Let Q be a quadratic algebra or a quaternion algebra over F, or the trace free part of a quaternion algebra over F with quadratic form q given by the minus of the reduced norm. Fix Haar measure on dx on Q which is self-dual with respect to $\psi(\langle\ ,\ \rangle)$, where $\langle x,y\rangle = q(x+y)-q(x)-q(y)$ is the bilinear form associated to q. Fix Haar measure on $Q^\times$ defined by

$$d^\times x = \begin{cases} L(1,1_F)\frac{dx}{|q(x)|}, & \text{if } Q \text{ is a quadratic extension of } F\\ L(1,1_F)\frac{dx}{|q(x)|^2}, & \text{if } Q \text{ is a quaternion algebra.}\end{cases}$$

Take Haar measure on $F^\times\backslash Q^\times$ to be the quotient measure. For $G_0 := \mathrm{SL}_2$, we take Haar measure to be $\frac{dxdydz}{|x|}$ for the coordinate $\begin{pmatrix} x & y\\ z & *\end{pmatrix}$.

For $\delta\in F^\times$, let η_δ be the quadratic character of $F^\times$ corresponding to extension $F(\sqrt{\delta})/F$, let $\psi_\delta(\cdot) = \psi(\cdot)$.

2.1. Inner products on local theta lifting.

Let B be a quaternion algebra over F and $V = B^{\mathrm{tr}=0}$ with quadratic form given by the minus of the reduced norm. Let $H = \mathrm{PB}^\times$ and identified with $\mathrm{SO}(V)$ via its conjugate action on V. Let $G = \widetilde{\mathrm{SL}_2}$ be the metaplectic covering of SL_2. Let $(w_\psi, \mathcal{S}(V))$ be the Weil-representation of $H\times G$ associated to ψ. Here, if F is non-Archimedean, $\mathcal{S}(V)$ is the space of Schwartz functions on V and if F is Archimedean, $\mathcal{S}(V)$ is the Fock model related to ψ, which is a certain subspace of Schwartz functions on V and stable under Hecke algebra [37]. Let π be an unitary irreducible admissible representation of H and $(\ ,\ )$ be an invariant positive definite hermitian pairing on π. Then $(\ ,\ )$ is a basis of $\mathrm{Hom}_{\Delta H}(\pi\boxtimes\overline{\pi},\mathbb{C})$ [16], where $\Delta: H\to H\times H$ is the diagonal embedding. Let $\Theta = \theta_\psi(\pi)$ be the theta correspondence of π. Then Θ is an unitary irreducible admissible representation of G and Θ is the unique one such that

$$\mathrm{Hom}_{H\times G}(w_\psi, \pi\boxtimes\Theta)$$

is one dimensional ([37], [9]).

Remark 2.1. If $H = \mathrm{PGL}_2$ over F, we only consider unitary irreducible admissible representations that are infinitely dimensional and for $G = \widetilde{\mathrm{SL}_2}$, we only consider unitary irreducible admissible representations that are not even Weil representations. The reason is that these representations are enough for global application and their matrix coefficients have good properties such that certain linear functionals, like doubling zeta integral, are well defined.

In the following, we consider relations between two explicit constructions of inner products on local theta liftings, one construction is via Waldspurger's explicit local theta lifting [37] involve toric models on π and Whittaker models defined as below and the other is given by local doubling zeta integral.

Let $0 \neq x \in V$ and $\delta = q(x)$. Let $T \subset H$ be the stabilizer of x and $N = \left\{ n(y) = \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} \,\middle|\, y \in F \right\} \subset G_0 = \mathrm{SL}_2$ the unipotent subgroup viewed as a subgroup of G. We have that

$$\dim_{\mathbb{C}} \mathrm{Hom}_T(\pi, \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{Hom}_N(\Theta, \psi_\delta) \leq 1.$$

It was shown in [37] separately that there always exists δ such that $\mathrm{Hom}_T(\pi, \mathbb{C}) \neq 0$ (resp. $\mathrm{Hom}_N(\Theta, \psi_\delta) \neq 0$). Then π admits an unique model $\mathcal{V}_x$ contained in the space of functions on $T\backslash H$, called T-model, and Θ admits a unique model $\mathcal{W}_\delta$ contained in $\{f : G \to \mathbb{C} \,\big|\, \varphi(n(y)g) = \psi_\delta(y)\varphi(g)\}$, called ψ_δ whittaker model. For example, let $0 \neq P \in \mathrm{Hom}_T(\pi, \mathbb{C})$ be a basis, then $\mathcal{V}_x = \{P(\cdot f) \mid f \in \pi\}$, and similar for the construction of the Whittaker model.

From now on, assume that $\mathrm{Hom}_T(\pi, \mathbb{C}) \neq 0$ (equivalently, $\mathrm{Hom}_N(\Theta, \psi_\delta) \neq 0$).

From H to G:

We have $\mathcal{W}_\delta$ consists of following Whittaker functions constructed from local theta lifting

$$\Theta^f_\phi(g) = \int_{T\backslash H} w_\psi(g)\phi(h^{-1} \circ x)\overline{f(h)}dh, \qquad \phi \in \mathcal{S}(V), f \in \mathcal{V}_x, h \in H, g \in G$$

here $\circ$ means the conjugate action of H on V and $w_\psi(h)\phi(\cdot) = \phi(h^{-1} \circ \cdot)$. The integral is absolutely convergent [37]. Note the above local theta lifting depends on x and ψ.

Let $(\ ,\ )$ be an invariant positive definite hermitian pairing on Θ.

We have construct a basis

$$(\theta^{f_1}_{\phi_1}, \theta^{f_2}_{\phi_2})$$

of the one dimensional space

$$\mathrm{Hom}_{(H\times G)^2}(w_\psi \otimes \overline{\pi} \boxtimes (\overline{w_\psi} \otimes \pi), \Theta \boxtimes \overline{\Theta}) \bigotimes \mathrm{Hom}_{\Delta G}(\Theta \boxtimes \overline{\Theta}, \mathbb{C}),$$

depends on an identification of π and Θ with their models introduced as above. Here H acts trivially on Θ and G acts trivially on π.

One the other hand, we have another basis: Let $(\ ,\ )$ be the L^2 norm on $\mathcal{S}(V)$ with the measure introduced in Notations.

Lemma 2.2. *The doubling zeta integral*

$$Z(\phi_1, \phi_2, f_1, f_2) := \int_H (h\phi_1.\phi_2)\overline{(hf_1, f_2)}dh$$

is absolutely convergent and is a basis of

$$\mathrm{Hom}_{(H\times G)^2}(w_\psi \otimes \overline{\pi} \boxtimes (\overline{w_\psi} \otimes \pi), \Theta \boxtimes \overline{\Theta}) \bigotimes \mathrm{Hom}_{\Delta G}(\Theta \boxtimes \overline{\Theta}, \mathbb{C}).$$

Proof. The proof for the absolutely convergence is essential down in Lemma 9.5 (ii) of [10] by using estimation of matrix coefficients of π (for example, see [30]) and estimation of matrix coefficients of Weil representation (for example, see [22] for relevant discussion). Note that although the result in [10] is for tempered representation, but the result is more strong and the same analysis is enough for general unitary π to see the doubling zeta integral is absolutely convergent. Since under our convention (See Remark 2.1), the only non-tempered unitary representation is the non-tempered unitary principle series, which is very close to the related tempered unitary principle series.

Note that the doubling zeta integral lies in the space

$$\mathrm{Hom}_{(H\times G)^2, \Delta G}(w_\psi \otimes \overline{\pi} \boxtimes (\overline{w_\psi} \otimes \pi), \mathbb{C}).$$

Under our convention, the maximal π isotropic quotient of $\mathcal{S}(V)$ is isomorphic to $\Theta \boxtimes \pi$, thus

$$\mathrm{Hom}_{(H\times G)^2, \Delta G}(w_\psi \otimes \overline{\pi} \boxtimes (\overline{w_\psi} \otimes \pi), \mathbb{C})$$

equals to

$$\mathrm{Hom}_{(H\times G)^2}(w_\psi \otimes \overline{\pi} \boxtimes (\overline{w_\psi} \otimes \pi), \Theta \boxtimes \overline{\Theta}) \bigotimes \mathrm{Hom}_{\Delta G}(\Theta \boxtimes \overline{\Theta}, \mathbb{C})$$

and hence one dimensional. □

Now we consider relations between these two basis. It turns out that they are closely related to the following linear functionals.

- the one dimensional space $\mathrm{Hom}_{T^2}(\pi \boxtimes \overline{\pi}, \mathbb{C})$ has two generators

$$f_1(1)\overline{f_2(1)}, \qquad \alpha(f_1, f_2) := \int_T (tf_1, f_2)dt, \quad f_i \in \mathcal{V}_x.$$

- the one dimensional space $\mathrm{Hom}_{N^2}(\Theta \boxtimes \overline{\Theta}, \psi_\delta \boxtimes \psi_{-\delta})$ has generators

$$\varphi_1(1)\overline{\varphi_2(1)}, \qquad \beta(\varphi_1, \varphi_2) := \int_F (n(y)\varphi_1, \varphi_2)\psi_\delta(-y)dy, \quad n(y) = \left(\begin{pmatrix} 1 & y \\ & 1 \end{pmatrix}, 1\right) \in N \subset G, \quad \varphi_i \in \mathcal{W}_\delta.$$

Here we give several explainations of the linear function introduced above: (1) α is absolutely convergent[38]. (2) We view function $(n(\cdot)\varphi_1, \varphi_2)$ of F as a distribution on $\mathcal{S}(F)$. The $\psi_{-\delta}$ Fourier transformation of distribution $(n(\cdot)\varphi_1, \varphi_2)$:

$$\phi \mapsto \int_F (n(y)\varphi_1, \varphi_2) \int_F \phi(z)\psi_{-\delta}(zy)dzdy$$

is represented by a smooth function t_δ on $F^\times$ [27]. Define the Whittaker-Fourier period

$$\int_F (n(y)\varphi_1, \varphi_2)\psi_\delta(-y)dy := t_\delta(1).$$

The following result is proved via Harmonic analysis on toric and unipotent subgroup and together with above multiplicity one results.

Theorem 2.3. *For any $f_i \in \mathcal{V}_x$, $\varphi_i = \theta_{\phi_i}^{f_i} \in \mathcal{W}_\delta$ with $\phi_i \in \mathcal{S}(V)$, $i = 1, 2$, we have*

$$Z(\phi_1, \phi_2, f_1, f_2) = (\varphi_1, \varphi_2) \cdot \overline{\frac{\alpha(f_1, f_2)}{f_1(1)\overline{f_2(1)}}} \cdot \frac{\varphi_1(1)\overline{\varphi_2(1)}}{\beta(\varphi_1, \varphi_2)} \cdot |8\delta|^{1/2}$$

Remark 2.4. In fact, $\frac{\alpha(f_1,f_2)}{(f_1,f_2)}$ is closely related to $L(1/2, \pi_{F(\sqrt{\delta})})$, $\frac{\beta(\varphi_1,\varphi_2)}{(\varphi_1,\varphi_2)}$ is closely related to $L(1/2, \pi \otimes \eta_\delta)$ and $\frac{Z(\phi_1,\phi_2,f_1,f_2)}{(f_1,f_2)}$ is closely related to $L(1/2, \pi)$.

Denoted by $V_a = \{x \in V \mid q(x) = a\}$. There exists a unique H-invariant measure $d_a v$ on V_a for each $a \in F^\times$ such that for each $\phi \in \mathcal{S}(V)$,

$$\int_V \phi(v)dv = \int_F \int_{V_a} \phi(v)d_a v da.$$

For $y \neq 0$ and T_y be the stablizer of y, then under identification $T_y \backslash H \simeq V_{q(y)}$, the H invariant measure on $T_y \backslash H$ induced from $V_{q(y)}$ is $\frac{1}{|8q(y)|^{1/2}}$ times the quotient measure on $T_y \backslash H$ [40].

We have that the local theta lifting from H to G connects ψ_a quotient of w_ψ under action of N, with $\mathrm{Ind}_T^G 1_T$ in the following way:

Lemma 2.5. *The maximal quotient of $\mathcal{S}(V)$ such that N acts by ψ_a is $\mathcal{S}(V_a) \subset \mathrm{Ind}_T^G 1_T$ and equals to $c - \mathrm{Ind}_T^G 1_T$ if F is non-Archimedean.*

Define

$$\int_F Z(n(y)\phi_1, \phi_2, f_1, f_2)\psi_\delta(-y)dy$$

in the same way as β.

Proof of Theorem 2.3. By multiplicity one of $\mathrm{Hom}_{(H\times G)^2}(w_\psi \otimes \overline{\pi} \boxtimes (\overline{w_\psi} \otimes \pi), \Theta \boxtimes \overline{\Theta}) \bigotimes \mathrm{Hom}_{\Delta G}(\Theta \boxtimes \overline{\Theta}, \mathbb{C})$, we have that

$$(\varphi_1, \varphi_2)$$

and

$$Z(\phi_1, \phi_2, f_1, f_2).$$

are differed by a scalar. We now understand their relation via consider Fourier coefficients.

Denoted by $\delta_1 \in \mathcal{S}'(F)$ the delta distribution. We have for $\phi_n \to \delta_1$,

$$\begin{aligned}
&\int_F Z(n(y)\phi_1, \phi_2, f_1, f_2)\psi_\delta(-y)dy \\
&= \lim_{m\to\infty} \int_F Z(n(y)\phi_1, \phi_2, f_1, f_2) \int_F \phi_m(z)\psi_\delta(-zy)dzdy \\
&= \int_H \overline{(hf_1, f_2)} \lim_{m\to\infty} \int_F (n(y)h\phi_1, \phi_2) \int_F \phi_m(z)\psi_\delta(-zy)dzdydh
\end{aligned}$$

Note that the $\psi_{-\delta}$ Fourier transformation of distribution $(n(\cdot)\phi_1, \phi_2)$ is represented by continuous function $f(y) = \int_{V_{y\delta}} \phi_1(v)\overline{\phi_2(v)} d_{y\delta}v$ on $F^\times$. By estimation of matrix coefficients of weil representation, the function $\int_{V_a} \phi_1(v)\overline{\phi_2(v)} d_a v$ as function of $a \in F^\times$ is continuous and in $L^1(F)$ and its Fourier transformation is also in $L^1(F)$, thus the Fourier inverse formula holds. Thus

$$\int_F Z(n(y)\phi_1, \phi_2, f_1, f_2)\psi_\delta(-y)dy = \int_H \int_{V_\delta} h\phi_1(v)\overline{\phi_2(v)} d_\delta v \cdot \overline{(hf_1, f_2)} dh$$

Since $\int_T |(thf_1, h'f_2)|dt$ is moderate growth as function of $(h, h') \in (T\backslash H)^2$ and $h\phi_1(x)$, $h'\phi_2(x)$ is rapidly decay as function of h, h' respectively, the following integral is absolutely convergent

$$|8\delta|^{1/2} \int_{T\backslash H} \overline{(h'\phi_2)(x)} \int_{T\backslash H} (h\phi_1)(x) \int_T \overline{(thf_1, h'f_2)} dt dh dh' \quad (*)$$

and hence

$$\begin{aligned}(*) = & |8\delta|^{1/2} \int_H \int_{T\backslash H} (h'h\phi_1)(x)\overline{(h'\phi_2)(x)(hf_1, f_2)} dh' dh \\ = & \int_H \int_{V_\delta} h\phi_1(v)\overline{\phi_2(v)} d_\delta v \cdot \overline{(hf_1, f_2)} dh\end{aligned},$$

where extra factor $\frac{1}{|8\delta|^{1/2}}$ comes from the comparison between two H-invariant measures on $T\backslash H$.

One the other hand, by multiplication one of $\mathrm{Hom}_T(\pi, \mathbb{C})$, $\mathrm{Hom}_N(\Theta, \psi_\delta)$, exists nonzero c_1, c_2 such that

$$f_1(1)\overline{f_2(1)} \cdot c_1 = \alpha(f_1, f_2)$$

$$W_1(1)\overline{W_2(1)} \cdot c_2 = \beta(W_1, W_2).$$

It follows that

$$\int_F Z(n(y)\phi_1, \phi_2, f_1, f_2)\psi_\delta(-y)dy = \frac{|8\delta|^{1/2}\overline{c_1}}{c_2}\beta(\varphi_1, \varphi_2).$$

□

From G to H:

One may also consider the theta lifting from G to H. Similarly, $\mathcal{V}_x$ consists of functions

$$\theta_\phi^\varphi(h) = \int_{N\backslash G_0} w_\psi(g)\phi(h^{-1} \circ x)\overline{\varphi(g)} dg, \qquad \phi \in \mathcal{S}(V), \varphi \in \mathcal{W}_\delta, h \in H, g \in G_0 = SL_2$$

where the integral is absolutely convergent [37]. Moreover, the one dimensional space

$$\mathrm{Hom}_{(H\times G)^2}(w_\psi \otimes \overline{\Theta} \boxtimes (\overline{w_\psi} \otimes \Theta), \pi \boxtimes \overline{\pi}) \bigotimes \mathrm{Hom}_{\Delta H}(\pi \boxtimes \overline{\pi}, \mathbb{C})$$

has two generators

$$(\theta_{\phi_1}^{\varphi_1}, \theta_{\phi_2}^{\varphi_2}), \qquad Z(\phi_1, \phi_2, \varphi_1, \varphi_2) := \int_G (g\phi_1.\phi_2)\overline{(g\varphi_1, \varphi_2)} dg, \quad \varphi_i \in \mathcal{W}_\delta.$$

In fact, parallel to Lemma 2.2, we have

Lemma 2.6. *The doubling zeta integral $Z(\phi_1, \phi_2, \varphi_1, \varphi_2)$ is absolutely convergent and is a basis of*

$$\mathrm{Hom}_{(H\times G)^2}(w_\psi \otimes \overline{\Theta} \boxtimes (\overline{w_\psi} \otimes \Theta), \pi \boxtimes \overline{\pi}) \bigotimes \mathrm{Hom}_{\Delta H}(\pi \boxtimes \overline{\pi}, \mathbb{C}).$$

In the following, we compare these two basis.

The following lemma is parallel to Lemma 2.5, which connects the T invariant quotient of Weil representation with $\mathrm{Ind}_N^G \psi_{q(x)}$.

Lemma 2.7. *Fix $0 \neq x \in K^{\mathrm{tr}=0}$,*

$$\int_T (t\phi_1, \phi_2)dt = |2q(x)|^{1/2} \int_{N\backslash G_0} g\phi_1(x)\overline{g\phi_2(x)} dg.$$

Proof. The equality essentially follows from the Fourier inversion formulae.

The integral on both sides are absolutely convergent. Note that $N\backslash G_0 = T_{\mathrm{diag}}wN \sqcup T_{\mathrm{diag}}$ and the measure becomes $dg = |a|^{-3}dady$ on $N\backslash NT_{\mathrm{diag}}wN$, where $g = d(a)wn(y)$, $d(a) = \begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}$. We have

$$\begin{aligned}&\int_{N\backslash G_0} g\phi_1(x)\overline{g\phi_2(x)}dg\\ =&\int_{F^\times}\int_F |a|^3 d(a)wn(y)\phi_1(x)\overline{d(a)wn(y)\phi_1(x)}dyda \quad (1)\end{aligned}$$

Let da be the self-dual measure on F with respect to ψ, and du be the self-dual measure on $K^{\mathrm{tr}=0}$ with respect to $\psi\circ\langle\ ,\ \rangle|_{K^{\mathrm{tr}=0}}$, then for $u = a\cdot x \in K^{\mathrm{tr}=0}$,

$$da = \frac{1}{|2q(x)|^{1/2}}d'ax.$$

We have

$$(1) = \frac{1}{|2q(x)|^{1/2}}\int_{K^{\mathrm{tr}=0}}\int_F wn(y)\phi_1(u)\overline{wn(y)\phi_2(u)}dydu \quad (2).$$

Note that the fiber of $q: V\to F$ at a is V_a and the fiber of restriction map $V_a \to K^{\mathrm{tr}=0}$ at u is an orbit of T given by $(u, \iota(t)u_0) \in K^{\mathrm{tr}=0}\oplus K^\perp$, here $\iota(t) = t/\bar{t}$, $u_0 \in K^\perp$ is any element such that $q(u,\iota(t)u_0) = a$. We have decomposition of the measure on V: $dv = d_a v da = dtduda$.

Thus (2) becomes:

$$\begin{aligned}=&\frac{1}{|2q(x)|^{1/2}}\int_{K^{\mathrm{tr}=0}}\int_F\Bigg(\int_F\psi(a_1a)\int_{V_{a_1}}\psi(\langle u,v_1\rangle)\phi_1(v_1)d_{a_1}v_1da_1\cdot\\ &\overline{\int_F\psi(a_2a)\int_{V_{a_2}}\psi(\langle u,v_2\rangle)\phi_2(v_2)d_{a_2}v_2da_2}\Bigg)dadu, (3)\\ =&\frac{1}{|2q(x)|^{1/2}}\int_{K^{\mathrm{tr}=0}}\int_F\Bigg(\int_F\psi(a_1a)\int_{K^{\mathrm{tr}=0}}\psi(\langle u,u_1\rangle)\int_T\phi_1(u_1,\iota(t_1)u_{1,0})dt_1du_1da_1\cdot\\ &\overline{\int_F\psi(a_2a)\int_{K^{\mathrm{tr}=0}}\psi(\langle u,u_2\rangle)\int_T\phi_1(u_2,\iota(t_2)u_{2,0})dt_2du_2da_2}\Bigg)dadu, (4)\end{aligned}$$

where $u_{i,0}\in K^\perp$ such that $q(u_i,u_{i,0}) = a_i$. Now the key is to applying Plancherel identity for Fourier transformation that

$$\int_F\int_F\psi(-a_1a)f_1(a_1)da_1\overline{\int_F\psi(-a_2a)f_2(a_2)da_2}da = \int_y f_1(a)\overline{f_2}(a)da,$$

where the measure are self-dual with respect to ψ holds whenever $f_i\in L^1(F)\cap L^2(F)$, where da is the self-dual measure with respect to ψ. Since the $f_i(a,u) = \int_{V_a}\psi(\langle u,x\rangle)\phi_i(x)d_ax$, $a\in F$, $u\in K^{tr=0}$ is in $L^1(F)\cap L^2(F)$, we have

$$\begin{aligned}(3) =&\frac{1}{|2q(x)|^{1/2}}\int_{K^{\mathrm{tr}=0}}\int_F\int_F\psi(aa_1)f_1(a_1,u)da_1\int_F\overline{\psi(aa_2)f_1(a_2,u)}da_2dadu\\ =&\frac{1}{|2q(x)|^{1/2}}\int_{K^{\mathrm{tr}=0}}\int_F f_1(a,u)\overline{f_1(a,u)}dadydu\\ =&\frac{1}{|2q(x)|^{1/2}}\int_{K^{\mathrm{tr}=0}}\int_F\Bigg(\int_{K^{\mathrm{tr}=0}}\psi(\langle u,u_1\rangle)\int_T\phi_1(u_1,\iota(t_1)u_{1,0})dt_1du_1\cdot\overline{\int_{K^{\mathrm{tr}=0}}\psi(\langle u,u_2\rangle)\int_T\phi_2(u_2,\iota(t_2)u_{2,0})dt_2du_2}\Bigg)dadu\\ &(4)\end{aligned}$$

here $q(u_i,u_{i,0}) = a$. We have that the outer integral in (3) of $u\in K^{\mathrm{tr}=0}$ and $a\in F$ are commutes, and also note that the function $g_i(a,u_i) = \int_T\phi_i(u_i,\iota(t_1)u_{i,0})dt_i$, $q(u_i,\iota(t)u_{i,0}) = a$ of u_i are in $L^1(K^{\mathrm{tr}=0})\cap L^2(K^{\mathrm{tr}=0})$.

Thus

$$(4) = \frac{1}{|2q(x)|^{1/2}} \int_F \int_{K^{\mathrm{tr}=0}} \left(\int_{K^{\mathrm{tr}=0}} \psi(\langle u, u_1 \rangle) \int_T \phi_1(u_1, \iota(t_1) u_{1,0}) dt_1 du_1 \cdot \overline{\int_{K^{\mathrm{tr}=0}} \psi(\langle u, u_2 \rangle) \int_T \phi_2(u_2, \iota(t_2) u_{2,0}) dt_2 du_2} \right) duda$$

(By Plancherel identity)

$$= \frac{1}{|2q(x)|^{1/2}} \int_F \int_{K^{\mathrm{tr}=0}} \int_T \phi_1(u, \iota(t_1) u_0) dt_1 \cdot \overline{\int_T \phi_2(u, \iota(t_2) u_0) dt_2} duda$$

(absolutely convergent)

$$= \frac{1}{|2q(x)|^{1/2}} \int_T \left(\int_F \int_{K^{\mathrm{tr}=0}} \phi_1(u, \iota(t_1 t_2) u_0) \cdot \overline{\int_T \phi_2(u, \iota(t_2) u_0)} dudadt_2 \right) dt_1$$

$$= \frac{1}{|2q(x)|^{1/2}} \int_T (t\phi_1, \phi_2) dt$$

□

Theorem 2.8. *For any $\varphi_i \in \mathcal{W}_\delta$, $f_i = \theta_{\phi_i}^{f_i} \in \mathcal{V}_x$ with $\phi_i \in \mathcal{S}(V)$, $i = 1, 2$, we have*

$$Z(\phi_1, \phi_2, \varphi_1, \varphi_2) = (f_1, f_2) \cdot \frac{\overline{\beta(\varphi_1, \varphi_2)}}{\varphi_1(1)\overline{\varphi_2(1)}} \cdot \frac{f_1(1)\overline{f_2(1)}}{\alpha(f_1, f_2)} \cdot |2q(x)|^{1/2}$$

Proof of Theorem 2.8. In the same way as the proof of Theorem 2.3, we want to show the

$$\int_T \int_{G_0} (tg\phi_1, \phi_2)\overline{(g\varphi_1, \varphi_2)} dgdt = |2q(x)|^{1/2} \int_{N\backslash G_0} \overline{g'\phi_2(x)} \int_{N\backslash G_0} g\phi_1(x) \overline{\int_F (n(y)g\varphi_1, g'\varphi_2)\psi_\delta(-y)dy} dgdg' \quad (*),$$

here $\int_F (n(y)g\varphi_1, g'\varphi_2)\psi_\delta(-y)dy$ is the regularized Whittaker functional defined as before.

Then by multiplication one of $\mathrm{Hom}_T(\pi, \mathbb{C})$, $\mathrm{Hom}_N(\Theta, \psi_\delta)$, exists nonzero c_1, c_2 such that

$$f_1(1)\overline{f_2(1)} \cdot c_1 = \alpha(f_1, f_2)$$

$$\varphi_1(1)\overline{\varphi_2(1)} \cdot c_2 = \beta(\varphi_1, \varphi_2).$$

Thus the above formula becomes

$$\begin{aligned} & |2q(x)|^{1/2}\overline{c_2} \int_{N\backslash G_0} \overline{g'\phi_2(x)} \int_{N\backslash G_0} g\phi_1(x)\overline{\varphi_1(g)}\varphi_2(g')dgdg' \\ =& |2q(x)|^{1/2}\overline{c_2}\theta_{\phi_1}^{\varphi_1}(1)\overline{\vartheta_{\phi_2}^{\varphi_2}(1)} \\ =& \frac{|2q(x)|^{1/2}\overline{c_2}}{c_1} \int_T (tf_1, f_2)dt \end{aligned}$$

It follows that

$$Z(\phi_1, \phi_2, \varphi_1, \varphi_2) = |2q(x)|^{1/2} \frac{\overline{c_2}}{c_1} (f_1, f_2).$$

We now focus on the proof of $(*)$.

The integral

$$\int_T \int_{G_0} (tg\phi_1, \phi_2)\overline{(g\varphi_1, \varphi_2)} dgdt$$

is absolutely convergent and thus by Lemma 2.7 equals to

$$|2q(x)|^{1/2} \int_{G_0} \int_{N\backslash G_0} g'g\phi_1(x)\overline{g'\phi_2(x)} dg' \overline{(g\varphi_1, \varphi_2)} dg \quad (2).$$

The integral may not absolutely convergent, so we can not commutes the position of integral directly. Let K be the maximal compact subgroup of $G_0 = \mathrm{SL}_2$, then

$$\int_{N\backslash G_0} \int_K |g'kg\phi_1(x)\overline{g'\phi_2(x)(kg\varphi_1, \varphi_2)}| dkdg' < \infty.$$

Thus we can interchange the position of K and G_0:

$$\begin{aligned} & \int_K \left(\int_{F^\times} \int_K d(a)k'kg\phi_1(x)\overline{d(a)k'\phi_2(x)(kg\varphi_1, \varphi_2)} dk' \frac{da}{|a|^3} \right) dk (\text{let } k = k'^{-1}k, \text{ then}) \\ =& \int_K \left(\int_{F^\times} \int_K d(a)kg\phi_1(x)\overline{d(a)k'\phi_2(x)(kg\varphi_1, k'\varphi_2)} dk' \frac{da}{|a|^3} \right) dk \end{aligned},$$

here dk is the measure on K such that $\frac{da}{|a|^3}dk$ gives dg. Thus

$$(2) = |2q(x)|^{1/2}\int_{G_0}\int_{F^\times}\int_K d(a)g\phi_1(x)\overline{d(a)k'\phi_2(x)(g\varphi_1, k'\varphi_2)}dk'|a|^{-3}dadg$$
$$= |2q(x)|^{1/2}\int_{N\backslash G_0}\int_F\int_{F^\times}\int_K d(a)n(y)g\phi_1(x)\overline{d(a)k'\phi_2(x)(n(y)g\varphi_1, k'\varphi_2)}dk'|a|^{-3}dadydg$$
$$= \int_{N\backslash G}\int_F\int_{K^{\mathrm{tr}=0}}\psi(yq(u))\int_K g\phi_1(u)\overline{k'\phi_2(u)(n(y)g\varphi_1, k'\varphi_2)}dk'dudydg$$
$$= \int_{N\backslash G}\int_K\left(\int_F\left(\int_{K^{\mathrm{tr}=0}}\psi(yq(u))g\phi_1(u)\overline{k'\phi_2(u)}du\right)\overline{(n(y)g\varphi_1, k'\varphi_2)}dy\right)dk'dg \quad (3)$$

here $K^{\mathrm{tr}=0} = F(x)^{\mathrm{tr}=0}$ and du is the self-dual measure on $K^{\mathrm{tr}=0}$ with respect to $\psi(\langle\ ,\ \rangle_{K^{\mathrm{tr}=0}})$. Applying the Plancherel identity for quadratic Fourier transformation (For example, see Lemma 3.5 of [27]) for the integral in bracket

$$(3) = \int_{N\backslash G}\int_K\left(\int_{K^{\mathrm{tr}=0}} g\phi_1(u)\overline{k'\phi_2(u)}\overline{\left(\int_F\psi(-yq(u))(n(y)g\varphi_1, k'\varphi_2)dy\right)}du\right)dk'dg \quad (4)$$

Here

$$\int_F\psi(-yq(u))(n(y)g\varphi_1, k'\varphi_2)dydu$$

is defined by distribution as before. We have

$$(4) = |2q(x)|^{1/2}\int_{N\backslash G}\int_K\left(\int_{F^\times} d(a)g\phi_1(x)\overline{d(a)k'\phi_2(x)}\overline{\left(\int_F\psi(-ya^2q(x))(n(y)g\varphi_1, k'\varphi_2)dy\right)}|a|^{-3}da\right)dk'dg(u=ax)$$

and also note that

$$\int_K\int_{F^\times}\int_{N\backslash G}|d(a)g\phi_1(x)\overline{d(a)k'\phi_2(x)}||\overline{\left(\int_F\psi(-ya^2q(x))(n(y)g\varphi_1, k'\varphi_2)dy\right)}|dg|a|^{-3}dadk' < \infty,$$

Thus we can make the change of position of $N\backslash G$, K, $F^\times$ in the integral in (4), it follows that

$$(4) = |2q(x)|^{1/2}\int_{N\backslash G}\int_{N\backslash G} g\phi_1(x)\overline{g'\phi_2(x)}\overline{\int_N\psi(-y)(n(y)g\varphi_1, g'\varphi_2)dy}dgdg'.$$

We have proved $(*)$. ☐

Local index In the following, we consider index of local theta liftings and the relation of indexes defined by various local theta liftings.

Let π, Θ are unitary irreducible admissible representation of H, G respectively such that they are local theta correspondence to each other, i.e.

$$\mathrm{Hom}_{H\times G}(w_\psi, \pi\boxtimes\Theta) \neq 0.$$

Definition and Proposition 2.9.

(1) *(abstract theta lifting)An abstract theta lifting is an equivalent class of triple* $0 \neq (\theta_\cdot, (\ ,\ )_\pi, (\ ,\ )_\Theta)$ *with* $\theta_\cdot \in \mathrm{Hom}_{H\times G}(w_\psi, \pi\boxtimes\Theta)$ *and* $(\ ,\ )_\pi$, $(\ ,\ )_\Theta$ *are nontrivial Hermitian pairings on* π, Θ *respectively. The triple gives an explicit theta lifting*

$$\theta_\phi^f := (\theta_\phi, f)_{\pi_v}, \quad \theta_\phi^\varphi := (\theta_\phi, \varphi)_{\Theta_v}, \quad f\in\pi_v, \varphi\in\Theta_v.$$

Here two triples $((\theta_\cdot, (\ ,\ )_{\pi_v}, (\ ,\ )_{\Theta_v}))$, $((\theta'_\cdot, (\ ,\ )'_{\pi_v}, (\ ,\ )'_{\Theta_v}))$ *are equivalent if then gives same explicit theta lifting, i.e. exists* $a\in\mathbb{R}_+^\times$ *such that*

$$\theta'_\cdot = a\theta_\cdot, \quad (\ ,\ )'_{\pi_v} = a^{-1}(\ ,\ )_{\pi_v}, \quad (\ ,\ )'_{\Theta_v} = a^{-1}(\ ,\ )_{\Theta_v}$$

(2) *Called one dimensional spaces* $V_1 \subset \pi$, $V_2 \subset \Theta$, $W \subset w_\psi$ *self-reflex if each basis* $\theta_\cdot \in \mathrm{Hom}_{H\times G}(w_\psi, \pi\boxtimes\Theta)$ *maps* W *onto* $V_1\otimes V_2$.

(3) *(local see-saw and equality of local index on both sides)Under abstract theta lifting, the following see-saw identity holds:*

$$(f, \theta_\phi^\varphi) = (\varphi, \theta_\phi^f).$$

In particular, if $(V_1, V_2; W)$ is self-reflex, then local index with respect to local theta lifting associated to $(\theta., (\ ,\ )_\pi, (\ ,\ )_\Theta)$ is

$$\mathrm{Ind}(V_1, V_2; W) := \frac{(\theta_\phi^f, \theta_\phi^f)_\Theta}{(f,f)_\pi(\phi,\phi)} = \frac{(\theta_\phi^\varphi, \theta_\phi^\varphi)_\pi}{(\varphi,\varphi)_\Theta(\phi,\phi)}, \quad f \in V_1, \phi \in W, \varphi \in V_2,$$

which only depends on $(V_1, V_2; W)$.

The Waldspurger's explicit local theta lifting ${}_x\theta$ for both sides of theta lifting corresponds to an abstract theta lifting and the index with respect to $(V_1, V_2; W)$ is denoted by $\mathrm{Ind}_{q(x)}(V_2, V_2; W)$.

There is also another explicit local theta lifting defined by doubling zeta integral. The local doubling zeta integral for each sides gives an abstract theta lifting. For example, consider the doubling zeta integral

$$Z(\phi_1, \phi_2, f_1, f_2) := \int_H (h\phi_1, \phi_2)\overline{(hf_1, f_2)}dh, \quad \phi_i \in w_\psi, f_i \in \pi$$

from H to G. There is an abstract theta lifting $(\theta., (\ ,\ )_\pi, (\ ,\ )_\Theta)$ with $(\ ,\ )_\pi = (\ ,\ )$ on π such that the corresponding explicit theta lifting satisfies:

$$(\theta_{\phi_1}^{f_1}, \theta_{\phi_2}^{f_2})_\Theta = Z(f_1, f_2, \phi_1, \phi_2), \quad \phi_i \in w_\psi, f_i \in \pi.$$

Similar for the other direction.

The two abstract theta liftings obtained from two directions of local doubling zeta integral are in fact equivalent.

Let $m_i \in F^\times$ such that the Haar measure on H and $G_0 = \mathrm{SL}_2(F)$ introduced in notation is $|m_i|$ times the one in [15]. For Global application, these $|m_i|$ will harmless since the product measure will give the Tamagawa measure, since product of these measures induces Tamagawa measure if the groups are considered as defined over number field.

Proposition 2.10. *Assume F is real if it is Archimedean. There exists $\theta \in \mathrm{Hom}_{H\times G}(w_\psi, \pi \boxtimes \theta)$ and invariant Hermitian pairings $(\ ,\ )_\pi$, $(\ ,\ )_\Theta$ on π, θ respectively such that the corresponding local theta lifting*

$$\theta_\phi^f := (\theta_\phi, f)_\pi, \quad \theta_\phi^\varphi := (\theta_\phi, \varphi)_\Theta$$

satisfies

$$Z(\phi_1, \phi_2, f_1, f_2) = |m_1|(\theta_{\phi_1}^{f_1}, \theta_{\phi_2}^{f_2})_\Theta, \quad Z(\phi_1, \phi_2, \varphi_1, \varphi_2) = |2m_2| \cdot (\theta_{\phi_1}^{\varphi_1}, \theta_{\phi_2}^{\varphi_2})_\pi,$$

here we use same Hermitian pairing in local doubling zeta integral.

Remark 2.11. Using globalization methods and Rallis inner product formulae, the assumption on F is not necessary. The about proposition is a result of Qiu, which is a consequence of (1) of Theorem A in [30] together with Theorem C in [29] on equality of formal degrees under local theta correspondence. The key is that there is a doubling zeta integral involve matrix coefficients of w_ψ, π and θ. It connects with doubling zeta integrals for both directions of local theta liftings and the difference is given by formal degree of π and θ respectively.

Denoted by $\mathrm{Ind}_Z(V_1, V_2; W)$ for the theta lifting given by the above proposition, then

$$\mathrm{Ind}_Z(V_1, V_2; W) = |m_1|^{-1}\frac{Z(\phi,\phi,f,f)}{(f,f)(\phi,\phi)} = |2m_2|^{-1}\frac{Z(\phi,\phi,\varphi,\varphi)}{(\varphi,\varphi)(\phi,\phi)}, \quad 0 \neq f \in V_1, 0 \neq \varphi \in V_2, 0 \neq \phi \in w_\psi.$$

In the following theorem, identify π with its T_x models and identify Θ with its $\psi_{q(x)}$ models (related to $x \in V$).

Theorem 2.12. *Let*

$$(\theta, (\ ,\ )_\pi, (\ ,\ )_\Theta)$$

be abstract theta lifting such that Proposition 2.10, then

$$\left(\left|\frac{2q(x)}{m_1m_2}\right|^{-1/4}\theta, (\ ,\ )_\pi, (\ ,\ )_\Theta\right)$$

is a abstract theta lifting corresponds to Waldspurger explicit theta lifting ${}_x\theta$. In particular, for self-reflex $(V_1, V_2; W)$,

$$\mathrm{Ind}_Z(V_1, V_2; W) = \left|\frac{2q(x)}{m_1m_2}\right|^{1/2}\mathrm{Ind}_\delta(V_1, V_2; W).$$

Proof. Let $(\ \ )_\pi$ and $(\ ,\ )_\Theta$ be Hermitian pairing on π and Θ such that

$$\frac{|m_1|}{|2|}\left(\frac{f(1)\overline{f}(1)}{\int_{T_{\delta,v}}(tf_1,f_2)_\pi dt}\frac{\int_N(n\varphi,\varphi)_\Theta\psi_{\delta,v}(-n)dn}{\varphi_1(1)\overline{\varphi_2}(1)}\right)=|2m_2|\left(\frac{\int_{T_{\delta,v}}(tf_1,f_2)_\pi dt}{f_2(1)\overline{f_1}(1)}\frac{\varphi_1(1)\overline{\varphi_2}(1)}{\int_N(n\varphi_1,\varphi_2)_\Theta\psi_{\delta,v}(-n)dn}\right).$$

For example, we may take $(\ ,\ )_\pi$, $(\ ,\ )_\Theta$ with $\frac{\int_{T_{\delta,v}}(tf_1,f_2)_\pi dt}{f(1)\overline{f}(1)}=1$, and $\frac{\int_N(n\varphi_1,\varphi_2)_\Theta\psi_{\delta,v}(-n)dn}{\varphi_1(1)\overline{\varphi_2}(1)}=|4m_2/m_1|^{1/2}$.
denoted by A to be this nonzero number. Then by Theorem local period relation,

$$\frac{Z(\phi_1,\phi_2,f_2,f_2)}{(f_1,f_2)_\pi}=\left|\frac{2q(x)}{m_1m_2}\right|^{1/2}\cdot|m_1|\frac{({}_x\theta_{\phi_1}^{f_1},{}_x\theta_{\phi_2}^{f_2})_\Theta}{(f_1,f_2)_\pi}$$

$$\frac{Z(\phi_1,\phi_2,\varphi_1,\varphi_2)}{(\varphi_1,\varphi_2)_\Theta}=\left|\frac{2q(x)}{m_1m_2}\right|^{1/2}\cdot|2m_2|\frac{({}_x\theta_{\phi_1}^{\varphi_1},{}_x\theta_{\phi_2}^{\varphi_2})_\pi}{(\varphi_1,\varphi_2)_\Theta}$$

Let $(\theta,B(\ ,\ )_\pi,C(\ ,\ )_\Theta)$ be the abstract theta lifting given in Proposition 2.10, let $(a\theta,b(\ ,\ )_\pi,c(\ ,\ )_\Theta)$ be the explicit local theta lifting corresponding to explicit local theta lifting ${}_x\theta$, then

$$BC=\left|\frac{2q(x)}{m_1m_2}\right|^{1/2}a^2b^2$$

$$BC=\left|\frac{2q(x)}{m_1m_2}\right|^{1/2}a^2c^2.$$

Thus we may take $a=\left|\frac{2q(x)}{m_1m_2}\right|^{-1/4}$, $B=b$, $C=c$. □

For arithmetic applications we need to study test vectors for linear forms α, β, $\theta_\phi^f(1)$, Z for the direction from H to G, and even their explicit and family behaviour under quadratic twists. Since α is well studied in [3] and Z follows from other ones by the Theorem 2.3 above, in the following we will focus on the Whittaker functional $\theta_\phi^f(1)$ and the properties of β will follow as well.

2.2. Test vector for Whittaker functional of local theta lifting.

Notations. Denoted by ϖ a uniformizer of F. Define $V_\delta=\{x\in V\ \big|\ q(x)=\delta\}$, $\delta\in F$. We will add subscript x (resp. δ) for T, θ_f^ϕ (resp. α, β) and so on to emphasize the dependence on x (resp. $q(x)$).

Let π_0 be an unitary irreducible admissible representation of H. Let $\mathfrak{X}_1$, $\mathfrak{X}_2\subset F^\times$ be one of the following two type equivalent classes:

Case (I). The residue field of F has odd characteristic and exists a quadratic character η_{δ_0} either trivial or ramified such that $\pi_0\otimes\eta_{\delta_0}$ is unramified. Let $\mathfrak{X}_1=\mathfrak{X}_2=F^\times$.

Case (II). $\mathfrak{X}_i\subset F^\times$ is a coset of $F^\times/F^{\times 2}$ such that $\epsilon(\pi_0\otimes\eta_{\delta_1})\eta_{\delta_1}(-1)=\epsilon(B)\epsilon(\pi_0\otimes\eta_{\delta_2})\eta_{\delta_2}(-1)$, where $\epsilon(\pi_0\otimes\eta_{\delta_i})$ is the root number of $\pi_0\otimes\eta_{\delta_i}$ and $\epsilon(B)$ the Hasse invariant of B.

Fix a non-trivial additive character ψ_0 of F. Let $\pi=\pi_0\otimes\eta_{\delta_1}$ and $\psi=\psi_{0,\delta_1^{-1}}$ (depends on δ_1) for $\delta_1\in\mathfrak{X}_1$. It follows from the property of Waldspurger packet that $\theta:=\theta_\psi(\pi)$ only depends on $\mathfrak{X}_1$. By result of Tunnell-Saito, for each $x\in V_{\delta_1\delta_2}$ with $\delta_2\in\mathfrak{X}_2$, $T_x\subset H$ the group of stablizers of x,

$$\dim_{\mathbb{C}}\mathrm{Hom}_{T_x}(\pi,\mathbb{C})=1,(\text{equivalently }\dim_{\mathbb{C}}\mathrm{Hom}_N(\theta,\psi_{\delta_1\delta_2})=1).$$

Let $\delta_i\in\mathfrak{X}_i$ varies, in the following, we will give uniform construction of (f,ϕ) with f a test vector of $\mathrm{Hom}_{T_x}(\pi,\mathbb{C})$ and ${}_x\theta_\phi^f\in\mathcal{W}_{q(x)}$ is a test vector for $\mathrm{Hom}_N(\theta,\psi_{\delta_1\delta_2})$. Here a test vector for $\mathrm{Hom}_{T_x}(\pi,\mathbb{C})$ means a vector in π such that a basis of $\mathrm{Hom}_{T_x}(\pi,\mathbb{C})$ takes non-zero value on it, similar for $\mathrm{Hom}_N(\theta,\psi_{\delta_1\delta_2})$. We add a left-subscript x for local theta lifting θ_ϕ^f to emphasis the dependence of x. Furthermore,

- They varies uniformly under normalized linear forms associated to α, β, ${}_x\theta_f^\phi(1)$,
- In the case (II)., ${}_x\theta_\phi^f$ is not a test vector of ψ_δ Whittaker functional for any other coset of δ that different from $\mathfrak{X}_1\cdot\mathfrak{X}_2$.
- Theta lifting twice of f with respect to ϕ still lies in $\mathbb{C}\cdot f$,
- In the non-archimedean case and $\delta_1=1$, ϕ is closely related to lattice.
- In the archimedean case, they have good algebraic properties for algebraic regular representations.

Remark 2.13. In the case (II), $\mathcal{K} := F(x)$ only depends on $\mathfrak{X}_1$, $\mathfrak{X}_2$, we have the following relation on different choice of $x \in \mathcal{K}^{\mathrm{tr}=0}$:

For $u \in F^\times$, note that

$$_{ux}\theta_\phi^f(\cdot) = |u|^{-3/2}\chi_\psi(u)^{-1}{}_x\theta_\phi^f\left(\left(\begin{pmatrix} u & \\ & u^{-1}\end{pmatrix}, 1\right)\cdot\right),$$

where $|\cdot|$ is the normalized valuation on $F^\times$ and χ_ψ is the genuine character associated to ψ. We have isomorphism of G-models

$$\mathcal{W}_{q(x)} \simeq \mathcal{W}_{q(ux)}, \quad {}_x\theta_\phi^f \mapsto {}_{ux}\theta_\phi^f.$$

Test vector for toric linear form Let π_0, $\mathfrak{X}_1$, $\mathfrak{X}_2$ be as before. For $\delta_i \in \mathfrak{X}_i$, let $K \subset B$ be subalgebra isomorphic to $F(\sqrt{\delta_1\delta_2})$ and let $T = K^\times/F^\times$.

In the following, we will construct a one dimensional space $V(\pi_0, \mathfrak{X}_1, \mathfrak{X}_2) \subset \pi_0$ depends on a certain compact subgroup of B such that for each $\delta_1 \in \mathfrak{X}_1$, a nonzero element in $V(\pi_0, \delta_1, \mathfrak{X}_2) := V(\pi_0, \mathfrak{X}_1, \mathfrak{X}_2) \otimes \eta_{\delta_1}$ is a test vector for $\mathrm{Hom}_T(\pi, \mathbb{C})$ for all $\delta_2 \in \mathfrak{X}_2$ whenever K has good relative position with the compact subgroup. And we also consider uniform bahaviour of these test vectors under α.

The construction is essential a special case of Cai-Shu-Tian's test vector theory we now recall: Let π' be an unitary admissible irreducible representation of H. Let $K' \subset B$ a quadratic subalgebra and $T' = K'^\times/F^\times \subset H$. Let χ' a quadratic character of T' comes from base change of a quadratic character of $F^\times$. Assume $\mathrm{Hom}_{T'}(\pi', \chi') \neq 0$. Let $n' \in \mathbb{Z}_{\geq 0}$ be the exponential conductor of Jacquet-Langlands correspondent of π', let $c' \in \mathbb{Z}_{\geq 0}$ be the exponential conductor of χ' and $\mathcal{O}_{\chi'}$ the order of K' with conductor c'.

Proposition 2.14. [3] *The following space is one dimensional and any nonzero element is a test vector for* $\mathrm{Hom}_{T'}(\pi', \chi')$.

- *F is non-Archimedean. Let R' be an admissible order for (π', χ') in the sense of* [3] *with discriminant equals to n' and $R' \cap K' = \mathcal{O}_{\chi'}$.*
  (a') *Assume π' is unramified, or K' splits, or $c' \geq n'$,*
  $$V(\pi', \chi') := \pi'^{R'^\times}.$$
  (b') *If K' is nonsplit and $0 = c' < n'$,*
  $$V(\pi', \chi') := \pi'^{\chi'} \subset \pi'^{R'^\times}.$$
- *F is Archimedean*
  (c') *Let U' be a maximal compact subgroup of H such that $U' \cap T'$ is the maximal compact subgroup of T'. Let*
  $$V(\pi', \chi') := \left\{f \in \pi' \;\middle|\; T' \cap U' \text{ acts by } \chi' \text{ and weight is minimal}\right\}.$$

Now come back to our situation.

Let $x \in V_{\delta_1\delta_2}$, $K = F(x)$, $\chi = \eta_{\delta_1} \circ N_{K/F}$. Observe that f is test vector for $\mathrm{Hom}_{T_x}(\pi, \mathbb{C})$ is equivalent to $f \otimes \eta_{\delta_1}$ is test vector for $\mathrm{Hom}_{T_x}(\pi_0, \chi)$ and furthermore

$$\alpha_{q(x)}(f_1, f_2) = \alpha_\chi(f_1 \otimes \eta_{\delta_1}, f_2 \otimes \eta_{\delta_1}) := \int_{T_x} (t(f \otimes \eta_{\delta_1}), f \otimes \eta_{\delta_1})\chi(t)dt, \quad f_i \in \pi$$

where we choose inner product on π_0 and π such that

$$(f_1, f_2) = (f_1 \otimes \eta_{\delta_1}, f_2 \otimes \eta_{\delta_1}), \quad f_i \in \pi.$$

It is enough to construct test vector for $\mathrm{Hom}_{T_x}(\pi_0, \chi) \neq 0$.

Let $\chi_1 = \eta_{\delta_1\delta_0} \circ N_{K/F}$, where we choose $\mathrm{ord}_F(\delta_0) = 1$ in the case (I). and η_{δ_0} ramified, and $\delta_0 = 1$ otherwise. Denoted by $\mathcal{O}$ (resp. $\mathcal{O}_K$) the ring of integers of F (resp. K) if F is non-archimedean. Let n be the exponential conductor of the Jacquet-Langlands correspondent of π_0, and let c (resp. c_1) be exponential conductor of χ (resp. χ_1) and $\mathcal{O}_\chi \subset \mathcal{O}_K$ (resp. $\mathcal{O}_{\chi_1} \subset \mathcal{O}_K$) be the order with conductor c (resp. c_1).

Theorem 2.15. *The following space $V(\pi_0, \mathfrak{X}_1, \mathfrak{X}_2) \subset \pi_0$ is one dimensional and nonzero vectors in it is test vectors for nonzero linear form in $\mathrm{Hom}_{T_x}(\pi_0, \chi)$ whenever K and R has relative position below:*

Case (I). *Let R be a maximal order of B. Let*

$$V(\pi_0, \mathfrak{X}_1, \mathfrak{X}_2) := V(\pi_0 \otimes \eta_{\delta_0}, \chi_1) \otimes \eta_{\delta_0} = \{f \in \pi_0 \mid R^\times \text{ acts via } \eta_{\delta_0} \circ \det\}.$$

Let $\delta_i \in \mathfrak{X}_i$ and $x \in V_{\delta_1\delta_2}$ such that $\mathcal{O}_{\chi_1} = K \cap R$.

Case (II). *Fix $x \in V_{\delta_1\delta_2}$, then $\mathcal{K} := F(x)$, T_x only depend on $\mathfrak{X}_i$.*

(a). *If F is non-Archimedean such that either $\mathcal{K}$ splits or $c \geq n$,*

$$V(\pi_0, \mathfrak{X}_1, \mathfrak{X}_2) := V(\pi_0, \chi) = \pi_0^{R^\times}.$$

where R is an admissible order for (π_0, χ) in the sense of [3] with discriminant equals to n and $R \cap \mathcal{K} = \mathcal{O}_\chi$.

(b). *If F is non-Archimedean, $\mathcal{K}$ is a field and $c < n$. Let R be an order with discriminant equals to conductor of π and $R \cap \mathcal{K} = \mathcal{O}_\mathcal{K}$.*

(i) *If π is ramified,*

$$V(\pi_0, \mathfrak{X}_1, \mathfrak{X}_2) := V(\pi_0, \chi) = V(\pi, 1_\mathcal{K}) \otimes \eta_{\delta_1} = \left\{f \in \pi_0 \;\middle|\; T_x \text{ acts by } \chi\right\}.$$

We have $R^\times$ acts by $\eta_{\delta_1} \circ \det$.

(ii) *If π is unramified,*

$$V(\pi_0, \mathfrak{X}_1, \mathfrak{X}_2) := V(\pi, 1_\mathcal{K}) \otimes \eta_{\delta_1} = \left\{f \in \pi_0 \;\middle|\; R^\times \text{ acts by } \eta_{\delta_1} \circ \det\right\}.$$

(c) *If F is Archimedean, let U be a maximal compact subgroup of H such that $U \cap T_x$ is the maximal compact subgroup of T_x. Let*

$$V(\pi_0, \mathfrak{X}_1, \mathfrak{X}_2) := V(\pi_0, \chi) = V(\pi, 1_K) \otimes \eta_{\delta_1} = \left\{f \in \pi_0 \;\middle|\; T_x \cap U \text{ acts by } \chi \text{ and weight is minimal}\right\}.$$

Remark 2.16. In the case (I). and η_{δ_0} is ramified, we modified the Cai-Shu-Tian test vector for (π_0, χ) such that is does not depends on different δ_i. In the case (II).(b).(ii)., we modified the Cai-Shu-Tian test vector so that the level of test vector is clear. These modifications are convenient for our purpose.

From a nonzero vector f_0 in $V(\pi_0, \mathfrak{X}_1, \mathfrak{X}_2)$, we get test vectors $f := f_0 \otimes \eta_{\delta_1} \in V(\pi_0, \delta_1, \mathfrak{X}_2) := V(\pi_0, \mathfrak{X}_1, \mathfrak{X}_2) \otimes \eta_{\delta_1}$ for $\mathrm{Hom}_{T_x}(\pi, \mathbb{C})$ as $\delta_i \in \mathfrak{X}_i$ varies so that $K = F(x)$ has good relative position with R as in Theorem 2.15. We have the following uniform property of

$$\alpha^0_{q(x)}(f) = \frac{L(1, \eta_{q(x)})^2 L(1, \pi, ad)}{L(2, 1_F) L(1/2, \pi_{F(x)})} \frac{\int_{T_x} (tf, f) dt,}{(f, f)}$$

for the non-Archimedean case as $\delta_i \in \mathfrak{X}_i$ varies,

$$\alpha^0_{q(x)}(f)|D\delta|^{-1/2} \cdot |c_{\delta_1}|^{-1/2} = \begin{cases} L(1, \eta_{\delta_1\delta_2})^2 |\varpi^{c_1}|, & \text{Case (I) } c_1 > 0, \\ 1, & \text{Case (I) } c_1 = 0, \text{ or Case (II).(a). } c = n = 0, \\ L(1, \eta_{\delta_1\delta_2})^2 |\varpi^c|, & \text{Case (II).(a). } c > 0 \text{ and } n = 0, \\ \frac{L(1,1_F)}{L(2,1_F)} L(1, \pi, ad)^{\delta_{\pi_0}}, & \text{Case (II).(a). } c = 0,\ n > 0,\ K \text{ splits}, \\ L(1, \eta_{\delta_1\delta_2})^2 |\varpi^c| \frac{L(1,1_F)}{L(2,1_F)} \frac{L(1,\pi,ad)^{\delta_{\pi_0}}}{L(1/2,\pi_K)}, & \text{Case (II).(a). } cn > 0, \\ e(1 - |\varpi|^e) \frac{L(1,\pi,ad)}{L(1/2,\pi_K)}, & \text{Case (II).(b).(i)}, \\ 1, & \text{Case (II).(b).(ii)}, \end{cases}$$

where $D\mathcal{O}_F$ is the relative discriminant of K/F, $\delta\mathcal{O}_F$ is the different ideal of F, $\delta_{\pi_0} = \begin{cases} 0, & \pi_0 \simeq \mathrm{St}(\mu) \text{ with } \mu \text{ unramified} \\ 1, & \text{otherwise}, \end{cases}$

c_{δ_1} such that $\psi(c_{\delta_1}^{-1} \cdot)$ is the standard additive character on F. e is the ramification index of $\mathcal{K}/F$.

Archimedean case is similar.

Test vector for Whittaker functional: non-Archimedean

Let R be as in Theorem 2.15 and $0 \neq f \in V(\pi_0, \delta_1, \mathfrak{X}_2) \subset \pi$ be as above. In the following, identify π with its T_x-models. We will construct $\phi \in \mathcal{S}(V)$ for each $\delta_i \in \mathfrak{X}_1$ from certain twist of ϕ_0 by η_{δ_1} in the case (I) such that for each $x \in V_{\delta_1\delta_2}$, $\delta_2 \in \mathfrak{X}_2$ such that whenever $K = F(x)$ has good relative position with R, ${}_x\theta^\phi_f(1) \neq 0$ and has further good and uniform properties introduced at the begging of this section.

Consider the following of choice of Schwartz function ϕ:

(I) Assume exists a quadratic character η_{δ_0} either trivial or ramified such that $\pi_0 \otimes \eta_{\delta_0}$ is unramified., and characteristic of F is odd, let $\phi_0 = 1_L$ with $L = R^{\mathrm{tr}=0}$. If π is unramified, let $\phi = \phi_0$. Fix $R \simeq M_2(\mathcal{O}_F)$. If π is ramified, let

$$\phi \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = \begin{cases} \eta_{\delta_1\delta_0}(-y) \quad (\text{resp. } \eta_{\delta_1\delta_0}(z)) & \text{if } g = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \in M_2(\mathcal{O}), \ \mathrm{ord}(\det(g)) \geq 1, \text{ and} \\ & \mathrm{ord}(y) = 0, \ \mathrm{ord}(z) > 0 \ (\text{resp. } \mathrm{ord}(z) = 0, \ \mathrm{ord}(y) > 0) \\ 0, & \text{otherwise.} \end{cases}$$

There is an equivalent description of ϕ in the case π ramified: $\mathrm{supp}(\phi) = R^\times \circ \begin{pmatrix} & \varpi\mathcal{O}_F \\ 1 & \end{pmatrix}$ and $\phi\left(r \circ \begin{pmatrix} & b \\ 1 & \end{pmatrix}\right) = \eta_{\delta_1\delta_0} \circ \det(r)$ for $r \in R^\times$ and $b \in \varpi\mathcal{O}_F$.

(II) In the case (II) of Proposition 2.15:

First consider the following case:

(*) π is ramified, and either $\mathcal{K}$ is a field and $c < n$, or $\mathcal{K}$ is split and $c = 0$.

Proposition 2.17 (Local obstruction)**.** *Under assumption* $(*)$ *We have* $V_{\mathcal{K}}(\pi)$ *is eigen under* $j \in B$ *with* $jkj^{-1} = \overline{k}$, $k \in \mathcal{K}$. *The action of* j *on* $V_{\mathcal{K}}(\pi)$ *is given by* $\epsilon(\pi)\epsilon(B)$

Proof. For K nonsplit, it follows from Theorem 4 of [24] and Tunnell-Saito condition; For v split, it follows from Theorem 3.2.2 of [32]. □

Suppose that $(*)$ holds and $\epsilon(\pi) = \epsilon(B)$, let $\boldsymbol{\delta} \subset \mathcal{O}_F$ be the ideal given by product of relative discriminants of quadratic field $\mathcal{K}_1$, $\mathcal{K}_2$, where $\mathcal{K}_i = F(\sqrt{\delta_i})$ with $\delta_i \in \mathfrak{X}_i$ only depends on $\mathfrak{X}_i$. Let

$$\phi(h \circ x) = \eta_{\delta_1} \circ \det(h) 1_{R^\times}(h) 1_{\mathcal{K}^{\mathrm{tr}=0}|_{q(\cdot)\in\boldsymbol{\delta}}}(x).$$

We will see later that ϕ is closely related a lattice relative to R whenever $\delta_1 = 1$, which will apply to get Tunnell type result.

In general, we introduce orientation.

Let $\mathcal{K}^o \subset \mathcal{K}^{\mathrm{tr}=0}|_{q(\cdot)\in\boldsymbol{\delta}}$ be an open compact subset given by

$$\begin{cases} \mathcal{K}^{\mathrm{tr}=0}|_{q(\cdot)\in\boldsymbol{\delta}}, & \text{if } \epsilon(\pi) = \epsilon(B), \ (*) \text{ holds} \\ \mathcal{K}^{\mathrm{tr}=0}|_{q(\cdot)\in\boldsymbol{\delta}} = \mathcal{K}^o \sqcup -\mathcal{K}^o, & \text{otherwise} \end{cases},$$

and $L^o = R^\times \circ \mathcal{K}^o$. Let ϕ be support on L^o given by

$$\phi(h \circ x) = \begin{cases} 1, & \text{case (II). (b).} \\ \eta_{\delta_1} \circ \det(h), & \text{case (II). (a).} \end{cases}$$

where $h \in R^\times, x \in \mathcal{K}^o$.

Theorem 2.18. *Let* ϕ *be the schwartz function as above, then for any* $\delta_i \in \mathfrak{X}_i$ *such that* $(\delta_1\delta_2) = D_{F(\delta_0\delta_1)/F} D_{F(\delta_0\delta_2)/F}$ *(here* $\delta_0 = 1$ *in case (II)) and* $x \in V_{\delta_1\delta_2}$ *such that* $K = F(x)$ *satisfies in the case (I),* $K \cap R = \mathcal{O}_{\chi_1}$ *and in the case (II)* $x \in \mathcal{K}^o$*, the following hold:*

$${}_x\theta_\phi^f(1)/\overline{f(1)} = \varepsilon \mathrm{vol}(R^\times, T_x \backslash H),$$

where $\epsilon = \begin{cases} 2, & \epsilon(\pi) = \epsilon(B), \ (*) \text{ holds and } R^\times \cap NK^\times \subset K^\times \\ 1, & \text{other wise} \end{cases}$*, where* $NK^\times$ *is the normalizer of* K *in* B*. Furthermore, twice theta lifting of* f *with respect to* ϕ *lines in* $\mathbb{C} \cdot f$*.*

Lemma 2.19 (Relation to lattice)**.** *Let* R *be an order containing* $\mathcal{O}_K$ *with discriminant* D_R*. let* $R^o \subset R$ *containing* $\mathcal{O}_K$ *be maximal such that for* $L = (\mathcal{O}_F + 2R^o)^{\mathrm{tr}=0}$*, we have*

$$L^{q=q(x)} = R^\times \cdot \{\pm x\}$$

for any $x \in K^{\mathrm{tr}=0}$ *such that* $(q(x)) = D_{K/F}$*. In fact, the discriminant of* R^0 *is* $D_R(D_R/D_B, D_{K/F})$*.*

Proof. We first give explicit description of R. There exists $j \in B^{\mathrm{tr}=0}$ such that $jkj^{-1} = \overline{k}$ for $\forall k \in K$ and

$$\mathrm{ord}(N(j)) = \begin{cases} \mathrm{ord}(D_B), & \text{if } K/F \text{ is inert,} \\ 0, & \text{if } K/F \text{ is not inert,} \end{cases}$$

If B is split and K is ramified, we further choose $j^2 = 1$; If $\mathrm{Char}(\mathcal{O}/\varpi) = 2$, B is ramified and K is ramified, we further choose $j^2 \equiv 1 \pmod{D_{K/F}/\varpi_F}$ not lies in norm of $K^\times$; If K is split, we further choose $j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to make the following holds. We have maximal order of B is given by

$$\mathcal{O}_B = \begin{cases} \mathcal{O}_K + \mathcal{O}_K j & , \quad \text{if } K \text{ is not ramified,} \\ \mathcal{O}_K + (D_B/D_{K/F})^{\frac{1}{2}} \mathcal{O}_K (1+j) & , \quad \text{if } K \text{ is ramified.} \end{cases}$$

and

$$R = \begin{cases} \mathcal{O}_K + (\mathrm{N}/D_B)^{\frac{1}{2}} \mathcal{O}_K j & , \quad \text{if } K/F \text{ is inert,} \\ \mathcal{O}_K + (\mathrm{N}/D_{K/F})^{\frac{1}{2}} \mathcal{O}_K (1+j) & , \quad \text{if } K \text{ is ramified,} \\ \mathcal{O}_K \oplus j \begin{pmatrix} \varpi^{\mathrm{ord}(\mathrm{N})} & 0 \\ 0 & 1 \end{pmatrix} \mathcal{O}_K = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathrm{N} & \mathcal{O} \end{pmatrix} & , \quad \text{if } K \text{ is split and } K \text{ diagonal embedded into } B \simeq M_2(F). \end{cases}$$

The result follows from explicit calculation based on the following fact:

Fact 2.20. *Let $j \in B^{\mathrm{tr}=0}$ such that $jkj^{-1} = \overline{k}$ for $k \in K$. Let $\mathcal{O}_{K,n} = \{x \in K \mid N(x) \in \varpi^n \mathcal{O}\}$. We have*

$$B^\times = (1 + j\mathcal{O}_{K,0})^{q(\cdot)\neq 0} K^\times \bigsqcup \left(1 + j\frac{\mathcal{O}_{K,1}}{N(j)}\right)^{q(\cdot)\neq 0} jK^\times.$$

□

Test vector for Whittaker functional: Archimedean

Let's recall some basic theory on Whittaker functions of an unitary irreducible admissible representation θ.

Let $\psi(\cdot) = e^{2\pi i \mathrm{tr}_{F/\mathbb{R}}(c\cdot)}$ be a character of F such that ψ-th Whittaker model of θ exists.

Assume first $F = \mathbb{R}$. Consider the complexified Lie algebra of $\mathfrak{g}_\mathbb{C}$ of G. The center of its universal enveloping algebra is generated by the Casimir element D. Denote λ be the eigen value of D on θ. By admissibility of θ, for each weight n appears in θ, the weight n vector is one dimensional. Choose $0 \neq W_n$ to be weight n. By Iwasawa decomposition, W_n is determined by

$$\varphi_n(t) := W_n\left(\begin{pmatrix} t^{1/2} & \\ & t^{-1/2} \end{pmatrix}, 1\right), t > 0.$$

Recall $\mathfrak{g} = \mathfrak{sl}_{2,\mathbb{C}} = M_2(\mathbb{C})^{\mathrm{tr}=0}$ and $\gamma \in M_2(\mathbb{R})$ acts on an elements in Whittaker model by

$$\gamma W(g, \epsilon) = \left(\frac{dt}{t} W((g,\epsilon)(e^{t\gamma}, 1))\right)|_{t=0}.$$

It follows from equation [16]

$$DW_n = \lambda W_n$$

that

$$\varphi_n(t)'' = \left(4\pi^2 c^2 - \frac{2\pi c n}{t} + \frac{\lambda}{2t^2}\right)\varphi_n(t), \quad (1).$$

Remark 2.21. Recall the classification of θ for $F = \mathbb{R}$:

Recall we have induced representation $\mathrm{Ind}(|t|^s\chi)$ consists of $\widetilde{\mathrm{SO}_2(\mathbb{R})}$-finite functions on G such that

$$f\left(\left(\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}, \epsilon\right) g\right) = \chi\left(\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}, \epsilon\right) |t|^{s+1} f(g),$$

where χ is the genuine character on $\left\{\left(\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}, \{\pm\}\right) \mid t \in \mathbb{R}^\times\right\}$ which factor through $\{(\pm 1, \pm 1)\}$ determined by

$$\chi\left(\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, 1\right) = e^{i\nu\pi},$$

$\nu \in \{\pm\frac{1}{2}\}$. May assume $\mathrm{Re}(s) \geq 0$. Now the irreducible representations are given by:

- Principle series: $\widetilde{\pi}(|t|^s\chi) = \mathrm{Ind}(|t|^s\chi)$, where $s \in i\mathbb{R}$, or $s \in (-\frac{1}{2}, 0) \sqcup (0, \frac{1}{2})$, consists of weight $\nu + 2\mathbb{Z}$.

- Holomorphic discrete series: $\widetilde{\sigma}(|t|^{k-1}\chi) \subset \mathrm{Ind}(|t|^{k-1}\chi)$, where $k-1 \in -\nu + 2\mathbb{Z}$, and consists of weight $k + 2\mathbb{Z}_{\geq 0}$ vectors. Simply denoted by $\widetilde{\sigma}_k^+$.
- Antiholomorphic discrete series: $\widetilde{\sigma}(|t|^{k-1}\chi) \subset \mathrm{Ind}(|t|^{k-1}\chi)$, where $k-1 \in \nu + 2\mathbb{Z}$, and consists of weight $k + 2\mathbb{Z}_{\leq 0}$ vectors. Simply denoted by $\widetilde{\sigma}_k^-$.

We have $\lambda = \frac{s^2-1}{2}$ if θ is an irreducible subrepresentation of $\widetilde{\pi}(|t|^s\chi)$. We have θ coming from theta lifting of algebraic regular representation if and only if $\theta = \widetilde{\sigma}_k^\epsilon$. If $n = k$ is the highest weight vector (resp. lowest weight vector) in $\widetilde{\sigma}_k^\epsilon$, the second order differential equation (1) degenerates to first order differential equation

$$2t\varphi_k'(t) = (-2\pi ct + \epsilon k)\varphi_k(t).$$

This can also be read from the weight lowing (resp, weight raising) operator,

We have similar differential equation for $F = \mathbb{C}$. Identify the complexified Lie-algebra $\mathfrak{g}_{\mathbb{C}}$ of G with $\mathfrak{sl}_{2,\mathbb{C}} \oplus \mathfrak{sl}_{2,\mathbb{C}}$ so that $\mathfrak{g}$ consists of $X \oplus \overline{X}$. In this case, the center of the universal enveloping algebra of complexified Lie algebra of G is generated by $D \otimes 1$ and $1 \otimes D$. Denote by λ_+, λ_- be the eigen value of $D \otimes 1$, $1 \otimes D$ on θ respectively. Let n be a SU_2 type appears in θ, i.e. exist irreducible SU_2 representation isomorphic to $\{\mathbb{C}X^iY^{n-i} \mid 0 \leq i \leq n\}$, where action is given by $gP(X,Y) = P((X,Y)g)$. Each type appears at most once in θ by the admissibility. Denote $W_{n,k}$ be the element corresponding to $X^{n/2+k}Y^{n/2-k}$, $-\frac{n}{2} \leq k \leq \frac{n}{2}$, $k \equiv \frac{n}{2} \pmod 2$. By Iwasawa decomposition again, W_n and $W_{n,k}$ are determined by their restriction on $\left\{ \left(\begin{pmatrix} t^{1/2} & \\ & t^{-1/2} \end{pmatrix}, 1 \right) \;\middle|\; t \in \mathbb{R}_{>0} \right\}$, denoted by $\varphi_n(t)$, $\varphi_{n,k}(t)$. We have [16], [23]

$$\varphi_{n,\epsilon\frac{n}{2}}''(t) - (1+n)\frac{\varphi_{n,\epsilon\frac{n}{2}}'}{t} - \left(16\pi^2|c| + \frac{2\lambda_\epsilon + 1 - (1+\frac{n}{2})^2}{t^2}\right)\varphi_{n,\epsilon\frac{n}{2}}, \quad (2).$$

And all other $\varphi_{n,k}$ can be find recursively by the following:

$$\varphi_{n,k}'' - (1-2k)\frac{\varphi_{n,k}'}{t} - (16\pi^2|c| + \frac{2\lambda_+ + 1 - (1-k)^2}{t^2})\varphi_{n,k}(t) = -8\pi i(n/2+k)\frac{\varphi_{n,k-1}}{t}$$

$$\varphi_{n,k}'' - (1+2k)\frac{\varphi_{n,k}'}{t} - (16\pi^2|c| + \frac{2\lambda_- + 1 - (1-k)^2}{t^2})\varphi_{n,k}(t) = 8\pi i(n/2-k)\frac{\varphi_{n,k-1}}{t}.$$

Remark 2.22. We have classification of θ for $F = \mathbb{C}$:

- $\widetilde{\pi}(|z|^s(z/\sqrt{|z|})^m)$, where $s \in i\mathbb{R}$, $m \in \mathbb{Z}$, or $s \in (-\frac{1}{2}, 0) \sqcup (0, \frac{1}{2})$ and $m = 0$: coming from restriction of principal series on $\mathrm{GL}_2(\mathbb{C})$ with character

$$\begin{cases} (|z|^{s/2}(z/\sqrt{|z|})^{m/2}, |z|^{-s/2}(z/\sqrt{|z|})^{-m/2}), & \text{if } 2|m \\ (|z|^{s/2}(z/\sqrt{|z|})^{(m+1)/2}, |z|^{-s/2}(z/\sqrt{|z|})^{-(m-1)/2}), & \text{if } 2\nmid m \end{cases}$$

and consists of SU_2 type $|m| + 2\mathbb{Z}_{\geq 0}$, here type n means a representation of SU_2 consists of two variable homogeneous polynomials of degree $n+1$ with action by

$$gP(X,Y) = P((X,Y)g).$$

We have $\lambda_\epsilon = \frac{1}{2}((s - \epsilon m/2)^2 - 1)$ if $\theta = \widetilde{\pi}(|z|^s(z/\sqrt{|z|})^m)$. We have $\widetilde{\pi}(|z|^s(z/\sqrt{|z|})^m)$ coming from theta lifting of algebraic regular representation if and only if $s = 0$ and $2 \nmid m$.

The two kinds of differential equations (1), (2) are essentially of the same type, both are special cases of the following differential equation satisfied by the classical whittaker function. Let's recall some basics. Let $\alpha \in \mathbb{R}$ and $\nu \in \mathbb{C}$, the Whittaker function related to (δ, ν), denoted by $\mathcal{W}_{\alpha,\nu}$ is the unique solution of Whittaker's differential equation

$$\mathcal{W}_{\alpha,\nu}'' = \left(\frac{1}{4} - \frac{\alpha}{t} + \frac{\nu^2 - 1/4}{t^2}\right)\mathcal{W}_{\alpha,\nu}$$

which is rapid decay when $t \to \infty, t > 0$. Another linear independent solution of the above differential equation over $\mathbb{C}$ is $\mathcal{W}_{-\alpha,\nu}(-t)$. The differential equation has 0 as regular singular point and ∞ as irregular singular point.

It has the following properties

- (Non-vanishingness) [12]By the asymptotic behavior
$$\mathcal{W}_{\alpha,\nu}(t) \sim t^{\alpha} e^{-t/2}\left(1+\sum_{k=1}^{\infty}\frac{1}{k!t^k}\prod_{j=1}^{k}\left(\nu^2-\left(\alpha+\frac{1}{2}-\ell\right)^2\right)\right), \quad t\to\infty$$
for ν pure imaginary, we have $\mathcal{W}_{\alpha,\nu}(t)\neq 0$ for $t\gg 0$.
For $\mathrm{Re}(\nu-\alpha) > -\frac{1}{2}$,
$$\mathcal{W}_{\alpha,\nu} = \frac{t^{\nu+1/2}e^{-t/2}}{\Gamma(\nu-\alpha+1/2)}\int_0^{\infty} e^{-yt}y^{\nu-\alpha-1/2}(1+y)^{\nu+\alpha-1/2}dy.$$
Thus if ν real with $\nu-\alpha > -\frac{1}{2}$, $\mathcal{W}_{\alpha,\nu}$, has no zeros on $\mathbb{R}_{>0}$. It can be also showed that for $\nu\notin i\mathbb{R}$, $\mathcal{W}_{0,\nu}$ has no zeros on $\mathbb{R}_{>0}$ and for each $t>0$, exists $\nu\in i\mathbb{R}$ such that $\mathcal{W}_{0,\nu}(t)=0$.
- (Algebraicity) Whenever one of $\frac{1}{2}+\alpha\pm\nu$ is a positive integer, $\mathcal{W}_{\alpha,\nu}(t)\in\mathbb{C}P_{\alpha,\nu}(t^{1/2})e^{-t/2}$, for some $P_{\alpha,\nu}(t^{1/2})\in\mathbb{Q}[t^{-1/2}]$. Furthermore,
$$\mathcal{W}_{\alpha,\pm(\alpha-\frac{1}{2})}(t) = t^{\alpha}e^{-\frac{t}{2}}$$
$$\mathcal{W}_{1-\alpha,\pm(\alpha-\frac{1}{2})}(t) = t^{1-\alpha}e^{-\frac{t}{2}}.$$

By above analysis, one have:

Lemma and Definition 2.23.

- *If $F=\mathbb{R}$,*
$$W_n\left(\begin{pmatrix} t^{1/2} & \\ & t^{-1/2}\end{pmatrix},1\right)\in\mathbb{C}\mathcal{W}_{\mathrm{sign}(c)n/2,s/2}(4\pi|ct|), t>0.$$
  - *If $\theta=\widetilde{\sigma}_k^{\epsilon}$, for $\psi(\cdot)=e^{2\pi c\cdot}$, the ψ-th Whittaker model of θ exists if and only if* $\mathrm{sign}(c)=\epsilon$. *If this is the case, choose*
$$W_k\left(\begin{pmatrix} t^{1/2} & \\ & t^{-1/2}\end{pmatrix},1\right) = t^{\epsilon k/2}e^{-2\pi|ct|}, t>0.$$
  - *If $\theta=\widetilde{\pi}(|t|^s\chi)$, then ψ-th Whittaker model always exists. (i). for each n appears in weight of θ, W_n only has finitely many zeros on*
$$\left(\begin{pmatrix} t^{1/2} & \\ & t^{-1/2}\end{pmatrix},1\right),\quad t>0,$$
(ii). for each $t>0$, exists n such that $W_n\left(\begin{pmatrix} t^{1/2} & \\ & t^{-1/2}\end{pmatrix},1\right)\neq 0$.
- *If $F=\mathbb{C}$ and $\theta=\widetilde{\pi}(|z|^s(z/\sqrt{|z|})^m)$, the ψ-th Whittaker model always exists. We have*
$$W_{n,\epsilon\frac{n}{2}}\left(\begin{pmatrix} t^{1/2} & \\ & t^{-1/2}\end{pmatrix},1\right)\in\mathbb{C}t^{\frac{n+1}{2}}\mathcal{W}_{0,s-\epsilon\frac{m}{2}}(8\pi\sqrt{|c|}t),\quad t>0,$$
and if n is the minimal type, for each $\frac{-n}{2}\le k\le\frac{n}{2}$ with $k\equiv\frac{n}{2}\pmod 2$,
$$W_{n,k}\left(\begin{pmatrix} t^{1/2} & \\ & t^{-1/2}\end{pmatrix},1\right)\in\mathbb{C}t^{\frac{n+1}{2}}\mathcal{W}_{0,s-k}(8\pi\sqrt{|c|}t),\quad t>0.$$
Thus we can take the following
  - *If $m\neq 0$, or s real and $m=0$, for each $n\in|m|+2\mathbb{Z}_{\ge 0}$, then $W_{n,\epsilon\frac{n}{2}}$ always non-vanishing on*
$$\left(\begin{pmatrix} t^{1/2} & \\ & t^{-1/2}\end{pmatrix},1\right),\quad t>0.$$
  - *In the left case, $W_{n,\epsilon\frac{n}{2}}$ only has finitely many zeros. Furthermore, for each $t>0$, exists some $W_{n,k}$ such that* $W_{n,k}\left(\begin{pmatrix} t^{1/2} & \\ & t^{-1/2}\end{pmatrix},1\right)\neq 0$.
  - *In particular, if θ is algebraic regular and n is minimal type, then we can take* $W_{n,\epsilon\frac{1}{2}}\left(\begin{pmatrix} t^{1/2} & \\ & t^{-1/2}\end{pmatrix},1\right) = t^{\frac{n+1}{2}}e^{-4\pi\sqrt{|c|}t}$, $t>0$.

Let w be the exponential of t in the choice of Whittaker function in the above lemma.

Now by the Lemma 2.23 and the relation between ${}_x\theta^f_\phi$ and ${}_{ux}\theta^f_\phi$ introduced before we have the following.

Let θ be the theta lifting of π with respect to ψ. Fix $f \in V(\pi_0, \delta_1, \mathfrak{X}_2)$ as before. Let $F^{\times o}$ be the identity component of $F^\times$ and let $\mathcal{K}^o \subset K^{\mathrm{tr}=0}\backslash\{0\}$ be an orbit of $F^{\times 0}$.

Theorem 2.24.

(1). *If $F = \mathbb{R}$, then*

(i). *If $\theta = \widetilde{\sigma}^\epsilon_k$, exists $\phi \in \mathcal{S}(V)$ of weight k (under action of the standard maximal compact subgroup of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$) such that for any $x \in \mathcal{K}^o$,*

$$ {}_x\theta^f_\phi(1)/\overline{f(1)} = |q(x)|^{\frac{\epsilon k}{2} - \frac{3}{4}} e^{-2\pi|cq(x)|}. $$

(ii). *If $\theta = \widetilde{\pi}(|t|^s\chi)$, then (a). for each n appears in weights of θ, exist $\phi_n \in \mathcal{S}(V)$ of weight n (under action of maximal compact subgroup of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$) such that for each each $x \in \mathcal{K}^o$,*

$$ {}_x\theta^f_{\phi_n}(1)/\overline{f(1)} = |q(x)|^{\frac{-3}{4}} \mathcal{W}_{\mathrm{sign}(c)n/2, s/2}(4\pi|cq(x)|), $$

which is nonzero for $|q(x)|$ sufficient large. (b). for each $x \in \mathcal{K}^o$, exists n and $\phi_n \in \mathcal{S}(V)$ of weight n such that ${}_x\theta^f_{\phi_n}(1) \neq 0$.

(2). *If $F = \mathbb{C}$ and $\theta = \widetilde{\pi}(|z|^s(z/\sqrt{|z|})^m)$, then*

– *In the case θ is algebraic regular, i.e. $s = 0$ and $2 \nmid m$, for n be the minimal type m, $\epsilon \in \{\pm 1\}$, exists $\phi_{n,\epsilon\frac{1}{2}}$ such that for each $x \in \mathcal{K}^o$, ${}_x\theta^f_{\phi_{n,\epsilon\frac{1}{2}}}$ corresponding to $W_{n,\epsilon\frac{1}{2}}$-vector in the $\psi_{q(x)}$-Whittaker model of θ and*

$$ {}_x\theta^f_{\phi_{n,\epsilon\frac{1}{2}}}(1)/\overline{f(1)} = |q(x)|^{\frac{n-2}{4}} e^{-4\pi\sqrt{|cq(x)|}} e^{i\epsilon\theta_x} \neq 0, $$

where x corresponding to $(|q(x)|^{1/4}e^{i\theta_x}, -|q(x)|^{1/4}e^{i\theta_x})$ in $K^{\mathrm{tr}=0} = \{(u,-u) \mid u \in \mathbb{C}\}$.

– *If $m \neq 0$, or s real and $m = 0$, for each n appears in the type of θ and $\epsilon \in \{\pm 1\}$, there exists $\phi_{n,\epsilon\frac{n}{2}} \in \mathcal{S}(V)$ such that for each $x \in \mathcal{K}^o$, ${}_x\theta^f_{\phi_{n,\epsilon\frac{n}{2}}}$ corresponding to $W_{n,\epsilon n/2}$-vector in the $\psi_{q(x)}$-Whittaker model of θ and*

$$ {}_x\theta^f_{\phi_{n,\epsilon\frac{n}{2}}}(1)/\overline{f(1)} = |q(x)|^{\frac{n-2}{4}} \mathcal{W}_{0, s-\epsilon\frac{m}{2}}(8\pi\sqrt{|cq(x)|}) e^{in\theta_x\epsilon} \neq 0. $$

– *For each $x \in \mathcal{K}^o$, exists $\phi_{n,k}$ for some n, k such that ${}_x\theta^f_{\phi_{n,k}}$ corresponding to $W_{n,k}$ in the $\psi_{q(x)}$-th Whittaker model of θ and ${}_x\theta^f_{\phi_{n,k}}(1) \neq 0$.*

We may denoted $C_{q(x)} = ({}_x\theta^f_\phi(1)/\overline{f(1)})/|q(x)|^{w/[F:\mathbb{R}]-3/4}$ for $f \in V(\pi_0, \delta_1, \mathfrak{X}_2)$ and ϕ be fixed one of the above choice.

Remark 2.25. The local theta correspondence with respect to ψ is given by the following: Denote χ_ψ be the genus character of diagonal torus of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ associated to ψ if F is real.

- If $F = \mathbb{R}$,

$$
\begin{array}{rrcll}
 & \mathrm{Irr}(\mathrm{PGL}_2(\mathbb{R})) & \hookrightarrow & \mathrm{Irr}(\widetilde{\mathrm{SL}_2(\mathbb{R})}) & \\
\text{Principle series:} & \pi(\mu,\mu^{-1}) & \leftrightarrow & \widetilde{\pi}(\mu\chi_\psi), & (\mu(t) = |t|^s\mathrm{sign}^\eta, s \notin 1/2 + \mathbb{Z}_{\geq 0}, \mathrm{Re}(s) \geq 0, \eta \in \{\pm 1\}) \\
\text{Discrete series} & \sigma(\mu,\mu^{-1}) & \leftrightarrow & \widetilde{\sigma}(\mu\chi_\psi), & (\mu(t) = |t|^s\mathrm{sign}^{s+1/2}, s \in 1/2 + \mathbb{Z}_{\geq 0}, \mathrm{Re}(s) \geq 0, \epsilon \in \{\pm 1\}), \\
 & \mathrm{Irr}(\mathbb{R}^\times\backslash\mathbb{H}^\times) & \hookrightarrow & \mathrm{Irr}(\widetilde{\mathrm{SL}_2(\mathbb{R})}) & \\
 & \rho_n, n \in 2\mathbb{Z}_{\geq 0} & \leftrightarrow & \widetilde{\sigma}(\mu_n\chi_\psi), & (\mu_n(t) = |t|^{(n+1)/2}\mathrm{sign}^{n/2})
\end{array}
$$

  where ρ_n is the unique irreducible representation of H with dimensional n.
- If $F = \mathbb{C}$, the genuine irreducible representations of $\widetilde{\mathrm{SL}_2(\mathbb{C})} = \mathrm{SL}_2(\mathbb{C}) \times \{\pm 1\}$ are the same as irreducible representations of $\widetilde{\mathrm{SL}_2(\mathbb{C})}$

$$
\begin{array}{rcl}
\mathrm{Irr}(\mathrm{PGL}_2(\mathbb{C})) & \leftrightarrow & \mathrm{Irr}(\mathrm{SL}_2(\mathbb{C})) \\
\pi(\mu,\mu^{-1}) & \leftrightarrow & \widetilde{\pi}(\mu)
\end{array}
$$

Remark 2.26. In the case $F = \mathbb{R}$, K/F splits and π is weight $2k$ holomorphic series, the CST's test vector are closely related to the holomorphic vector, if we choose the holomorphic vector f_{2k}, one can construct ϕ such that θ_ϕ^f is holomorphic weight $3/2$ and twice of theta lifting of f_{2k} lies in $\mathbb{C}\cdot f_{2k}$.

Test vector for Whittaker functional (II)

Let $\delta_i \in \mathfrak{X}_i$ such that $\delta = \delta_1\delta_2$ generates $D_{F(\delta_0\delta_1)/F}D_{F(\delta_0\delta_2)/F}$ (here $\delta_0 = 1$ in case (II)) and $\psi = \psi_{0,\delta_1^{-1}}$. Let $x \in V_\delta$, f, ϕ be as in Subsection 2.2 and $W = {}_x\theta_\phi^f \in \mathcal{W}_\delta$. In the following, we study the uniform properties of Whittaker functional $\beta(W, W)$.

The main result of this subsection is to get explicit formula for

$$\beta_\delta^0(\theta) = \frac{L(1, \pi\otimes\eta_\delta, ad)}{L(1/2, \pi\otimes\eta_\delta)L(2, 1_F)}\frac{\int_F (n(y)\theta, \theta)\psi_\delta(-y)dy}{(\theta,\theta)}, \quad n(y) = \left(\begin{pmatrix}1 & y\\ & 1\end{pmatrix}, 1\right) \in N.$$

Here dy is the Haar measure on F which is self dual with respect to ψ. Let u_0 such that ψ_{0,u_0} is unramified.

To understand β^0, several construction of Hermitian invariant pairing for θ is crucial.

Lemma 2.27 (local invariant Hermitian pairing I)**.** [27] *For any set of representations δ_i of $F^\times/F^{\times 2}$, there exists ψ_{δ_i} Whittaker functions $W_{\psi_{\delta_i}}$ of θ such that*

$$(\varphi_1, \varphi_2)_1 = \sum_{\delta_i}\int_{F^\times} W_{\psi_{\delta_i}}(d(a)\varphi_1)\overline{W_{\psi_{\delta_i}}(d(a)\varphi_2)}d^\times a$$

and

$$\int_F (n(y)\varphi_1, \varphi_2)_1\psi_{\delta_i}(-y)dy = \frac{2}{|2\delta_i|}W_{\psi_{\delta_i}}(\varphi_1)\overline{W_{\psi_{\delta_i}}(\varphi_2)}, \quad \varphi_i \in \theta.$$

Remark 2.28. If either

- exists only one coset, say represented by δ, in $F^\times/F^{\times 2}$ such that θ admits ψ_δ-th Whittaker model or
- exists only one coset, say represented by δ, δ in $F^\times/F^{\times 2}$ such that φ_i are test vectors of ψ_δ-th Whittaker functional, then

$$(\varphi_1, \varphi_2)_1 = \int_{F^\times} W_{\psi_\delta}(d(a)\varphi_1)\overline{W_{\psi_\delta}(d(a)\varphi_2)}d^\times a.$$

Lemma 2.29 (local invariant Hermitian pairing II)**.** *Let W_{ψ_δ} be a nonzero ψ_δ-th Whittaker functional on θ, then the following pairing is a nonzero Hermitian pairing:*

$$(\varphi_1, \varphi_2)_2 = \begin{cases}\int_{F^\times} W_{\psi_\delta}(d(a)\varphi_1)\overline{W_{\psi_\delta}(d(a)\varphi_2)}d^\times a, & \text{if } F = \mathbb{C}, \text{ or } F = \mathbb{R} \text{ and } \varphi_1,\ \varphi_2 \text{ are of same weight}\\ 0, & \text{otherwise.}\end{cases}$$

If $F = \mathbb{R}$ and θ has nonzero ψ_δ Whittaker functional. let c_δ be the nonzero constant such that $(\ ,\ )_1 = c_\delta(\ ,\ )_2$, where the ψ_δ Whittaker functional in definite of $(\ ,\ )_i$ are the same. Then c_δ only depend on the class $\mathfrak{X}$ of δ in $F^\times/F^{\times 2}$, may denoted by $d_{\mathfrak{X}}$ And we have $c_\delta = 1$ if and only if $F = \mathbb{C}$, or $F = \mathbb{R}$ and θ is discrete series.

For Principle series, have another Hermitian invariant pairing.

Lemma 2.30. [27] *If $\theta \simeq \widetilde{\pi}(\chi_\psi\mu)$ is a principle series, then the following pairing on $\widetilde{\pi}(\chi_\psi\mu)\otimes\widetilde{\pi}(\chi_{\psi_{-1}}\mu^{-1})$ is G-invariant:*

$$(\varphi_1, \varphi_2) = \int_N \varphi_1(wn(y))\varphi_2(wn(y))dy,,$$

where $w = \left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, 1\right)$. In particular,

- *If μ is unitary, then*

$$(\varphi_1, \varphi_2)_3 = \int_N \varphi_1(wn(y))\overline{\varphi_2(wn(y))}dy$$

  is a Hermitian pairing on θ.
- *If $\mu = |\cdot|^s\chi$ with χ quadratic and $s \in (-\frac{1}{2}, 0)\sqcup(0, \frac{1}{2})$, let $M_\mu : \widetilde{\pi}(\chi_\psi\mu) \mapsto \widetilde{\pi}(\chi_\psi\mu^{-1})$ be the intertwining operator given by*

$$M_\mu\varphi(g) = \int_F \varphi(wn(y)g)dy$$

Then the following is a Hermitian pairing on θ:

$$(\varphi_1, \varphi_2)_3 = \int_F M_\mu \varphi_1(wn(y))\overline{\varphi_2(wn(y))}dy.$$

Theorem 2.31. *We have the following uniform property of W:*

$$\beta_\delta^0(W) = \begin{cases} \frac{1}{|\delta_1'||c_{\delta_1}|^{1/2}|d_F|^{1/2}}, & \text{Case (I).} \\ \frac{L(1,\pi\otimes\eta_\delta,ad)}{L(1/2,\pi\otimes\eta_\delta)L(2,1_F)} \frac{2^n L(1,1_F)}{|2\delta||c_{\delta_1}|^{1/2}|d_F|^{1/2}}, & \text{Case (II). non-Archimedean} \\ \frac{L(1,\pi\otimes\eta_\delta,ad)}{L(1/2,\pi\otimes\eta_\delta)L(2,1_F)} \frac{2W(1)\overline{W}(1)|\delta|^{1/2}}{c_{\mathfrak{X}_1\cdot\mathfrak{X}_2}|2||c_{\delta_1}|^{1/2-2w/[F:\mathbb{R}]}(W^o,W^o)_2}, & \text{Archimedean} \end{cases}$$

c_{δ_1} is such that $\psi_{c_{\delta_1}^{-1}}$ is the standard additive character, $(\delta_1') = D_{F(\delta_1\delta_0)/F}$ and

$$n = \begin{cases} 1, & \text{if } (*) \text{ holds and } \epsilon(\pi) = \epsilon(B), \text{ or } R^\times \cap NK^\times \not\subseteq K^\times \\ 2, & \text{otherwise} \end{cases},$$

$W^o = |c_{\delta_1}|^{w/[F:\mathbb{R}]-1/4}|\delta|^{3/4}W$ in the ψ_δ Whittaker model of θ has the property that $(W^0, W^0)_2 = \int_{F^\times} W^0(d(a))\overline{W}^0(d(a))d^\times a$ does not depend on δ and δ_1, w corresponds to the the exponential of t in formula of test vector in Lemma 2.23.

Remark 2.32. For F Archimedean, explicit formula for $W(1)$ is given in 2.24. For algebraic regular representations, $c_{\mathfrak{X}_1\cdot\mathfrak{X}_2} = 1$. If $F = \mathbb{R}$, $\theta \simeq \widetilde{\sigma}_k^\epsilon$, and W is of weight k, then

$$(W^o, W^o)_2 = \frac{\Gamma(\epsilon k)}{(4\pi)^{\epsilon k}}.$$

If $F = \mathbb{C}$, $\theta \simeq \widetilde{\pi}((z/\sqrt{|z|})^m)$ with $2 \nmid m$ and W corresponds to vector $W_{|m|,\epsilon\frac{1}{2}}$, then

$$(W^0, W^0)_2 = \frac{2\pi\Gamma(|m|+1)}{(8\pi)^{|m|+1}}.$$

We now consider proof of Theorem 2.31. The case (I). follows from the following Proposition. By definition of Weil representation, the automorphism

$$\iota_u : \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \epsilon\right) \mapsto \begin{cases} \left(\begin{pmatrix} a & ub \\ u^{-1}c & d \end{pmatrix}, \epsilon\right), & \text{if } c \neq 0 \\ \left(\begin{pmatrix} a & ub \\ 0 & d \end{pmatrix}, \epsilon\chi_u(d)\right), & \text{if } c = 0 \end{cases}$$

on G induces isomorphism

$$\widetilde{\pi}(\chi_\psi \mu) \simeq \widetilde{\pi}(\chi_{\psi_u}\mu), \quad \varphi \mapsto \varphi \circ \iota_u.$$

The result follows from the following proposition.

Proposition 2.33. *Let $\pi = \pi(\mu)$ be unramified principle series. Let $\widetilde{K}$ be the standard maximal compact subgroup of G. Let $u \in F^\times$ such that ψ_u is unramified and hence $\widetilde{\pi}(\chi_{\psi_u}\mu)$ is unramified, then the subspace*

$$\widetilde{\pi}(\chi_\psi \mu)^{\iota_u(\widetilde{K})}$$

is one dimensional. For any nonzero $\varphi \in \widetilde{\pi}(\chi_\psi\mu)^{\iota_u(\widetilde{K})}$ with $\operatorname{ord}(\delta) = 0, 1$ and $\delta \in F^\times$, we have

$$\frac{L(1, \pi \otimes \eta_\delta, ad)}{L(1/2, \pi \otimes \eta_\delta)L(2, 1_F)} \frac{\int_F (n(y)\varphi, \varphi)\psi_\delta(-y)dy}{(\varphi, \varphi)} = |u|^{1/2},$$

here dy is self dual with respect to ψ.

Proof. The proof could be reduces to the case $\widetilde{\pi}(\chi_\psi\mu)$ is unramified via ι_u introduced above. Then it follows from explicit G-invariant Hermitian pairing $(\ ,\ )_3$ in Lemma 2.30 and explicit description of spherical vector, for example in [27].

□

For the case (II). and F is non-Archimedean, the result follows directly from Lemma 2.27 and the property of θ that it is not the test vector of $\psi_{\delta'}$-th for any other δ' which represents different coset in $F^\times/F^{\times 2}$ as $q(x)$.

For the Archimedean case, the results follows from Lemma and Lemma 2.27 and Lemma 2.29.

3. Rallis inner product formulae

In this section, we give a new proof of the Rallis inner product formulae for the dual pair $(\mathrm{SO}_3, \widetilde{\mathrm{SL}_2})$ over a number field. The proof is via considering relations among several periods formulae associated to toric periods, Whittaker-Fourier periods, and Peterson inner products. For the theta lifting from SO_3 to $\widetilde{\mathrm{SL}_2}$, the keys are the comparison of two different Whittaker-Fourier periods formulae (See Theorem 3.1 and Theorem 3.2) and the local comparison result (See Theorem 2.3). The first Whittaker-Fourier period formula is established by considering relation between Whittaker-Fourier periods and toric periods. In fact, the ratio of Whittaker-Fourier period by toric period is given by product of local Whittaker functionals constructed by Waldspurger's explicit local theta lifting. The second formula of Whittaker-Fourier coefficients involves local Whittaker functional constructed from matrix coefficients. There are parallel story for the other direction of theta lifting. For both directions of theta liftings, the role of Whittaker-Fourier periods and toric periods are interchanged, .

Notations Let F be a number field, $\mathcal{O}$ be its ring of integers and $\mathbb{A}$ be its ring of adéles . For G an algebraic group over F, also denoted by G the group of its F points. For v a place of F, denoted by G_v the set of F_v points of G, which is a locally compact topological group.

For global version of algebraic groups introduced in Notations of the local theory, we take Haar measure on the group of adelic points to be the one induced by the product measure with local measure given in local theory. We choose global additive character of $\mathbb{A}$ to be trivial on F and then the measures on adelic points of these algebraic groups are the Tamagawa measures.

For $\delta \in F^\times$ and ψ a nontrivial additive character of $F\backslash\mathbb{A}$, denoted $\psi(\delta\cdot)$ by $\psi_\delta(\cdot)$. Denoted by η_δ the quadratic character of $F^\times\backslash\mathbb{A}^\times$ associated to the quadratic extension $F(\sqrt{\delta})/F$.

3.1. **Theta lifting.** Let B/F be a quaternion algebra and let $V = B^{\mathrm{tr}=0}$ be the quadratic space with quadratic form q given by minus of the reduced norm. Let $H = \mathrm{SO}(V)$ and identify it with $\mathrm{PB}^\times$ via its conjugate action on V. Denoted by $\mathbb{G}$ the metaplectic double covering of $\mathrm{SL}_2(\mathbb{A})$. Fix a non-trivial additive character $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$. Let $(w_\psi, \mathcal{S}(V(\mathbb{A})))$ be the Weil representation of $H(\mathbb{A}) \times \mathbb{G}$ [9], $\mathcal{S}(V(\mathbb{A})) := \bigotimes_v \mathcal{S}(F_v)$ with $\mathcal{S}(F_v)$ defined in local theory.

Via Weil representation, one can construct cuspidal automorphic representations of one group in dual pair $(H(\mathbb{A}), \mathbb{G})$ from cuspidal automorphic representations of another group. Such process is called theta lifting we now recall.

For each $\phi \in \mathcal{S}(V(\mathbb{A}))$, the theta kernel function

$$\theta_\phi : (h, g) \mapsto \sum_{x\in V} (w_\psi(h, g)\phi)(x), \quad h \in H(\mathbb{A}), g \in \mathbb{G}$$

is an automorphic form on $H(\mathbb{A}) \times \mathbb{G}$ and hence induces an $H(\mathbb{A}) \times \mathbb{G}$-equivalent map

$$\mathcal{S}(V) \to \mathcal{A}(H(\mathbb{A}) \times \mathbb{G}), \quad \phi \mapsto \theta_\phi,$$

where $\mathcal{A}(H(\mathbb{A}) \times \mathbb{G})$ is the space of automorphic forms on $H(\mathbb{A}) \times \mathbb{G}$. Similarly denoted by $\mathcal{A}(\cdot)$ (resp. $\mathcal{A}_0(\cdot)$) the space of (resp. cuspidal automorphic forms) automorphic forms on $\cdot$ for $\cdot = H(\mathbb{A}), \mathbb{G}$.

Given an irreducible $\pi \subset \mathcal{A}_0(H(\mathbb{A}))$, its theta lifting $\theta_\psi(\pi) \subset \mathcal{A}_0(\mathbb{G})$ consists of

$$\theta_f^\phi(g) = \int_{H\backslash H(\mathbb{A})} \theta_\phi(h, g)\overline{f(h)}dh, \quad f \in \pi, \phi \in \mathcal{S}(V(\mathbb{A})),$$

where the measure on $H\backslash H(\mathbb{A})$ is the Tamagawa measure with total volume 2. In this paper we only consider cuspidal automorphic forms on $\mathbb{G}$ that are genuine and orthogonal to elementary theta functions (See [9]).

For the other direction, given $\theta \subset \mathcal{A}_0(\mathbb{G})$ irreducible, its theta lifting $\pi := \theta_\psi(\theta)$ consists of

$$\theta_\phi^\varphi(h) = \int_{\mathrm{SL}_2\backslash\mathrm{SL}_2(\mathbb{A})} \theta_\phi(h, g)\overline{\varphi(g)}dg, \quad \varphi \in \theta, \phi \in \mathcal{S}(V(\mathbb{A})),$$

where the measure on $\mathrm{SL}_2\backslash\mathrm{SL}_2(\mathbb{A})$ is the Tamagawa measure with total volume 1.

Whenever $\theta := \theta_\psi(\pi) \neq 0$, $\theta \subset \mathcal{A}_0(\mathbb{G})$ is irreducible, $\theta_\psi(\theta) = \pi$, and similar for the other direction. We call (θ, π) a global theta correspondence. The theta correspondence has the following see-saw property:

$$(\theta_\phi^f, \varphi) = (\theta_\phi^\varphi, f), \quad f \in \pi, \phi \in \mathcal{S}(V(\mathbb{A})), \varphi \in \theta,$$

where (,) stands for the Petersson inner products on two groups respectively with the choice of Tamagawa measures.

Given a cuspidal automorphic irreducible representation Π of $H(\mathbb{A})$, or $\mathbb{G}$, A fundamental question if nonvanishingness of $\theta_\psi(\Pi)$. By Proposition 3.5, $\otimes_v \theta_{\psi_v}(\Pi_v)$ is cuspidal irreducible automorphic representation if and only if Π has global root number $+1$. This is a necessary condition for $\theta_\psi(\Pi)$ to be nonzero. In fact, there exists global obstruction for cuspidal automorprhic representation $\theta_\psi(\Pi)$ to be nonzero, which is given by central L-value due to the Rallis inner product formula, which connects Peterson inner product of lifted forms with central L-value. More precise statement, see Theorem 3.6 and Theorem 3.7, and also see Remark 3.8 for previous works.

In the following, we will give new proof of Rallis inner product formulae, via considering several periods formulae and their relations.

3.2. **Whittaker-Fourier periods formulae of theta liftings from SO_3 to $\widetilde{\mathrm{SL}_2}$.** In this subsection, we consider the relations between Whittaker-Fourier periods and L-values via decomposition. Fix irreducible $\pi \subset \mathcal{A}_0(H(\mathbb{A}))$, nontrivial additive character $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ and let $\theta := \theta_\psi(\theta)$ be its theta lifting.

The group $\mathbb{G}$ splits over the unipotent subgroup $N(\mathbb{A}) = \left\{ n(y) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \;\middle|\; y \in \mathbb{A} \right\}$ of $\mathrm{SL}_2(\mathbb{A})$, hence we may view $N(\mathbb{A})$ as a subgroup of $\mathbb{G}$. For $\delta \in F^\times$, consider ψ_δ-th Whittaker-Fourier coefficient

$$W_{\psi_\delta} : \theta \to \mathbb{C}, \quad \varphi \mapsto \int_{F\backslash\mathbb{A}} \varphi(n(y))\psi(-\delta y)dy.$$

Recall the action of $N(\mathbb{A})$ on $\mathcal{S}(V(\mathbb{A}))$ is given by

$$w_\psi(n(y))\phi(x) = \psi(yq(x))\phi(x)$$

and the action of $H(\mathbb{A})$ on $\mathcal{S}(V(\mathbb{A}))$ is given by $w_\psi(h)\phi(x) = \phi(h^{-1} \circ x)$, where $\circ$ means conjugate action. We have

$$W_{\psi_\delta}(\theta_f^\phi) = \int_{H\backslash H(\mathbb{A})} \left(\sum_{x \in V_\delta} \phi(h^{-1} \circ x) \right) \overline{f(h)} dh,$$

where $V_\delta = \{x \in V \mid q(x) = \delta\}$.

Consider the space of ψ_δ Whittaker functionals $\bigotimes \mathrm{Hom}_{N_v}(\theta_{\psi_v}(\pi_v), \psi_{\delta,v})$. If $\theta \neq 0$ and the ψ_δ-th Whittaker-Fourier coefficient on θ is nonzero, then it gives a basis of this space. Unlike the theory of automorphic forms on $\mathrm{GL}_2(\mathbb{A})$, even if $\theta \neq 0$ and the space of ψ_δ-Whittaker functionals is nonzero, it is not necessary comes from the ψ_δ-th Whittaker-Fourier coefficient. Will see this global obstruction is given by twist L-values under $\theta \neq 0$.

In the following, we introduce two formulae for Whittaker-Fourier periods which connect to quadratic twists L-values. The local Whittaker functionals appeared in the two formulae are from different sources.

In the following type (I), (II) formulae, we fix an irreducible $\pi \subset \mathcal{A}_0(H(\mathbb{A}))$ with $\epsilon(1/2, \pi) = 1$ and $\theta := \theta_\psi(\pi)$ its theta lifting.

Type (I) formulae

In the following, we introduce a formulae for Whittaker-Fourier period which connect to base change L-value. The key observation is that the toric period on π appears in the Whittaker-Fourier period on θ and their difference is given by product of local Whittaker functionals constructed from explicit local theta liftings. Hence we connects ψ_δ-th Whittaker-Fourier period of θ to base change L-value $L(1/2, \pi_{F(\sqrt{\delta})})$ via Waldspurger formula for toric periods. In particular, the global obstruction for Whittaker-Fourier period of θ to be nonzero are given by base change central L-value.

Let $x \in V_\delta$ and $K = F(x)$. Denoted by $T_x := F^\times\backslash K^\times \subset H$ be the group of stablizer of x. Fix a decomposition $\pi = \bigotimes_v \pi_v$.

Assume that

(a_1) $\dim_\mathbb{C} \bigotimes_v \mathrm{Hom}_{N_v}(\theta_{\psi_v}(\pi_v), \psi_{\delta,v}) = 1$, equivalently, $\dim_\mathbb{C} \bigotimes_v \mathrm{Hom}_{T_{x,v}}(\pi_v, \mathbb{C}) = 1$,
(a_2) $f = \otimes f_v \in \pi$ such that f_v is a test vector for $\mathrm{Hom}_{T_{x,v}}(\pi_v, \mathbb{C})$ for all places v.

The above assumptions only depend on coset of δ in $F^\times/F^{\times 2}$. The existence of x satisfying (a_1) is a local problem we have introduced in local theory which is proved by Waldspurger in [37].

Whenever $\theta \neq 0$, we have an isomorphism

$$\theta \simeq \bigotimes \mathcal{W}_{\delta,v}, \quad \theta_\phi^f \mapsto \otimes_v \frac{{}_x\theta_{\phi_v}^{f_v}}{f_v(1)}, \quad \forall \phi = \otimes_v \phi_v \in \mathcal{S}(V(\mathbb{A}))$$

where $\mathcal{W}_{\delta,v}$ is the $\psi_{\delta,v}$ Whittaker model of $\theta_{\psi_v}(\pi_v)$ introduced in local theory. Here we view f_v in the $T_{x,v}$ model $\mathcal{V}_{x,v}$ of π_v and we add subscript x for explicit local theta lifting (in Subsection 2.1) to emphasis its dependence on x.

In the degenerate case, i.e. $\delta \in F^{\times 2}$, the toric period on π is directly related to L-value $L(1/2,\pi)$ via Whittaker theory of PGL_2. Since we are interested in quadratic twist L-values, in the following theorem, we only consider $\delta \notin F^{\times 2}$, i.e. the case $T_x(\mathbb{A})$ is compact. For $f \in \pi$, its toric period along T_x is defined by

$$P_{T_x}(f) = \int_{T_x\backslash T_x(\mathbb{A})} f(t)dt.$$

Theorem 3.1. *Under assumption (a_1) and $\delta \notin F^{\times 2}$:*

(1). *Let $f = \otimes f_v \in \pi$ be as in (a_2). The equality holds in $\mathrm{Hom}_{N(\mathbb{A})}(\theta, \psi_\delta)$:*

$$W_{\psi_\delta}(\theta^f_\phi) = \overline{P_{T_x}(f)} \cdot \prod_v \frac{{}_x\theta^{f_v}_{\phi_v}(1)}{\overline{f_v(1)}}, \quad \forall \phi = \otimes\phi_v \in \mathcal{S}(V(\mathbb{A})).$$

(2). *Let $f_i = \otimes f_{i,v} \in \pi$, $i = 1,2$ be as in (a_2). The equality holds in $\mathrm{Hom}_{N(\mathbb{A})}(\theta, \psi_\delta) \bigotimes \overline{\mathrm{Hom}_{N(\mathbb{A})}(\theta, \psi_\delta)}$:*

$$W_{\psi_\delta}(\theta^{f_1}_{\phi_1})\overline{W_{\psi_\delta}(\theta^{f_2}_{\phi_2})} = \frac{L(1/2,\pi_K)L(2,1_F)}{2L(1,\eta_\delta)^2L(1,\pi,ad)} \cdot \prod_v \alpha^0_{\delta,v}(f_{2,v}, f_{1,v}) \frac{{}_x\theta^{f_{1,v}}_{\phi_{1,v}}(1)}{\overline{f_{1,v}(1)}} \frac{\overline{{}_x\theta^{f_{2,v}}_{\phi_{2,v}}(1)}}{f_{2,v}(1)}.$$

$$\forall \phi_i = \otimes\phi_{i,v} \in \mathcal{S}(V(\mathbb{A})), \quad i = 1,2,$$

where $\alpha^0_{\delta,v}$ is the normalized basis of $\mathrm{Hom}_{T_{x,v}}(\pi_v, \mathbb{C}) \otimes \overline{\mathrm{Hom}_{T_{x,v}}(\pi_v, \mathbb{C})}$ defined by

$$\alpha^0_{\delta,v}(f_{2,v}, f_{1,v}) := \frac{L(1,\eta_{\delta,v})^2L(1,\pi_v,ad)}{L(2,1_{F_v})L(1/2,\pi_{K_v})} \int_{T_{x,v}} (tf_{2,v}, f_{1,v})_v dt,$$

$L(s,1_{F_v})$ is the local factor of Dedekind zeta function of F at v and $L(s,\pi_v,ad)$ is the local factor of adjoint L-function of π at v, $(\ ,\ )_v$ is a H_v invariant Hermitian pairing on $\pi_v \times \pi_v$ for each v such that their product gives Petersson inner product on $\pi \times \pi$.

As a consequence, for $\delta \in F^\times$ satisfies condition (a_1), the ψ_δ-th Whittaker-Fourier periods on θ is nonzero if and only if $L(1/2,\pi_K) \neq 0$. And by the non-vanishing of quadratic twists [7], $\theta \neq 0$ if and only if $L(1/2,\pi) \neq 0$.

Proof. Recall the ψ_δ-th Whittaker-Fourier coefficient is given by

$$W_{\psi_\delta}(\theta^f_\phi) = \int_{H\backslash H(\mathbb{A})} \overline{f(h)} \sum_{x \in V_\delta} \phi(h^{-1} \circ x)dh.$$

By Witt theorem,

$$W_{\psi_\delta}(\theta^f_\phi) = \int_{T_x(\mathbb{A})\backslash H(\mathbb{A})} \phi(h^{-1} \circ x)\overline{P_{T_x}(hf)}dh.$$

Recall the Waldspurger formulae for toric periods [38](See also Theorem 3.4) say that under assumption (a_1), (a_2), $P_{T_x}(f) = 0$ if and only if $L(1/2,\pi_K) = 0$.

Thus if $L(1/2,\pi_K) = 0$, then $P_{T_x}(hf) = 0$ for all h and hence $W_{\psi_\delta}(\theta^\phi_f) = 0$. Now assume $L(1/2,\pi_K) \neq 0$. Since for each v, f_v is test vector for $\mathrm{Hom}_{T_{x,v}}(\pi_v, \mathbb{C})$, we have $P_{T_x}(f) \neq 0$. Since the space $\mathrm{Hom}_{T_x(\mathbb{A})}(\pi, \mathbb{C})$ is one dimensional, say generated by P_v, we have the identity in $T_x(\mathbb{A})$-model of π

$$\frac{P_{T_x}(hf)}{P_{T_x}(f)} = \prod_v \frac{P_v(h_vf_v)}{P_v(f_v)}$$

for all $h \in H(\mathbb{A})$. Under identification of π_v with its $T_{x,v}$ model, $\frac{P_v(h_vf_v)}{P_v(f_v)} = \frac{f_v(h_v)}{f_v(1)}$ for all v. Thus

$$W_{\psi_\delta}(\theta^f_\phi) = \overline{P_{T_x}(f)} \cdot \prod_v \frac{{}_x\theta^{f_v}_{\phi_v}(1)}{\overline{f_v(1)}},$$

the second part of the theorem follows from Waldspurger formula for toric periods. ∎

Type (II) formulae

Now we introduce another Whittaker-Fourier period formula which connects Whittaker-Fourier period to quadratic twist L-value, Petersson inner product and product of local Whittaker functional β constructed from matrix coefficients. The following theorem is due to [27] for split B and $\delta = 1$ and the general case could be easily reduced to this essential case. See also work of Baruch-Mao [1].

Fix decomposition $\theta \simeq \bigotimes_v \theta_v$ and choose a local Hermitian invariant pairing $(\ ,\ )_v$ for each v such that the product gives Petersson inner product

$$(\theta_1, \theta_2) = \int_{\mathrm{SL}_2 \backslash \mathrm{SL}_2(\mathbb{A})} \theta_1(g)\overline{\theta_2(g)}dg$$

for pure tensors $\theta_i = \otimes \theta_{i,v}$.

Theorem 3.2. *Under assumption (a_1), we have the following equality in $\mathrm{Hom}_{N(\mathbb{A})}(\theta, \psi_\delta) \bigotimes \overline{\mathrm{Hom}}_{N(\mathbb{A})}(\theta, \psi_\delta)$: For pure tensors $\varphi_i \in \theta$,*

$$W_{\psi_\delta}(\varphi_1)\overline{W_{\psi_\delta}(\varphi_2)} = \frac{L(1/2, \pi \otimes \eta_\delta)L(2, 1_F)}{2L(1, \pi, ad)} \prod_v \beta^0_{\delta,v}(\varphi_{1,v}, \varphi_{2,v}),$$

where $\beta^0_{\delta,v}$ is the normalized basis of $\mathrm{Hom}_{N_v}(\theta_v, \psi_{\delta,v}) \bigotimes \overline{\mathrm{Hom}_{N_v}(\theta_v, \psi_{\delta,v})}$ given by

$$\beta^0_{\delta,v}(\varphi_{1,v}, \varphi_{2,v}) = \frac{L(1, \pi_v, ad)}{L(1/2, \pi_v \otimes \eta_{\delta,v})L(2, 1_{F_v})} \int_{F_v} (n(y)\varphi_{1,v}, \varphi_{2,v})_v \psi_{\delta,v}(-y)dy.$$

In fact, this theorem has nothing to do with SO_3 side and is a result on $\widetilde{\mathrm{SL}_2}$ side.

As a consequence, for $\delta \in F^\times$ satisfies condition (a_1), ψ_δ-th Whittaker-Fourier periods of θ is nonzero if and only if $\theta \neq 0$ and $L(1/2, \pi \otimes \eta_\delta) \neq 0$. Again, by the non-vanishing of quadratic twists, $\theta \neq 0$ if and only if $L(1/2, \pi) \neq 0$.

Proof. May assume $\theta \neq 0$. If $L(1/2, \pi \otimes \eta_\delta) = 0$, then both sides of the equality in the theorem is zero by Theorem 3.1. So we may also assume $L(1/2, \pi \otimes \eta_\delta) \neq 0$ and hence $\theta_{\psi_\delta}(\pi^{\mathrm{JL}} \otimes \eta_\delta) \neq 0$, where π^{JL} is the Jacquet-Langlands correspondence of π. Now the properties of Waldspurger packet [9] says that $\theta = \theta_{\psi_\delta}(\pi^{\mathrm{JL}} \otimes \eta_\delta)$ if and only the assumption (a_1) holds. Thus we reduces to the case B splits and $\delta = 1$. $\square$

3.3. **Toric periods formulae of theta liftings from $\widetilde{\mathrm{SL}_2}$ to SO_3.** Let $\theta \subset \mathcal{A}_0(\mathbb{G})$ be irreducible and $\pi := \theta_\psi(\theta)$ its theta lifting. Let $\delta \in F^\times$, $x \in V_\delta$, $K = F(x)$ and $T_x = F^\times \backslash K^\times \subset H$ its stablizer. Parallel to last subsection, in this subsection, we introduce two formulae on relation between toric periods of theta liftings

$$P_{T_x}(\theta^\varphi_\phi), \quad \varphi \in \theta, \phi \in \mathcal{S}(V(\mathbb{A}))$$

and L-values.

In the following type (I), (II), formulae, we fix an irreducible $\theta \subset \mathcal{A}_0(\mathbb{G})$ with $\epsilon(1/2, \theta, \psi) = +1$ and $\pi := \theta_\psi(\theta)$ its theta lifting.

These two type formulae are due to work of Waldspurger and Qiu.

Type (I) formulae

Fix a decomposition $\bigotimes_v \theta_v$ of θ.

Assume that

- (b_1) $\dim \bigotimes_v \mathrm{Hom}_{T_{x,v}}(\theta_{\psi_v}(\theta_v), \mathbb{C}) = 1$, equivalently, $\dim \bigotimes \mathrm{Hom}_{N_v}(\theta_v, \psi_{v,\delta}) = 1$,
- (b_2) $\varphi = \otimes \varphi_v \in \theta$ such that φ_v is a test vector for $\mathrm{Hom}_{N_v}(\theta_v, \psi_{\delta,v})$ for all v.

The above assumption only depends on coset of δ in $F^\times / F^{\times 2}$.

Whenever $\pi = \theta_\psi(\theta) \neq 0$, we have isomorphism

$$\pi \simeq \bigotimes_v \mathcal{V}_{x,v}, \quad \theta^\varphi_\phi \mapsto \otimes_v \frac{{}_x\theta^{\varphi_v}_{\phi_v}(1)}{\overline{\varphi_v(1)}L(1/2, \theta_v, \psi_v)}, \quad \forall \phi = \otimes_v \phi_v \in \mathcal{S}(V(\mathbb{A})),$$

where $\mathcal{V}_{x,v}$ is the $T_{x,v}$ model of $\theta_{\psi_v}(\theta_v)$ introduced in local theory. Here we view $\varphi_v \in \mathcal{W}_{\delta,v}$ in the $\psi_{\delta,v}$ Whittaker model of θ_v and we add subscript x for local theta lifting to emphasis its dependence on x.

Theorem 3.3. *Under assumption (b_1):*

(1). *Let $\varphi = \otimes\varphi_v \in \theta$ be as in (b_2). The equality holds in* $\mathrm{Hom}_{T_x(\mathbb{A})}(\pi, \mathbb{C})$*:*

$$P_{T_x}(\theta_\phi^\varphi) = L(1/2, \theta, \psi) \cdot \overline{W_{\psi_\delta}(\varphi)} \cdot \prod_v \frac{{}_x\theta_{\phi_v}^{\varphi_v}(1)}{\overline{\varphi_v(1)} L(1/2, \theta_v, \psi_v)}, \quad \forall \phi = \otimes\phi_v \in \mathcal{S}(V(\mathbb{A})).$$

(2). *Let $\varphi_i = \otimes\varphi_{i,v} \in \pi$, $i = 1, 2$ be as in (b_2). The equality holds in* $\mathrm{Hom}_{T_x(\mathbb{A})}(\pi, \mathbb{C}) \bigotimes \overline{\mathrm{Hom}_{T_x(\mathbb{A})}(\pi, \mathbb{C})}$*:*

$$P_{T_x}(\theta_{\phi_1}^{\varphi_1})\overline{P_{T_x}(\theta_{\phi_2}^{\varphi_2})} = \frac{L(1/2, \theta, \psi)^2 L(1/2, \theta, \psi_\delta) L(2, 1_F)}{2L(1, \pi, ad)} \prod_v \beta_{\delta,v}^0(\varphi_{2,v}, \varphi_{1,v}) \prod_v \frac{{}_x\theta_{\phi_v}^{\varphi_v}(1)}{\overline{\varphi_v(1)} L(1/2, \theta_v, \psi_v)} \frac{\overline{{}_x\theta_{\phi_v}^{\varphi_v}(1)}}{\varphi_v(1) L(1/2, \theta_v, \psi_v)}$$

$$\forall \phi_i = \otimes\phi_{i,v} \in \mathcal{S}(V(\mathbb{A})), \quad i = 1, 2,$$

where $\beta_{\delta,v}^0$ is the regularized Whittaker functional in Theorem 3.2 with $(\ ,\ )_v$ in $\beta_{\delta,v}^0$ is a invariant Hermitian pairing on $\theta_v \times \theta_v$ for each v such that its product gives Petersson inner product.

The proof of the first part is (4) of Lemma 45 of [37] . The second part follows from the first part and Theorem 3.2.

We have for each v, $L(s, \psi, \theta_v) = L(s, \theta_\psi(\theta_v))$ and under assumption (b_1), $L(s, \psi_{\delta,v}, \theta_v) = L(s, \theta_\psi(\theta_v) \otimes \eta_{\delta,v})$.

Type (II) formulae

The following formulae just the Waldspurger formula for toric periods [38].

Theorem 3.4. *Assume T_x is nonsplit. Let π be an irreducible cuspidal automorphic representation of $H(\mathbb{A})$. Under assumption (b_1), we have the following equality in $\mathrm{Hom}_{T_x}(\pi, \mathbb{C}) \otimes \overline{\mathrm{Hom}_{T_x}(\pi, \mathbb{C})}$: For pure tensor $f_i = \otimes f_{i,v} \in \pi$,*

$$P_{T_x}(f)\overline{P_{T_x}(f)} = \frac{L(1/2, \theta, \psi) L(1/2, \theta, \psi_\delta) L(2, 1_F)}{2L(1, \eta_\delta)^2 L(1, \pi, ad)} \cdot \prod_v \alpha_{\delta,v}^0(f_{1,v}, f_{2,v}), \quad f_i = \otimes f_{i,v}$$

where

$$\alpha_{\delta,v}^0(f_{1,v}, f_{2,v}) := \frac{L(1, \eta_{\delta,v})^2 L(1, \pi_v, ad)}{L(2, 1_{F_v}) L(1/2, \theta_v, \psi_v) L(1/2, \theta_v, \psi_{\delta,v})} \int_{T_{x,v}} (tf_{1,v}, f_{2,v})_v dt,$$

For the case T_x split, the toric period is related to L-value via Whittaker theory.

3.4. **Rallis inner product formulae and index formulae.** Let's first consider a necessary condition for the non-vanishingness of theta lifting: automorphy.

We first consider a general setting (including sign -1 case). Let $\mathbb{B}$ be a quaternion algebra over $\mathbb{A}$ which is either coherent or coherent. Let $\epsilon(\mathbb{B}) = \prod_v \epsilon(\mathbb{B}_v) = 1$ or -1 depending on $\mathbb{B}$ is coherent or incoherent respectively. Let $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ be a non-trivial additive character. Let $\otimes_v\pi_v$ and $\otimes_v\theta_v$ be irreducible representation of $\mathbb{H}$ and $\mathbb{G}$ respectively such that π_v and θ_v are local theta correspondence with respect to ψ_v for all v. Call $\otimes_v\pi_v$ cuspidal automorphic if ${\otimes_v\pi_v}^{\mathrm{JL}}$ is an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$.

Proposition 3.5. *$\pi := \otimes_v\pi_v$ is cuspidal automorphic and $\epsilon(1/2, \pi) = \epsilon(\mathbb{B})$ if and only if $\theta := \otimes_v\theta_v$ is cuspidal automorphic(necessary have $\epsilon(1/2, \theta, \psi) = \epsilon(\mathbb{B})$).*

Proof. If $\theta = \otimes\theta_v$ is cuspidal automorphic. There exists a a such that $\theta_{\psi_a}(\theta)$ is a nonzero cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$, say $\theta_{\psi_a}(\theta) = \sigma \otimes \eta_a$ for an irreducible cuspidal automorphic representation σ of $\mathrm{PGL}_2(\mathbb{A})$. Then $\epsilon(1/2, \sigma \otimes \eta_a) = +1$. By result of local theta correspondence [9], we must have

$$\pi_v^{\mathrm{JL}} = \sigma_v, \quad \epsilon(\pi_v)\epsilon(\pi_v \otimes \eta_{a,v})\eta_{a,v}(-1) = \epsilon(\mathbb{B}_v), \quad \text{for all } v,$$

In particular, $\pi = \otimes_v\pi_v$ is cuspidal automorphic and $\epsilon(\pi) = \epsilon(\mathbb{B})$. By property of central sign, $\epsilon(1/2, \theta, \psi) = \epsilon(\mathbb{B})$.

If $\pi = \otimes_v\pi_v$ is cuspidal automorphic and $\epsilon(\pi) = \epsilon(\mathbb{B})$, let $\sigma = \pi^{\mathrm{JL}}$. Then exists a [37] such that

$$\epsilon(\pi_v)\epsilon(\pi_v \otimes \eta_{a,v})\eta_{a,v}(-1) = \epsilon(\mathbb{B}_v), \quad \forall v$$

and further choose a such that $L(1/2, \sigma \otimes \eta_a) \neq 0$ [7]. We must have

$$\theta_v \simeq \theta_{\psi_{a,v}}(\sigma_v \otimes \eta_{a,v}), \quad \forall v.$$

And hence $\theta = \otimes_v\theta_v = \theta_{\psi_a}(\sigma \otimes \eta_a)$ is cuspidal automorphic.

□

Type (III) formulae Now we introduce Rallis inner product formulae and index formulae. Given π an irreducible cuspidal automorphic representation of $H(\mathbb{A})$ and θ be an cuspidal automorphic representation of $\mathbb{G}$ which is the theta lifting of π. As we have said, there always exists $\delta \in F^\times \backslash F^{\times 2}$ and $x \in V_\delta$ such that $\mathrm{Hom}_{T_x(\mathbb{A})}(\pi, \mathbb{C}) \neq 0$ and we can further choose δ such that $L(1/2, \pi \otimes \eta_\delta) \neq 0$ by non-vanishing of quadratic twist [7]. Together with the local comparison result 2.3, any two of the the following formulae

- Type (I): Theorem 3.1
- Type (II): Theorem 3.2
- Type (III): Rallis inner product formula from $H(\mathbb{A})$ to $\mathbb{G}$

implies the third one.

Fix decomposition $\pi \simeq \otimes_v \pi_v$. Whenever $\theta \neq 0$, we have $\theta \simeq \bigotimes \theta_v$ with θ_v the local theta lifting of π_v for each v. It follows that

Theorem 3.6 (Rallis inner product formula)**.** *Assume $\epsilon(1/2, \pi) = +1$. For pure tensors $f_1, f_2 \in \pi$ and $\phi_1, \phi_2 \in \mathcal{S}(V(\mathbb{A}))$. The the following equality holds in*

$$\bigotimes_v \left(\mathrm{Hom}_{(G_v \times H_v)^2}(w_{\psi_v} \otimes \overline{\pi_v} \boxtimes (\overline{w_{\psi_v}} \otimes \pi_v), \theta_v \boxtimes \overline{\theta_v}) \otimes \mathrm{Hom}_{\Delta G_v}(\theta_v \boxtimes \overline{\theta_v}, \mathbb{C})\right) :$$

$$(\theta_{\phi_1}^{f_1}, \theta_{\phi_2}^{f_2}) = \frac{L(1/2, \pi)}{L(2, 1_F)} \cdot \prod_v Z_v^0(\phi_{1,v}, \phi_{2,v}, f_{1,v}, f_{2,v}),$$

where $Z_v^0(\phi_{1,v}, \phi_{2,v}, f_{1,v}, f_{2,v}) = \dfrac{L(2, 1_{F_v})}{L(1/2, \pi_v)} \cdot \displaystyle\int_{H_v} (h\phi_{1,v}, \phi_{2,v})_v \overline{(hf_{1,v}, f_{2,v})_v} dh.$

Now consider the Rallis inner product formula of the converse direction: Given an irreducible automorphic cuspidal representation θ of $\mathbb{G}$ and π an irreducible automorphic cuspidal representation of $H(\mathbb{A})$ which is the theta lifting of θ. Parallel to the above analysis, by local comparison result 2.8, the following three theorem are equivalent:

- Type (I): Theorem 3.3
- Type (II): Theorem 3.4
- Type (III): Rallis inner product form $\mathbb{G}$ to $H(\mathbb{A})$

We have:

Theorem 3.7 (Rallis inner product formula)**.** *Assume $\epsilon(1/2, \theta, \psi) = 1$. For pure tensors $\varphi_1, \varphi_2 \in \theta$ and $\phi_1, \phi_2 \in \mathcal{S}(V(\mathbb{A}))$. The the following equality holds in*

$$\bigotimes_v \left(\mathrm{Hom}_{(H_v \times G_v)^2}(w_{\psi_v} \otimes \overline{\theta_v} \boxtimes (\overline{w_{\psi_v}} \otimes \theta_v), \pi_v \boxtimes \overline{\pi_v}) \otimes \mathrm{Hom}_{\Delta H_v}(\pi_v \boxtimes \overline{\pi_v}, \mathbb{C})\right) :$$

$$(\theta_{\phi_1}^{\varphi_1}, \theta_{\phi_2}^{\varphi_2}) = \frac{L(1/2, \theta, \psi)}{L(2, 1_F)} \cdot \prod_v Z_v^0(\phi_{1,v}, \phi_{2,v}, \varphi_{1,v}, \varphi_{2,v}),$$

where $Z_v^0(\phi_{1,v}, \phi_{2,v}, \varphi_{1,v}, \varphi_{2,v}) = \dfrac{L(2, 1_{F_v})}{L(1/2, \theta_v, \psi_v)} \cdot \displaystyle\int_{G_{0,v}} (h\phi_{1,v}, \phi_{2,v})_v \overline{(h\varphi_{1,v}, \varphi_{2,v})_v} dh.$

By Proposition 3.5, the assumption on root number in the above two Theorems is necessary for the non-vanishingness of theta lifting.

Remark 3.8. The relation between non-vanishingness of theta lifting and non-vanishingness central L-value is considered by Waldspurger [35], [37] via relation between toric period, whittaker-Fourier period and L-value. Both Rallis inner inner product formulae for the two direction could be proved by Siegel-Weil and doubling methods, see [28], [42].

Now we introduce index formulae, which connects Rallis inner product formulae for two direction of theta liftings. As before, consider $\pi \subset \mathcal{A}_0(H(\mathbb{A}))$ with $\epsilon(\pi) = +1$ and $\Theta := \otimes \theta_{\psi_v}(\pi_v) \subset \mathcal{A}_0(\mathbb{G})$.

Definition 3.9.

- *Call one dimensional pure tensor spaces $V_1 \subset \pi$, $V_2 \in \Theta$, $W \subset \mathcal{S}(V(\mathbb{A}))$ self-reflex if for each v and basis $\theta_v \in \mathrm{Hom}_{G_v \times H_v}(\mathcal{S}(V_v), \pi_v \boxtimes \Theta_v)$, one have $\theta_v(W_v) = V_{1,v} \otimes V_{2,v}$.*
- *Let $(V_1, V_2; W)$ be self-reflex lines.*
    - *The global index*

$$\mathrm{Ind}(V_1, V_2; W) := \frac{\theta_\phi^{\theta_\phi^f}}{f(\phi, \phi)} = \frac{\theta_\phi^{\theta_\phi^\varphi}}{\varphi(\phi, \phi)}, \quad 0 \neq f \in V_1, 0 \neq \varphi \in V_2, 0 \neq \phi \in W.$$

– *For each v, let $\delta \in F_v^\times$, $x \in V_v$ with $q(x) = \delta$ and $T_{\delta,v} = \mathrm{Stab}(x) \subset H_v$. The local index*

$$\mathrm{Ind}_\delta(V_{1,v}, V_{2,v}; W_v) := \frac{{}_x\theta_\phi^{x\theta_\phi^f}}{f(\phi,\phi)_v} = \frac{{}_x\theta_\phi^{x\theta_\phi^\varphi}}{\varphi(\phi,\phi)_v}, \quad 0 \neq f \in V_{1,v}, 0 \neq \varphi \in V_{2,v}, 0 \neq \phi \in W_v.$$

Remark 3.10. As a consequence of two rallis inner product formulae and globalization, Proposition 2.10 also holds for general Archimedean local field. And if 2.10 holds for general local field, then the two Rallis inner product for the two directions of theta lifting are equivalent to each other as a consequence of (local and global) see-saw, multiplicity one together with 2.10 (Also see [30]).

Take δ, x to be global and the local measure such that the product measure induces Tamagawa measures.

The following Theorem is a consequence of one side of Rallis inner product formulae, and the relation between local index defined by Waldspurger's explicit local theta lifting and index of normalized local lifting defined by local doubling zeta integral 2.12. If 2.10 holds for general local field, then one could use index formulae to deduce either side of Rallis inner product formulae via multiplicity one, relation between two indexes.

Theorem 3.11. *The following index formulae holds*

$$\mathrm{Ind}(V_1, V_2; W) = \frac{L(1/2,\pi)}{L(2,1)} \prod_v \frac{L(2,1_v)}{L(1/2,\pi_v)} \mathrm{Ind}_\delta(V_{1,v}, V_{2,v}; W_v).$$

4. Arithmetic Rallis inner product formula

In this section, we consider arithmetic Rallis inner product formula for $(\mathrm{SO}_3, \widetilde{\mathrm{SL}_2})$, which lifts cuspidal irreducible automorphic representations of sign -1. The theory of arithmetic theta liftings over totally real base field for parallel weight 2 representations were accomplished by Yuan-Zhang-Zhang's work on modularity of CM points on Shimura curves. For arithmetic theta lifting from SO_3 to $\widetilde{\mathrm{SL}_2}$, we will see that parallel to last Section, the Rallis inner product formulae follows from comparison of two formulae of arithmetic Whittaker-Fourier periods and a local comparison result 2.3.

We also get arithmetic Rallis inner product formula for the converse direction, via showing the equivalence of the two arithmetic Rallis inner product formulae. There are also relations among arithmetic toric periods formulae for the arithmetic theta lifting from SO_3 to $\widetilde{\mathrm{SL}_2}$. As a byproduct, we get a new formulae of arithmetic toric periods, which independent of Gross-Zagier formulae.

In this section, F stands for a totally real field. For an abelian group G and a ring R, denoted $G \otimes_{\mathbb{Z}} R$ by G_R.

4.1. **Arithmetic theta lifting.** Let $\mathbb{B}$ be a totally definite quaternion algebra over $\mathbb{A}$ and B/F be the quaternion algebra which ramified exactly at all except one archimedean place ι of F together with an isomorphism $B(\mathbb{A}_{\mathrm{fin}}) \simeq \mathbb{B}_{\mathrm{fin}}$. Let $\mathbb{H} = \mathbb{A}^\times \backslash \mathbb{B}^\times$.

For each open compact subgroup U of $\mathbb{H}_{\mathrm{fin}}$, denoted by X_U/F the Shimura curve associated to $\mathbb{H}$ of level U with complex uniformization $B^\times \backslash \mathcal{H}^\pm \times \mathbb{H}_{\mathrm{fin}}/U \sqcup \{\mathrm{cusps}\}$, where $B^\times$ acts on upper/lower half-plane $\mathcal{H}^\pm$ via $\iota : B \to B_\iota \simeq M_2(\mathbb{R})$ and fractional linear transformation. Denoted by $\mathrm{Ch}^1(X_U)$ the Chow group of codimensional 1 cycles on X_U and let $\mathrm{Ch}^1(X)_{\mathbb{Q}} = \varinjlim_U \mathrm{Ch}^1(X_U)_{\mathbb{Q}}$, where the inductive system is with respect to pull back maps. There is a natural action of $\mathbb{H}$ on $\mathrm{Ch}^1(X)_{\mathbb{Q}}$ with $\mathbb{H}_\infty$ acts trivially and H_{fin} acts via Hecke correspondence.

Fix a non-trivial additive character $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$. Let $\mathbb{V} = \mathbb{B}^{\mathrm{tr}=0}$. Recall there is a Weil representation w_ψ of $\mathbb{H} \times \mathbb{G}$ acts on $\mathcal{S}(\mathbb{V})$. Denoted by $\mathcal{A}_{\psi,3/2}(\mathbb{G}) \subset \mathcal{A}(\mathbb{G})$ the subspace consists of irreducible automorphic representations θ such that $\theta_{\psi_\infty}(\theta_\infty)$ is the trivial representation of $\mathbb{H}_\infty$. Generalizing the work of Kohnen-Gross-Zagier on modularity of Heegner points, Yuan-Zhang-Zhang [41] constructed a $\mathbb{H} \times \mathbb{G}$ equivalent map

$$\vartheta : \mathcal{S}(\mathbb{V}) \to \mathcal{A}_{\psi,3/2}(\mathbb{G}) \otimes_{\mathbb{Q}} \mathrm{Ch}^1(X)_{\mathbb{Q}},$$
$$\phi \mapsto \vartheta_\phi$$

which we now recall. View $V = B^{\mathrm{tr}=0}$ as quadratic subspace of $\mathbb{V}_{\mathrm{fin}}$. Let $V^- = \{x \in V \mid q(x) \text{ is totally negative}\}$

For $x \in V^-$, let $T = \mathit{Stab}(x) \subset \mathrm{SO}(V)$. Let $z_x^\pm \in \mathcal{H}^\pm$ be the unique fixed point of T_x. We have associated CM cycle

$$z_x := \frac{1}{2}([z_x^+, 1] + [z_x^-, 1]),$$

where c is the complex conjugation. Similar, we have Hecke action

$$z_x \cdot h := \frac{1}{2}([z_x^+, h] + [z_x^-, h]).$$

Fix an identification of V^- as a subset of $\mathbb{V}$ with totally negative norm.

The arithmetic theta kernel is

$$\vartheta_\phi(g) := -2\xi w_\psi(g)\phi(0) + \int_{h \in (H\backslash \mathbb{H}_{\mathrm{fin}})\cdot \mathbb{H}_\infty} \left(\sum_{x \in V^-} w_\psi(g,h)\phi(x) z_x \cdot h \right) dh,$$

where the ξ is the normalized Hodge class on the Shimura curve associated to $\mathbb{H}$ with degree 1. If ϕ is fixed by U, the above arithmetic theta kernel could be described in terms of cycles on level U Shimura curve, which is equivalent to the one in Proposition 4.8 of [40].

Let $\mathcal{A}_0(\mathbb{H})$ be the direct sum of isomorphic classes of irreducible admissible representations π of $\mathbb{H}$ such that the Jacquet-Langlands correspondence of π are cuspidal automorprhic representations of discrete series of parallel weight 2 at infinity. The space $\mathcal{A}_0(\mathbb{H})$ has a $\mathbb{Q}$ structure $\mathcal{A}_0(\mathbb{H}, \mathbb{Q})$ [40]. For irreducible representation $\pi \in \mathcal{A}_0(\mathbb{H}, \mathbb{Q})$ with $M = \mathrm{End}(\pi)_\mathbb{Q}$, we have decomposition

$$\pi_\mathbb{C} = \bigoplus_{\iota: M \to \mathbb{C}} \pi_\iota$$

as $\mathbb{H} \times M$ modules, where $\pi_\iota = \pi \otimes_{M,\iota} \mathbb{C}$.

Let $\mathcal{A}_{0,\psi,3/2}(\mathbb{G}) \subset \mathcal{A}_{\psi,3/2}(\mathbb{G})$ be the subspace of cusp forms. Similar as $\mathcal{A}(\mathbb{H})$, let $\mathcal{A}_{0,\psi,3/2}(\mathbb{G}, \mathbb{Q})$ be direct sum of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ orbits of irreducible representations in $\mathcal{A}_{0,\psi,3/2}(\mathbb{G})$, i.e. consists of $\theta := \bigoplus_{\iota: M \to \mathbb{C}} \theta_{0,\iota}$ as $\mathbb{G} \times M$ representation, here $\theta_{0,\iota} \subset \mathcal{A}_{0,\psi,3/2}(\mathbb{G})$ is irreducible with Hecke field $\iota(M)$ and $\theta_{0,\iota}$ are conjugate to each other.

The semisimplicity of $\mathcal{A}_0(\mathbb{G})$ is well known. Let $\mathrm{Ch}^1_s(X)_\mathbb{Q} \subset \mathrm{Ch}^1(X)_\mathbb{Q}$ be sub-$\mathbb{H}$-module generated by Hodge cycle and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ orbits of CM cycles introduced before, let's say something on automorphy and semisimplicity of $\mathrm{Ch}^1_s(X)_\mathbb{Q}$.

Let ξ be the normalized Hodge cycle with degree 1 and $\mathrm{Ch}^1_s(X)^0$ is the cohomological trivial (equivalently, degree 0) part. By definition, $\mathbb{Q}\xi$ is a trivial $\mathbb{H}$ module and the cohomological trivial part $\mathrm{Ch}^1_s(X)^0_\mathbb{Q}$ is also $\mathbb{H}$ stable.

As $\mathbb{H}$ module, $J_X(F) \simeq \bigoplus_{[A]} A(F)_\mathbb{Q} \otimes_{\mathrm{End}(A)_\mathbb{Q}} \pi_A$ [40], where $[A]$ runs over all simple abelian parameterized by X. Thus the $\mathbb{H}$ submodule $\mathrm{Ch}^1_s(X)^0$ of $J_X(F)$ is automorphic and semisimple.

The space $\mathrm{Ch}^1_s(X)^0_\mathbb{Q}$ also has multiplicity one [39] in the sense that for each π_A, if $\epsilon(A) = +1$, then $\dim_M \mathrm{Hom}_\mathbb{H}(\mathrm{Ch}^1_s(X)^0_\mathbb{Q}, \pi_A) = 0$ and ≤ 1 if $\epsilon(A) = -1$. Such multiplicity one and condition on sign follows from the multiplicity one of irreducible cuspidal automorphic representation of $\mathbb{G}$ and automorphy criterion of arithmetic theta lifting (See Proposition 3.5). Thus

Theorem 4.1 (Yuan-Zhang-Zhang)**.** *We have*

$$\mathrm{Ch}^1_s(X)^0_\mathbb{Q} \simeq \bigoplus_{\substack{[A] \\ \epsilon(A) = -1}} \mathrm{Ch}^1_s(X)^0_\mathbb{Q}[\pi_A],$$

where $\mathrm{Ch}^1_s(X)^0_\mathbb{Q}[\pi_A] \subset \mathrm{Ch}^1_s(X)^0_\mathbb{Q}$ *is the* π_A *component, which is either* 0 *or isomorphic to* π_A*.*

Arithmetic theta lifting from $\mathbb{H}$ to $\mathbb{G}$

Given a simple abelian variety A over F parameterized by X. One can construct a irreducible representation of $\mathbb{H}$ over $\mathbb{Q}$ in the following: Let $\xi_U \in \mathrm{Pic}(X_U)_\mathbb{Q}$ be the normalized Hodge class on X_U, which has degree 1 on each geometric connected component of X_U. We have

$$\pi_A := \varinjlim_U \mathrm{Hom}_{\xi_U}(X_U, A)_\mathbb{Q},$$

where $\mathrm{Hom}_{\xi_U}(X_U, A)_\mathbb{Q}$ are the morphisms in $\mathrm{Hom}(X_U, A)_\mathbb{Q}$ using ξ_U as a base point in the sense that if ξ_U is represented by a divisor $\sum a_i x_i$, then $f \in \mathrm{Hom}_{\xi_U}(X_U, A)$ if and only if $\sum_i a_i f(x_i) = 0$ in $A(\overline{F})_\mathbb{Q}$. Let $M := \mathrm{End}_F(A)_\mathbb{Q}$ which is a totally real field with $[M : \mathbb{Q}] = \dim A$, then $\mathrm{End}(\pi_A)_\mathbb{Q} = M$. The following spectral decomposition holds [40]

$$\mathcal{A}(\mathbb{H}, \mathbb{Q}) = \bigoplus_{[A]} \pi_A,$$

where $[A]$ runs over all isogeny classes of simple abelian varieties A over F parameterized by X.

Define the arithmetic theta lifting $\vartheta := \vartheta_\psi(\pi_{A,\mathbb{C}})$ of $\pi_{A,\mathbb{C}}$ to be the $\mathbb{G}\times M$-module

$$\{\vartheta_f^\phi := \overline{f}\circ\vartheta_\phi \in \mathcal{A}_{0,\psi,3/2}(\mathbb{G})\otimes_{\mathbb{Q}} A(F)_{\mathbb{Q}} \;\Big|\; f\in\pi_{A,\mathbb{C}}, \phi\in\mathcal{S}(\mathbb{V})\}.$$

Note here for $f\in\pi_A$, $\overline{f}=f$, since M is totally real. If we only consider $f\in\pi_A$, the lifting gives the same representation, we may also write as $\vartheta_\psi(\pi_A)$ to emphasis $f\in\pi_A$. There is a $\mathbb{C}\otimes_{\mathbb{Q}} M$ valued $\mathbb{H}$ invariant M-bilinear Hermitian pairing on ϑ:

$$\langle\varphi_1,\varphi_2\rangle_M := \int_{\mathrm{SL}_2\backslash\mathrm{SL}_2(\mathbb{A})}\langle\varphi_1(g),\varphi_2(g)\rangle_M dg,$$

where $\langle\ ,\ \rangle_M : A(F)_{\mathbb{Q}}\otimes_M A(F)_{\mathbb{Q}}\to\mathbb{C}\otimes_{\mathbb{Q}} M$ is the M-bilinear height pairing will such that $\mathrm{tr}_{M/\mathbb{Q}}\circ\langle\ ,\ \rangle_M$ is the Néron -Tate height pairing. We may extend $\langle\ ,\ \rangle_M$ to a Hermitian pairing from $A(F)_{\mathbb{C}}\otimes_{\mathbb{C}\otimes_{\mathbb{Q}}M} A(F)_{\mathbb{C}}$ to $\mathbb{C}\otimes_{\mathbb{Q}} M$ and still denoted by $\langle\ ,\ \rangle_M$. Fix a decomposition $\pi_A\simeq\bigotimes\pi_v$. We have $\pi_{v,\mathbb{C}}=\bigoplus_{\iota:M\to\mathbb{C}}\pi_{v,\iota}$, where $\pi_{v,\iota}$ is a irreducible representation of $\mathbb{H}_v$ over $\mathbb{C}$ on which M acts via embedding $\iota: M\to\mathbb{C}$. The local theta lifting $\theta_{\psi_v}(\pi_{v,\mathbb{C}})$ of $\pi_{v,\mathbb{C}}$ is a $M\times G_v$ module given by $\bigoplus_{\iota:M\to\mathbb{C}}\theta_{\psi_v}(\pi_{v,\iota})$.

Proposition 4.2. *Under assumption $\epsilon(A)=-1$, the $\mathbb{G}\times M$ representation $\bigotimes_v\vartheta_{\psi_v}(\pi_{v,\mathbb{C}})$ is cuspidal automorphic, and isomorphic to a unique θ in $\mathcal{A}_{0,\psi,3/2}(\mathbb{G},\mathbb{Q})$. The representation $\vartheta\neq 0$ if and only if the $\mathbb{C}\otimes_{\mathbb{Q}} M$ module $L\subset A(\mathbb{Q})_{\mathbb{C}}$ generated by Whittaker-Fourier coefficients of $\vartheta_\psi(\pi_A)$ is free of rank 1. In any case, $\vartheta_\psi(\pi_{A,\mathbb{C}})=\theta\otimes_{\mathbb{C}\otimes_{\mathbb{Q}}M} L$.*

Proof. The first claim follows from multiplicity one of $\mathcal{A}_0(\mathbb{G})$, the fact that theta lifting preserving Hecke field and action of Hecke field and Proposition 3.5.

Assume $0\neq\vartheta_\psi(\pi_{A,\mathbb{C}})$. Since $\mathcal{A}_0(\mathbb{G})$ is semisimple, the global arithmetic theta lifting factor through tensor product $\otimes_v\theta_{\psi_v}(\pi_{v,\mathbb{C}})$ of local theta liftings thus must induce an isomorphism $\otimes_v\theta_{\psi_v}(\pi_{v,\mathbb{C}})\simeq\vartheta_\psi(\pi_A)$.

Fix embedding $\iota: M\to\mathbb{C}$ and let $\pi_{A,\iota}:=\pi_A\otimes_{M,\iota}\mathbb{C}$ be the irreducible representation of $\mathbb{H}$ over $\mathbb{C}$. Since $\mathcal{A}_{0,\psi,3/2}(\mathbb{G})$ has multiplicity one, thus $\vartheta_\psi(\pi_{A,\iota})=\theta_\iota\otimes_{M,\iota} L$ an irreducible representation $\theta_\iota\subset\mathcal{A}_{0,\psi,3/2}(\mathbb{G})$ with Hecke field $\iota(M)$ and $L\otimes_{M,\iota}\mathbb{C}$ is one dimensional. Different ι are conjugate to each other thus

$\vartheta_\psi(\pi_A)=\theta\otimes_{\mathbb{C}\otimes_{\mathbb{Q}}M} L$ for some $\theta\subset\mathcal{A}_{0,\psi,3/2}(\mathbb{G})$. □

Arithmetic theta lifting from $\mathbb{G}$ to $\mathbb{H}$

Define the arithmetic theta lifting $\pi:=\vartheta_\psi(\theta)$ of $\theta\subset\mathcal{A}_{0,\psi,3/2}(\mathbb{G},\mathbb{Q})$ with Hecke field M be the $\mathbb{H}\times M$ module

$$\left\{\vartheta_\phi^\varphi=\sum_{\iota:M\to\mathbb{C}}\int_{\mathrm{SL}_2\backslash\mathrm{SL}_2(\mathbb{A})}\vartheta_\phi(g)\overline{\varphi_\iota(g)}dg\in\mathrm{Ch}_s^1(X)_{\mathbb{C}}\;\Big|\;,\quad\varphi\in\theta,\phi\in\mathcal{S}(\mathbb{V})\right\}$$

here φ_ι is the ι component of φ.

Lemma 4.3. *The arithmetic theta lifting lies in cohomological trivial part $\mathrm{Ch}_s^1(X)^0_{\mathbb{C}}$ of $\mathrm{Ch}_s^1(X)_{\mathbb{C}}$.*

Proof. We will prove each ι component lies in cohomological trivial part. The degree map on $\vartheta_\psi(\theta_\iota)$ gives an element in

$$\mathrm{Hom}_{M,\mathbb{H}\times\mathbb{G}}(\mathcal{S}(\mathbb{V})\boxtimes\overline{\theta_\iota},\mathbb{C}).$$

If it is nonzero, there exists a pure tensor $\phi_0\boxtimes\varphi_0$ such that $\deg(\vartheta_{\phi_0}^{\varphi_0})$ is nonzero. Take a place v such that the maximal G_v invariant quotient of $\mathcal{S}(\mathbb{V}_v)\boxtimes\overline{\theta_{\iota,v}}$ has no trivial representation of $\mathbb{H}_v$. Consider $\mathbb{H}_v\times\mathbb{G}_v$ equivalent embedding map

$$\mathcal{S}(\mathbb{V}_v)\boxtimes\overline{\theta_{\iota,v}}\to\mathcal{S}(\mathbb{V})\boxtimes\overline{\theta_\iota},\quad \phi_v\boxtimes\overline{\varphi_v}\mapsto\phi_v\otimes\phi_0^{(v)}\boxtimes\overline{\varphi_v\otimes\varphi_0^{(v)}},$$

the degree map give a nonzero elements in

$$\mathrm{Hom}_{M,\mathbb{H}_v\times G_v}(\mathcal{S}(\mathbb{V}_v)\boxtimes\overline{\theta_{\iota,v}},\mathbb{C}),$$

contradiction. □

We now introduce a $\mathbb{H}\times M$ invariant pairing $\langle\ ,\ \rangle_M$ on π.

There is a $\mathbb{H}$ invariant height pairing on $\mathrm{Ch}_s^1(X)^0_{\mathbb{Q}}$ defined by

$$\langle P,Q\rangle = 2\mathrm{vol}(X_U)^{-1}\langle P,Q\rangle_{\mathrm{NT},U},\quad P,Q\in\mathrm{Ch}_s^1(X_U)^0_{\mathbb{Q}},$$

where $\langle\ ,\ \rangle_{\mathrm{NT},U}$ is the Néron -Tate height pairing of level U, X_U is viewed as disjoint union of quotient of $\mathcal{H}^+$ and equipped each $\mathcal{H}^+$ with measure $\frac{dxdy}{4\pi y^2}$. May extend the height pairing to be a Hermitian pairing on $\mathrm{Ch}^1_s(X)^0_{\mathbb{C}}$, still denoted by $\langle\ ,\ \rangle$.

There is a $\mathbb{H}$ invariant M-bilinear Hermitian pairing on π

$$\langle\ ,\ \rangle_M : \pi \otimes_{\mathbb{C}\otimes_{\mathbb{Q}} M} \pi \to \mathbb{C} \otimes_{\mathbb{Q}} M$$

such that $\langle\ ,\ \rangle = \mathrm{tr}_{\mathbb{C}\otimes_{\mathbb{Q}} M/\mathbb{C}} \circ \langle\ ,\ \rangle_M$.

Proposition 4.4. *Under assumption $\epsilon(1/2, \theta, \psi) = -1$, the $\mathbb{H} \times M$ representation $\bigotimes_v \theta_{\psi_v}(\theta_v)$ is cuspidal automorphic, and isomorphic to a unique $\pi_{A,\mathbb{C}}$ for π_A in $\mathcal{A}_0(\mathbb{H}, \mathbb{Q})$. If $\vartheta_\psi(\theta) \neq 0$, then $\vartheta_\psi(\theta) = \mathrm{Ch}^1_s(X)^0_{\mathbb{C}}[\pi_{A,\mathbb{C}}] \neq 0$. In any case, $\vartheta_\psi(\theta) = \mathrm{Ch}^1_s(X)^0_{\mathbb{C}}[\pi_{A,\mathbb{C}}]$.*

The proof is parallel to the Proposition 4.2.

Remark 4.5. We will see from arithmetic see-saw that $\vartheta_\psi(\theta) = \mathrm{Ch}^1_s(X)^0_{\mathbb{C}}[\pi_{A,\mathbb{C}}]$ always holds.

Arithmetic see-saw Assume $\pi_A \subset \mathcal{A}_0(\mathbb{H}, \mathbb{Q})$ with $\varepsilon(1/2, \pi_A) = -1$ and $\theta \subset \mathcal{A}_{0,\psi,3/2}(\mathbb{G}, \mathbb{Q})$ such that they are correspondence to each other in the sense that locally they are theta lifting of each other. (See Proposition 3.5.) Let $L \subset A(F)_{\mathbb{Q}}$ be the M-submodule generated by image of CM points in $A(F)_{\mathbb{Q}}$ under elements in π_A. We have known that $\dim_M L \leq 1$ and $L \neq 0$ if and only if $\mathrm{Ch}^1_s(X)_{\mathbb{Q}}[\pi_A] \neq 0$. Assume

$$L \neq 0.$$

Note that there is a natural M linear pairing

$$\vartheta_\psi(\theta) \otimes_M \pi_A \otimes_M L \to \mathbb{C} \otimes_{\mathbb{Q}} M, \quad (C, f, P) \mapsto \langle f(C), P\rangle_M$$

induces a map $\iota : \vartheta_\psi(\vartheta) \to \pi_{A,\mathbb{C}} \otimes_M L$ via Riesz representation theorem, where the pairing on $\pi_A \otimes_M L$ is induced by intersection on L and the Hermitian invariant pairing on $\pi_{A,\mathbb{C}}$. The map ι preserve pairing on both side.

We have arithmetic see-saw:

$$\langle \vartheta^f_\phi, \varphi \otimes P\rangle_M = \langle \iota(\vartheta^\varphi_\phi), f \otimes P\rangle_M, \quad f \in \pi_A, \varphi \in \theta, \quad P \in L, \phi \in \mathcal{S}(\mathbb{V}).$$

In particular, $\mathrm{Ch}^1_s(X)_{\mathbb{Q}}[\pi_A] \neq 0$ implies that $\vartheta_\psi(\pi_A) \neq 0$ (via considering specifically choice of ϕ) and hence $\vartheta_\psi(\theta) \neq 0$. Thus they are all equivalent by Proposition 4.2, 4.4.

4.2. **Whittaker-Fourier periods formulae of arithmetic theta lifting from SO_3 to $\widetilde{\mathrm{SL}_2}$.** Consider the arithmetic theta lifting $\vartheta := \vartheta_\psi(\pi_A) \subset \mathcal{A}_{0,\psi,3/2}(\mathbb{G})$ of an irreducible representation $\pi_A \subset \mathcal{A}_0(\mathbb{H}, \mathbb{Q})$.

In this subsection, we consider two formulae on relation between Whittaker-Fourier periods

$$W_{\psi_\delta}(\varphi) := \int_{N\backslash N(\mathbb{A})} \varphi(n(y))\psi_\delta(-y)dy, \quad \varphi \in \vartheta, \delta \in F^\times$$

and central value of derivatives of L-functions of quadratic twists. In particular, the quadratic twist L-value gives global obstruction for arithmetic Whittaker-Fourier periods to be nonzero global Whittaker functional, whenever arithmetic theta lifting is nonzero and local $\psi_{\delta,v}$ Whittaker functionals exist for all v.

In the following type (I'), (II') formulae, we fix irreducible $\pi_A \subset \mathcal{A}_0(\mathbb{H}, \mathbb{Q})$ with sign $\epsilon(1/2, \pi_A) = -1$. Let $\theta \subset \mathcal{A}_{0,\psi,3/2}(\mathbb{G}, \mathbb{Q})$ corresponding to π_A as in Proposition 4.2, let $\vartheta := \vartheta_\psi(\pi_A)$ its arithmetic theta lifting.

Type (I') formulae Let $\delta \in F$ be totally negative, $x \in V^-_\delta := \{x \in V^- \mid q(x) = \delta\}$ and $K = F(x)$. Recall we also view x as element in $\mathbb{V}_\delta$. Let $T_x, \mathbb{T}_x$ be stablizer of x in H, $\mathbb{H}$ respectively. Note that

$$\begin{aligned} W_{\psi_\delta}(\vartheta_\phi) &= \int_{N\backslash N(\mathbb{A})} \vartheta_\phi(n(y))\psi_\delta(-y)dy \\ &= \int_{h\in(H\backslash\mathbb{H}_{\mathrm{fin}})\cdot\mathbb{H}_\infty} \left(\sum_{x\in V^-_\delta} \phi(h^{-1}\circ x)z_x \cdot h\right) dh \\ &= \int_{\mathbb{T}_x\backslash\mathbb{H}} \phi(h^{-1}\circ x) \int_{T_x\backslash T_x(\mathbb{A})} z_x \cdot thdtdh \end{aligned}$$

It follows that

$$W_{\psi_\delta}(\vartheta_\phi^f) = \int_{\mathbb{T}_x\backslash\mathbb{H}} \phi(h^{-1}\circ x)P_{T_x}^+(hf)dtdh,$$

where $P_{T_x}^+ \in \mathrm{Hom}_{\mathbb{T}_x,M}(\pi_A, A(F)_\mathbb{Q})$ given by $P_{T_x}^+(f) = \frac{1+c}{2}P_{T_x}(f)$, with $P_{T_x}(f) = \int_{T_x(F)\backslash T_x(\mathbb{A})} f([z_x, t_{\mathrm{fin}}])dt$.

Fix decomposition $\pi_A = \bigotimes_v \pi_v$. The local theta lifting $\theta_{\psi_v}(\pi_{v,\mathbb{C}})$ of $\pi_{v,\mathbb{C}}$ is the $G_v \times M$ representation given by $\bigoplus_{\iota:M\to\mathbb{C}} \theta_{\psi_v}(\pi_{v,\iota})$. Assume that

(a_1') $\mathrm{rank}_{\mathbb{C}\otimes_\mathbb{Q}M} \bigotimes_{\mathbb{C}\otimes_\mathbb{Q}M,v} \mathrm{Hom}_{N_v,M}(\theta_{\psi_v}(\pi_{v,\mathbb{C}}), \psi_{v,\delta}\otimes_\mathbb{Q} M) = 1$, equivalently
$\mathrm{rank}_{\mathbb{C}\otimes_\mathbb{Q}M} \bigotimes_{\mathbb{C}\otimes_\mathbb{Q}M,v} \mathrm{Hom}_{\mathbb{T}_{x,v},M}(\pi_{v,\mathbb{C}}, \mathbb{C}\otimes_\mathbb{Q} M) = 1$,

(a_2') $f = \otimes f_v \in \pi_A$ pure tensor so that f_v is a test vector for $\mathrm{Hom}_{\mathbb{T}_{x,v},M}(\pi_v, M)$.

Whenever $\vartheta \neq 0$, we have

$$\vartheta \simeq \bigotimes_{\mathbb{C}\otimes_\mathbb{Q}M,v} \mathcal{W}_{\delta,v}, \quad \theta_\phi^f \mapsto \otimes\theta_{\phi_v}^{f_v}(1)/f_v(1), \quad \forall\phi = \otimes\phi_v \in \mathcal{S}(\mathbb{V}),$$

where for each v, (i) $\mathcal{V}_{x,v}$ is the $\mathbb{T}_{x,v}$ model of π_v which is a subspace of M-valued functions on $\mathbb{T}_{x,v}\backslash\mathbb{H}_v$ and we view f_v as in $\mathcal{V}_{x,v}$; (ii) $\mathcal{W}_{\delta,v}$ is the $\psi_{\delta,v}\otimes_\mathbb{Q} M$ Whittaker model of $\theta_{\psi_v}(\pi_{v,\mathbb{C}})$ which is s subspace of $\mathbb{C}\otimes_\mathbb{Q} M$-valued functions on G_v with action of N_v by $\psi_{\delta,v}$. Note here $f = \overline{f}$ and $\overline{f_v} = \overline{f_v}$ for each v, since M is totally real. We add subscript x for explicit local theta lifting to emphasis its dependence on x and note here the local theta lifting maps $\mathcal{V}_{x,v}$ to $\mathcal{W}_{\delta,v}$

We have the following equality of Whittaker functionals:

$$W_{\psi_\delta}(\vartheta_\phi^f) = P_{T_x}^+(f)\prod_v \frac{{}_x\theta_{\phi_v}^{f_v}}{f_v(1)} \in A(F)_\mathbb{C}, \quad \phi = \otimes\phi_v \in \mathcal{S}(\mathbb{V}).$$

We now consider the self intersection of the Whittaker-Fourier coefficients.

There exists a global $\mathbb{H}$-invariant M-linear pairing $\pi_A \otimes_M \pi_A \to M$ given by

$$(f_1, f_2) = 2\mathrm{vol}(X_U)^{-1} f_{1,U}\circ f_{2,U}^\vee,$$

where $f_i \in \pi_A^U$ for some open compact subgroup $U \subset \mathbb{H}_{\mathrm{fin}}^\times$, $f_{i,U} \in \mathrm{Hom}(\mathrm{Jac}(X_U), A)_\mathbb{Q}$ corresponds to f_i. For each place v of F, fix a M-bilinear local invariant Hermitian pairing $(\ ,\ )_v : \pi_v \otimes_M \pi_v \to M$ such that the product gives global one. Let $\alpha_{\delta,v}^0$ be the basis of $\mathrm{Hom}_{\mathbb{T}_{x,v},M}(\pi_{v,\mathbb{C}}, \mathbb{C}\otimes_\mathbb{Q} M) \otimes_{\mathbb{C}\otimes_\mathbb{Q}M} \overline{\mathrm{Hom}_{T_{x,v},M}(\pi_{v,\mathbb{C}}, \mathbb{C}\otimes_\mathbb{Q} M)}$ the same as in Waldspurger formula:

$$\alpha_{\delta,v}^0(f_{1,v}, f_{2,v}) := \frac{L(1,\eta_{\delta,v})^2 L(1,\pi_v, ad)}{L(2,1_{F_v})L(1/2,\pi_{v,K_v})}\int_{\mathbb{T}_{x,v}} (tf_{1,v}, f_{2,v})_v dt, \quad f_{i,v} \in \pi_{v,\mathbb{C}}.$$

Since we have relation between arithmetic Whittaker-Fourier periods and arithmetic toric periods, as a consequence of Gross-Zagier formula [40],

Theorem 4.6. *Under assumption (a_1'), and let $f_i = \otimes f_{i,v}$ be as in (a_2'). The equality holds in*

$$\mathrm{Hom}_{N(\mathbb{A}),M}(\vartheta, \psi_\delta\otimes_\mathbb{Q} M) \bigotimes_{\mathbb{C}\otimes_\mathbb{Q}M} \overline{\mathrm{Hom}_{N(\mathbb{A})}(\vartheta, \psi_\delta\otimes_\mathbb{Q} M)} :$$

$$\langle W_{\psi_\delta}(\vartheta_{\phi_1}^{f_1}), W_{\psi_\delta}(\vartheta_{\phi_2}^{f_2})\rangle_M = \frac{L'(1/2,\pi_{A,K})L(2,1_F)}{2L(1,\eta_\delta)^2 L(1,\pi_A, ad)}\prod \alpha_{\delta,v}^0(f_{1,v}, f_{2,v})\frac{{}_x\theta_{\phi_{1,v}}^{f_{1,v}}(1)}{f_{1,v}(1)}\cdot\frac{\overline{{}_x\theta_{\phi_{2,v}}^{f_{2,v}}(1)}}{f_{2,v}(1)}, \quad \phi_i = \otimes\phi_{i,v} \in \mathcal{S}(\mathbb{V}).$$

The proof is the parallel to Theorem3.1 and we omit here.

Type (II') formulae

Now we introduce another decomposition formulae for arithmetic Whittaker-Fourier period. Recall there is a natural Hermitian pairing on ϑ induced by the Néron -Tate height pairing and Petersson inner product:

$$\langle\varphi_1, \varphi_2\rangle_M = \int_{\mathrm{SL}_2\backslash\mathrm{SL}_2(\mathbb{A})} \langle\varphi_1(g), \varphi_2(g)\rangle_M dg.$$

Consider the case $\vartheta \neq 0$, then we have $\vartheta = \theta\otimes_M L$. Observe that for $\varphi_i := \vartheta_{\phi_i}^{f_i} = \varphi_{i,0}\otimes y_i \in \theta\otimes_M L$,

$$(\varphi_{1,0}, \varphi_{2,0})\langle W_{\psi_\delta}(\varphi_1), W_{\psi_\delta}(\varphi_2)\rangle_M = \langle\varphi_1, \varphi_2\rangle_M \cdot W_{\psi_\delta}(\varphi_{1,0})\overline{W_{\psi_\delta}(\varphi_{2,0})}$$

as equality in $\mathbb{C}\otimes_\mathbb{Q} M$. Fix a decomposition of Petersson inner product on θ. It follows from the Type (II) formula 3.2 that

Theorem 4.7. *Under the assumption* (a_1')*, For* $\varphi_i \in \vartheta$ *pure tensor,*

$$\langle W_{\psi_\delta}(\varphi_1), W_{\psi_\delta}(\varphi_2)\rangle_M = \langle\varphi_1,\varphi_2\rangle_M \frac{L(1/2,\pi_A\otimes\eta_\delta)L(2,1_F)}{2L(1,\pi_A,ad)}\prod_v \frac{\beta^0_{\delta,v}(\varphi_{1,0,v},\varphi_{2,0,v})}{(\varphi_{1,0,v},\varphi_{2,0,v})_v}.$$

Note that whenever $\vartheta = 0$, both sides of the equation are 0.

4.3. **Toric periods formulae of arithmetic theta lifting from $\widetilde{\mathrm{SL}_2}$ to SO_3.** In the following type (I'), (II') formulae, we fix $\theta \subset \mathcal{A}_{0,3.2,\psi}(\mathbb{H},\mathbb{Q})$ with sign $\epsilon(1/2,\theta,\psi) = -1$. Let $\pi_A \subset \mathcal{A}_0(\mathbb{H},\mathbb{Q})$ corresponding to θ as in Proposition 4.2, let $\pi := \vartheta_\psi(\theta)$ its arithmetic theta lifting.

Type (I') formulae

Let δ x, T_x, $\mathbb{T}_x$ be the same as in the last subsection. Fix decomposition $\theta = \otimes\theta_v$.

Assume

- (b_1') $\mathrm{rank}_{M\otimes\mathbb{C}} \bigotimes_{\mathbb{C}\otimes_\mathbb{Q} M,v} \mathrm{Hom}_{\mathbb{T}_{x,v},M}(\theta_{\psi_v}(\theta_v), \mathbb{C}\otimes_\mathbb{Q} M) = 1$, equivalently $\mathrm{rank}_{M\otimes\mathbb{C}} \bigotimes_{\mathbb{C}\otimes_\mathbb{Q} M} \mathrm{Hom}_{N_v,M}(\theta_v, \psi_{v,\delta}\otimes_\mathbb{Q} M) = 1$,
- (b_2') $\varphi = \otimes_v\varphi_v \in \theta$ be a pure tensor such that φ_v is a local test vector for $\mathrm{Hom}_{N_v,M}(\theta_v, \psi_{v,\delta}\otimes_\mathbb{Q} M)$ for all v.

Whenever $\pi \neq 0$, we have

$$\pi \simeq \bigotimes_{\mathbb{C}\otimes_\mathbb{Q} M,v} \mathcal{V}_{x,v}, \quad \vartheta^\varphi_\phi \mapsto \otimes_v \frac{{}_x\theta^{\varphi_v}_{\phi_v}(1)}{\overline{\varphi_v(1)}L(1/2,\theta_v,\psi_v)}, \quad \forall\phi = \otimes_v\phi_v \in \mathcal{S}(\mathbb{V}),$$

here for each v, $\mathcal{V}_{x,v}$ is the $\mathbb{T}_{x,v}\times M$ model of $\theta_\psi(\theta_v)$ introduced in local theory consists of $M\otimes\mathbb{C}$ valued functions on $\mathbb{T}_{x,v}\backslash\mathbb{H}_v$, view $\varphi_v \in \mathcal{W}_{\delta,v}$ in the $\psi_{\delta,v}\times M$ model of θ_v and we add subscript x for local theta lifting to emphasis its dependence on x.

Let $Z_x = \int_{T_x(F)\backslash T_x(\mathbb{A})}[z_x^+, t]dt$, where $z_x^+ \in \mathcal{H}^+$ is the fixed point of T_x. Then $\langle\vartheta^\varphi_\phi, Z_x\rangle_M$ gives an element in $\mathrm{Hom}_{\mathbb{T}_x,M}(\pi, \mathbb{C}\otimes_\mathbb{Q} M)$. Fix a decomposition of Petersson inner product on θ.

Conjecture 4.8. *Under assumption* (b_1')*.*

- *Let* φ *be pure tensor as in* (b_2')*.The following equality holds in* $\mathrm{Hom}_{\mathbb{T}_x}(\pi, \mathbb{C}\otimes_\mathbb{Q} M)$*:*

$$\langle\vartheta^\varphi_\phi, Z_x\rangle_M = L'(1/2,\theta,\psi)\cdot\overline{W_{\psi_\delta}(\varphi)}\prod_v \frac{{}_x\theta^{\varphi_v}_{\phi_v}(1)}{\overline{\varphi_v(1)}L(1/2,\theta_v,\psi_v)}.$$

- *Let* φ_i *be pure tensor as in* (b_2')*,*

$$\langle\vartheta^{\varphi_1}_{\phi_1}, Z_x\rangle_M\overline{\langle\vartheta^{\varphi_2}_{\phi_2}, Z_x\rangle_M} = \frac{L'(1/2,\theta,\psi)^2L(1/2,\theta,\psi_\delta)L(2,1_F)}{2L(1,\pi_A,ad)}\prod_v \beta^0_{\delta,v}(\varphi_{2,v},\varphi_{1,v})\prod_v \frac{\theta^{\varphi_{1,v}}_{\phi_{1,v}}(1)}{\overline{\varphi_{1,v}(1)}L(1/2,\theta_v,\psi_v)}\frac{\overline{\theta^{\varphi_{2,v}}_{\phi_{2,v}}(1)}}{\varphi_{2,v}(1)L(1/2,\theta_v,\cdot$$

here $\beta^0_{\delta,v}$ *is the same as sign* $+1$ *case.*

Remark 4.9. The second part follows from the first part of conjecture and the Theorem 3.2. We will prove the first part of conjecture holds up to ±1 (See Theorem 4.13).

We have $L(s,\theta,\psi_\delta) = L(s,\pi_A\otimes\eta_\delta)$ for any $\delta \in F^\times$.

Type (II') formulae

We also have type (II') decomposition formulae, i.e. Gross-Zagier formulae:

We have $\iota : \pi \simeq \pi_A \otimes_{\mathbb{C}\otimes_\mathbb{Q} M} L$, $L \subset A(F)_\mathbb{C}$ with $\mathbb{C}\otimes_\mathbb{Q} M$ rank either 1 or 0. And if $\iota(f) = f_0\otimes P \in \pi_A\otimes_M L$, then

$$\langle P_{T_x}(f_0), P\rangle_M = \langle f, Z_x\rangle_M.$$

Theorem 4.10. [40] *Assume* $\mathrm{Hom}_{\mathbb{T}_{x,v}}(\pi_{A,v}, \mathbb{C}\otimes_\mathbb{Q} M) = 1$ *for all* v*. For each pure tensor* $f_i = f_{i,0}\otimes P_i \in \vartheta_\psi(\theta)$*,*

$$\langle f_1, Z_x\rangle_M\overline{\langle f_2, Z_x\rangle_M} = \langle f_1, f_2\rangle_M\cdot\frac{L'(1/2,\theta,\psi)L(1/2,\theta,\psi_\delta)L(2,1_F)}{2L(1,\eta_\delta)^2L(1,\pi_A,ad)}\prod_v \frac{\alpha^0_{\delta,v}(f_{1,0,v},f_{2,0,v})}{(f_{1,0,v},f_{2,0,v})}$$

here $\alpha^0_{\delta,v}$ *is the same as sign* $+1$ *case.*

4.4. **Arithmetic Rallis inner product formula and arithmetic index formulae.** The method is same as in the sign +1 case.

Arithmetic theta lifting from $\mathbb{H}$ to $\mathbb{G}$: Given $\pi \subset \mathcal{A}_0(\mathbb{H}, \mathbb{Q})$ irreducible with $\epsilon(\frac{1}{2}, \pi) = -1$ Recall there exists totally negative δ such that assumption (a_1') holds and $L(1/2, \pi_A \otimes \eta_\delta) \neq 0$. Together with local comparison result 2.3, have any two of the above formulae implies the third one:

- Type (I): Theorem 4.6
- Type (II): Theorem 4.7
- Type (III): Arithmetic Rallis inner product formula from $\mathbb{H}$ to $\mathbb{G}$.

Fix decomposition of π_A and decomposition of M equivalent Hermitian inner product $(\ ,\ ) = \otimes_v(\ ,\ )_v$ on $\pi_{A,\mathbb{C}}$.

Theorem 4.11 (type (III')). *For pure tensors $f_1, f_2 \in \pi_A$ and $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{V})$, the following equality holds:*

$$(\vartheta_{\phi_1}^{f_1}, \vartheta_{\phi_2}^{f_2}) = \frac{L'(1/2, \pi_A)}{L(2, 1_F)} \cdot \prod_v Z_v^0(\phi_{1,v}, \phi_{2,v}, f_{1,v}, f_{2,v}),$$

where $Z_v^0(\phi_{1,v}, \phi_{2,v}, f_{1,v}, f_{2,v}) = \dfrac{L(2, 1_{F_v})}{L(1/2, \pi_v)} \cdot \displaystyle\int_{\mathbb{H}_v} (h\phi_{1,v}, \phi_{2,v})_v \overline{(hf_{1,v}, f_{2,v})_v} dh$ *is the doubling zeta integral which valued in $\mathbb{C} \otimes_{\mathbb{Q}} M$.*

Arithmetic theta lifting from $\mathbb{G}$ to $\mathbb{H}$: Given $\theta \subset \mathcal{A}_{0,\psi,3/2}(\mathbb{G}, \mathbb{Q})$ with $\epsilon(1/2, \theta, \psi) = -1$. Then there exists totally negative δ such that assumption (b_1') holds and $L(1/2, \theta, \psi_\delta) \neq 0$. Together with local comparison result 2.8, have any two of the above formulae implies the third one:

- Type (I'): Conjecture 4.8
- Type (II'): Theorem 4.10
- Type (III'): Arithmetic Rallis inner product formula from $\mathbb{H}$ to $\mathbb{G}$.

By multiplicity one, global arithmetic see-saw and local see-saw 2.10, and Proposition 2.10, the arithmetic Rallis inner product formulae for the both sides are also equivalent. Thus we have the following:

Fix decomposition of θ and decomposition of M equivalent Hermitian inner product $(\ ,\ ) = \otimes_v(\ ,\ )_v$ on θ

Theorem 4.12. *Assume $\epsilon(1/2, \theta, \psi) = -1$. For pure tensors $\varphi_1, \varphi_2 \in \theta$ and $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{V})$, the following equality holds:*

$$(\vartheta_{\phi_1}^{\varphi_1}, \vartheta_{\phi_2}^{\varphi_2}) = \frac{L'(1/2, \theta, \psi)}{L(2, 1_F)} \cdot \prod_v Z_v^0(\phi_{1,v}, \phi_{2,v}, \varphi_{1,v}, \varphi_{2,v}),$$

where $Z_v^0(\phi_{1,v}, \phi_{2,v}, \varphi_{1,v}, \varphi_{2,v}) = \dfrac{L(2, 1_{F_v})}{L(1/2, \theta_v, \psi_v)} \cdot \displaystyle\int_{G_{0,v}} (g\phi_{1,v}, \phi_{2,v})_v \overline{(g\varphi_{1,v}, \varphi_{2,v})_v} dg$ *is the doubling zeta integral which valued in $\mathbb{C} \otimes_{\mathbb{Q}} M$.*

Theorem 4.13. *The first part of Conjecture 4.8 holds up to ± 1 and the second part of Conjecture 4.8 holds.*

In the same principle, the following arithmetic index theorem are equivalent to either side of Rallis inner product formulae.

Let π, Θ as before, $(V_1, V_2; W)$ be self-reflex lines (here similar in sign +1 case, but we equipped V_1, V_2 with M structure.) Let $x \in V^-$ such that $\mathrm{Hom}_{T_{x,v},M}(\pi_v, M \otimes_{\mathbb{Q}} \mathbb{C})$ is nonzero for all v.

Theorem 4.14.

$$\mathrm{Ind}(V_1, V_2; W) = \frac{L'(1/2, \pi)}{L(2, 1)} \prod_v \frac{L(2, 1_v)}{L(1/2, \pi_v)} \mathrm{Ind}_\delta(V_{1,v}, V_{2,v}; W_v)$$

Either arithmetic Rallis inner product formulae or Gross-Zagier formulae could implies the following spectral decomposition:

Corollary 4.15. *We have*

$$\mathrm{Ch}_s^1(X)_{\mathbb{Q}}^0 \simeq \bigoplus_{\substack{[A] \\ L'(A,1) \neq 0}} \mathrm{Ch}_s^1(X)_{\mathbb{Q}}^0[\pi_A],$$

with $\mathrm{Ch}_s^1(X)_{\mathbb{Q}}^0[\pi_A] \simeq \pi_A$.

By the BSD conjecture, one would like to replace the analytic rank by algebraic rank.

The case sign $+1$ Let $\mathcal{T}$ be an quadratic twist family of irreducible cuspidal automorprhic representations of $\mathrm{PGL}_2(\mathbb{A})$ over a number field F. Let $\Sigma_0 = \{v \mid v|2\infty \text{ or } \sigma_v \text{ is ramified for all } \sigma \in \mathcal{T}\}$. Let $\Sigma \supset \Sigma_0$ be any finite set of places of F. Let $\sigma_0 \in \mathcal{T}$ be any element. We call a fiber $\mathfrak{X}$ of the map

$$F^\times \to \prod_{v\in\Sigma} F_v^\times / F_v^{\times 2}.$$

a Σ-equivalent class. Identify $\delta \in \mathfrak{X}$ with quadratic twist $\sigma_0 \otimes \eta_\delta$ of σ_0. The sign of $\sigma_0 \otimes \eta_\delta$ only depends on $\mathfrak{X}$, denoted by $\epsilon(\sigma_0 \otimes \mathfrak{X})$.

Let $\mathfrak{X}_i$, $i = 1, 2$ be two Σ-equivalent classes. For each $v \in \Sigma$, the quadratic extension $\mathcal{K}_v = F_v(\sqrt{\delta_1\delta_2})$, with $\delta_i \in \mathfrak{X}_i$, of F_v only depends on $\mathfrak{X}_1$, $\mathfrak{X}_2$.

Assume that

$$\epsilon(\sigma_0 \otimes \mathfrak{X}_1) = +1, \epsilon(\sigma_0 \otimes \mathfrak{X}_2) = +1.$$

Let B be the quaternion algebra over F unramified outside Σ such that

$$\epsilon(B_v) = \epsilon(\sigma_{0,\mathcal{K}_v}), \quad \forall v \in \Sigma.$$

Let π_0 be the cuspidal automorphic irreducible representation over $H = \mathrm{PB}^\times$ that corresponds to σ_0 via Jacquet-Langlands.

Fix a non-trivial additive character ψ_0 of $F\backslash\mathbb{A}$. By property of Waldspurger packet [9], for $\psi = \psi_{0,\delta_1^{-1}}$, $\delta_1 \in \mathfrak{X}_1$ such that $L(1/2, \pi_0 \otimes \eta_{\delta_1}) \neq 0$, the representation

$$\theta := \theta_\psi(\pi_0 \otimes \eta_{\delta_1})$$

is nonzero and only depends on $\mathfrak{X}_1$.

We will give a uniform construction of $\varphi_{\delta_1} = \theta_{\phi_{\delta_1}}^{f_{\delta_1}} \in \theta_\psi(\pi)$ such that we have the uniform relation between

(I). $|W_{\psi_{\delta_1\delta_2}}(\varphi_{\delta_1})|^2$ and $L(1/2, \pi_0 \otimes \eta_{\delta_1})L(1/2, \pi_0 \otimes \eta_{\delta_2})$ as $\delta_i \in \mathfrak{X}_i$ varies;
(II). $|W_{\psi_{\delta_1\delta_2}}(\varphi_{\delta_1})|^2$ and $(\varphi_{\delta_1}, \varphi_{\delta_1})L(1/2, \pi_0 \otimes \eta_{\delta_2})$ as $\delta_i \in \mathfrak{X}_i$ varies;
(III). $(\varphi_{\delta_1}, \varphi_{\delta_1})$ and $L(1/2, \pi_0 \otimes \eta_{\delta_1})$ as $\delta_1 \in \mathfrak{X}_1$ varies.

The case sign -1 Assume F is totally real, and $\mathcal{T}$ be a quadratic twist family of irreducible cuspidal automorphic representations of $\mathrm{PGL}_2(\mathbb{A})$ over F and with infinite component given by discrete series of parallel weight 2. Define Σ, $\mathfrak{X}$, $\epsilon(\sigma_0 \otimes \mathfrak{X})$ in the same way as sign $+1$ case.

Assume that

$$\epsilon(\sigma_0 \otimes \mathfrak{X}_1) = -1, \epsilon(\sigma_0 \otimes \mathfrak{X}_2) = +1$$

and

$$\mathfrak{X}_1\mathfrak{X}_2$$

is totally negative. Let $\mathbb{B}$ be the incoherent totally definite quaternion algebra over $\mathbb{A}$ unramified outside Σ such that

$$\epsilon(\mathbb{B}_v) = \epsilon(\sigma_{0,\mathcal{K}_v}), \quad \forall v \in \Sigma.$$

Let X the Shimura curve associated to $\mathbb{H} = \mathbb{A}^\times\backslash\mathbb{B}^\times$ and let A/F be the simple abelian variety corresponds to π_0. Then the $\mathbb{H}$ representation $\pi_A = \varinjlim_U \mathrm{Hom}^0_{\xi_U}(X_U, A)$ corresponds to σ_0 via Jacquet-Langlands. Let $\pi_0 = \pi_{A,\mathbb{C}}$. In this arithmetic case, we will give a uniform construction of cycle valued automorphic forms $\varphi_{\delta_1} = \vartheta_{\phi_{\delta_1}}^{f_{\delta_1}} \in \vartheta_\psi(\pi)$, with $\psi = \psi_{0,\delta_1^{-1}}$, $\pi = \pi_0 \otimes \eta_{\delta_1}$, $\delta_1 \in \mathfrak{X}_1$, such that we have the uniform relation between

(I'). $\langle W_{\psi_{\delta_1\delta_2}}(\varphi_{\delta_1}), W_{\psi_{\delta_1\delta_2}}(\varphi_{\delta_1})\rangle_M$ and $L'(1/2, \pi_0 \otimes \eta_{\delta_1})L(1/2, \pi_0 \otimes \eta_{\delta_2})$ as $\delta_i \in \mathfrak{X}_i$ varies; Here $M = \mathrm{End}_F(A)_\mathbb{Q}$.
(II'). $\langle W_{\psi_{\delta_1\delta_2}}(\varphi_{\delta_1}), W_{\psi_{\delta_1\delta_2}}(\varphi_{\delta_1})\rangle_M$ and $(\varphi_{\delta_1}, \varphi_{\delta_1})L(1/2, \pi_0 \otimes \eta_{\delta_2})$ as $\delta_i \in \mathfrak{X}_i$ varies;
(III'). $(\varphi_{\delta_1}, \varphi_{\delta_1})$ and $\{L'(1/2, \pi_0 \otimes \eta_{\delta_1})\}$ as $\delta_1 \in \mathfrak{X}_1$ varies.

We will also consider the arithmetic application of these formulae.

5.1. **Test vector space.** Let's first focus on the choice of family of f_{δ_1}. For each $v \in \Sigma$ finite, let $\eta_{\mathfrak{X}_1,v} = \eta_{\delta_1,v}$, $\delta_i \in \mathfrak{X}_1$, $\chi_v = \eta_{\mathfrak{X}_1,v} \circ N_{\mathcal{K}_v/F_v}$, which only depends on $\mathfrak{X}_1$. Let

$$\mathcal{R} \subset \mathcal{B} := \begin{cases} B \\ \mathbb{B} \end{cases} \text{be an} \begin{cases} \mathcal{O}_F, & \text{sign } +1 \text{ case} \\ \widehat{\mathcal{O}}_F, & \text{sign } -1 \text{ case} \end{cases}$$

order of discriminant

$$\prod_{\substack{v \in \Sigma_{\text{fin}} \\ \mathcal{K}_v \text{ split or} \\ \text{ord}_v(\text{Cond}(\chi_v)) \geq \text{ord}_v(\text{Cond}(\sigma_{0,v}))}} \text{Cond}(\sigma_{0,v}) \prod_{\substack{v \in \Sigma_{\text{fin}} \\ \mathcal{K}_v \text{ nonsplit and} \\ \text{ord}_v(\text{Cond}(\chi_v)) < \text{ord}_v(\text{Cond}(\sigma_{0,v}))}} \text{Cond}(\sigma_{0,v} \otimes \eta_{\mathfrak{X}_1,v})$$

such that for $v \in \Sigma$, $\mathcal{R}_v \cap \mathcal{K}_v = \mathcal{O}_{\chi_v}$ if $\mathcal{K}_v$ split or $\text{ord}_v(\text{Cond}(\chi_v)) \geq \text{ord}_v(\text{Cond}(\sigma_{0,v}))$ and $\mathcal{R}_v \cap \mathcal{K}_v = \mathcal{O}_{\mathcal{K}_v}$ if $\mathcal{K}_v$ nonsplit and $\text{ord}_v(\text{Cond}(\chi_v)) < \text{ord}_v(\text{Cond}(\sigma_{0,v}))$. Here $\text{Cond}(\sigma_{0,v})$, $\text{Cond}(\chi_v)$ is the conductor of $\sigma_{0,v}$, χ_v respectively, $\mathcal{O}_{\chi_v} \subset \mathcal{O}_{\mathcal{K}_v}$ is the order with conductor equals to conductor of χ_v. Let $\mathcal{U}_\infty = \prod_{v|\infty} \mathcal{U}_v$ such that $\mathcal{U}_v$ is a compact subgroup of

$$\begin{cases} H_v, & \text{sign } +1 \text{ case} \\ \mathbb{H}_v, & \text{sign } -1 \text{ case} \end{cases}$$

such that $\mathcal{U}_v \cap \mathcal{K}_v^\times$ is the maximal compact subgroup of $\mathcal{K}_v^\times$.

Let

$$V(\pi_0, \mathfrak{X}_1, \mathfrak{X}_2)$$

be the tensor product of local test vector space defined in Theorem 2.15 relative to $\mathcal{R}_v$ if v is finite and $\mathcal{U}_v$ if $v|\infty$ (also $\mathcal{K}_v$ if $v \in \Sigma$), where the places outside Σ corresponds to the case (I). and the places inside Σ corresponds to the case (II). in Theorem 2.15. Note that in the sign -1 case, all the spaces equipped with an M structure.

Proposition 5.1. *$V(\pi_0, \mathfrak{X}_1, \mathfrak{X}_2)$ is rank one*

$$\begin{cases} \mathbb{C}, & \textit{sign } +1 \textit{ case} \\ \mathbb{C} \otimes_{\mathbb{Q}} M, & \textit{sign } -1 \textit{ case} \end{cases}$$

space.

We also identify element $\delta := \delta_1\delta_2$ with $(\delta_1, \delta_2) \in (\mathfrak{X}_1, \mathfrak{X}_2)$ as a quadratic algebra

$$\mathcal{K} = \begin{cases} F(\sqrt{\delta}), & \text{sign } +1 \text{ case} \\ \mathbb{A}(\sqrt{\delta}), & \text{sign } -1 \text{ case} \end{cases}$$

contained in $\mathcal{B}$. Let $T = F(\sqrt{\delta})^\times/F^\times$. For any $\mathcal{K} \in \mathfrak{X}_1\mathfrak{X}_2$, exists $h_{\delta_1,\mathcal{K}} \in \begin{cases} H(\mathbb{A}), & \text{sign } +1 \text{ case} \\ \mathbb{H}, & \text{sign } -1 \text{ case} \end{cases}$ such that $\mathcal{R}_{\mathcal{K}} = h_{\delta_1,\mathcal{K}}\mathcal{R}h_{\delta_1,\mathcal{K}}^{-1}$ and $\mathcal{K}$ satisfies good relative position in Theorem 2.15. For $\delta_i \in \mathfrak{X}_1$, let $\pi = \pi_0 \otimes \eta_{\delta_1}$.

Proposition 5.2. *Let $0 \neq f_0 \in V(\pi_0, \mathfrak{X}_1, \mathfrak{X}_2)$. For each $(\delta_1, \delta_2) \in (\mathfrak{X}_1, \mathfrak{X}_2)$, $f_{\delta_1,\mathcal{K}} = f_0^{h_{\delta_1,\mathcal{K}}} \otimes \eta_{\delta_1}$ is a test vector for $(\pi, \mathcal{K})$ in the sense that for each v, the local toric linear form defined in 2.2 is nonzero on local component of $f_{\delta_1,\mathcal{K}}$ at v.*

Recall Waldspurger/Gross-Zagier formula say that $P_T(f_{\delta_1,\mathcal{K}})$ is non-vanishing on $\pi_{\delta_1} := \pi_0 \otimes \eta_{\delta_1}$ if and only if the derivative base change central L-value/base change central L-value is non-vanishing. In the following, we consider uniform relation between toric period and L-values.

Similarly to $\mathcal{R}$, there exists a admissible order $\mathcal{R}_0$ (depends on δ_1) in the sense of [3] with discriminant equals to $\mathrm{N} = \text{Cond}(\sigma_0)$ such that for each embedding $\mathcal{K} \to \mathcal{B}$ and $h_{\mathcal{K}}$ above, $\mathcal{R}'_{\mathcal{K}} = h_{\mathcal{K}}\mathcal{R}_0 h_{\mathcal{K}}^{-1}$ is admissible order for (π_0, η_{δ_1}).

Denote $||\mathfrak{a}||$ the norm of an ideal $\mathfrak{a}$ of F. In the sign -1 case, simply denoted by $|Q|_M^2 = \langle Q, Q \rangle_M$ for $Q \in A(\overline{F})_{\mathbb{C}}$.

It follows from Explicit Waldspurger formula of Cai-Shu-Tian for $(\pi_0, \eta_{\delta_1} \circ N_{F(\sqrt{\delta_1\delta_2})/F})$ and its variation (see also Remark 2.16), we have: For $\delta_i \in \mathfrak{X}_1$, let $\pi = \pi_0 \otimes \eta_{\delta_1}$.

Theorem 5.3. *For each* $(\delta_1, \delta_2) \in (\mathfrak{X}_1, \mathfrak{X}_2)$,

$$L^{(c_{1,2})}(1, \eta_{\mathcal{K}})^2 ||c_{1,2}|| \cdot \frac{|D_{\mathcal{K}}|}{|D_F|} \cdot \begin{cases} |P_T(f_{\delta_1,\mathcal{K}})|^2 \\ |P_T(f_{\delta_1,\mathcal{K}})|_M^2 \end{cases} = C_{\mathfrak{X}_1,\mathfrak{X}_2} \cdot \frac{(f_0, f_0)}{(\phi^0, \phi^0)} \cdot \sqrt{|D_{\mathcal{K}/F}|} \cdot \begin{cases} L^{(\Sigma')}(1/2, \pi_{\mathcal{K}}), & \textit{sign } +1 \textit{ case} \\ L'^{(\Sigma')}(1/2, \pi_{\mathcal{K}}), & \textit{sign } -1 \textit{ case} \end{cases}$$

where

$$C_{\mathfrak{X}_1,\mathfrak{X}_2}$$

only depends on $\sigma_0, \mathfrak{X}_1, \mathfrak{X}_2$ *given by local factors mainly contributed from places in* Σ *and bad places of* σ_0*:*

$$C_{\mathfrak{X}_1,\mathfrak{X}_2} = |D_F| C_\infty{}^{-1} 2^{\#\left\{v \;\middle|\; \substack{\textit{Case (II).(b).(i),} \\ \mathcal{K}_v \textit{ is ramified}}\right\}} \prod_{\substack{v \notin \Sigma \\ \pi_{0,v} \textit{ is ramified}}} \frac{L(2, 1_{F_v})}{L(1, F_v) L(1, \pi_{0,v}, ad)} \prod_{v | N \cap \Sigma} \frac{\mathrm{vol}(U_0(N))_v}{\mathrm{vol}(\mathcal{R}_{0,v})} \prod_{Case(II).(b).(ii)} \frac{1}{(1 - q_v{}^{-e_v}) L(1, \pi_{0,v}, ad)},$$

with q_v *is the cardinality of residue field of* F_v*,* e_v *is the ramification index of* $\mathcal{K}_v/F_v$*,* C_∞ *is defined in* [3]*,* ϕ^0 *the normalized Hilbert new vector;*

$$c_{1,2} = \prod_{v \notin \Sigma} \sqrt{\frac{D_{F_v(\delta_{1,v}\delta_{0,v})/F_v} D_{F_v(\delta_{2,v}\delta_{0,v})/F_v}}{D_{F_v(\delta_{1,v}\delta_{2,v})/F_v}}} \prod_{v \in \Sigma, Case(II).(a).} \mathfrak{p}_v^{c_v},$$

which depends on δ_1, δ_2 *with* $\delta_{0,v} = 1$ *or a uniformizer of* F_v *such that* $\sigma_{0,v} \otimes \eta_{\delta_{0,v}}$ *is unramified for* $v \notin \Sigma$*,* c_v *the conductor of* $\chi_{\delta_1,v} := \eta_{\delta_{0,v}\delta_{1,v}} \circ N_{\mathcal{K}_v/F_v}$ *for* $v \in \Sigma$ *and* $\mathfrak{p}_v$ *the prime of* F_v*.*

$$\Sigma' = \left\{ v \in \Sigma \;\middle|\; \substack{v | (n_v, c_v D_{\mathcal{K}_v/F_v}) \textit{ if } v \textit{ is finite; if } v \parallel N, \textit{ then } \mathrm{ord}_v(c_v) \geq 1; \\ v \textit{ is not Case (II).(b).(ii); if } v | \infty, \textit{ then } \mathcal{K}_v \simeq \mathbb{C}} \right\},$$

where n_v *is the conductor of* $\pi_{0,v}$*.*

Now consider choice of family of ϕ_{δ_1}. We need to choose representative of $\delta_i \in \mathfrak{X}_i$ such that the Whittaker-Fourier coefficients and L-values have uniform relation.

Let $\mathfrak{a}$ be a fractional ideal of F prime to Σ. Let $\mathfrak{X}_{\mathfrak{a}} \subset \mathfrak{X}_1 \times \mathfrak{X}_2$ be subset consists of (δ_1, δ_2) such that $(\delta_1 \delta_2) = \mathfrak{a}^2 \prod_{v \text{ is finite}} D_{F_v(\sqrt{\delta_1 \delta_{0,v}})/F_v} D_{F_v(\sqrt{\delta_2 \delta_{0,v}})/F_v}$. Let Pic_F^+ be the narrow ideal class group of F. Note that we have surjection:

$$\bigsqcup_{[\mathfrak{a}] \in \mathrm{Pic}_F^+} \mathfrak{X}_{\mathfrak{a}} \to (\mathfrak{X}_1 \times \mathfrak{X}_2)/F^{\times 2}.$$

We normalize the Schwartz function by the following so that the theta lifting has good properties $\phi_{\delta_1} = \frac{\prod_{v|\infty} L(1, \eta_{\mathcal{K}_v})}{\mathrm{vol}(\widehat{\mathcal{R}}^\times, H(\mathbb{A}_{\mathrm{fin}})) / \prod_{v \text{ is finite}} |c_{\delta_1,v}|_v^{1/2}} \otimes_v \phi_{\delta_1,v}$, where $\widehat{\mathcal{R}}^\times = \begin{cases} \widehat{\mathcal{R}}^\times, & \text{sign } +1 \text{ case} \\ \mathcal{R}^\times, & \text{sign } -1 \text{ case} \end{cases}$, $\phi_{\delta_1,v}$ is defined in Section 2.2 relative to $\pi_v = \pi_{0,v} \otimes \eta_{\delta_1,v}$, $\mathcal{R}_v$ for all v and $\mathcal{K}_v$ for $v \in \Sigma$.

For each $\mathfrak{a}$, fix a generator $a \in \mathbb{A}_{\mathrm{fin}}$ of $\widehat{\mathfrak{a}}$. Let $\phi_{\delta_1,\mathfrak{a}}$ be the translation of ϕ by a in the sense that $\phi_{\delta_1,\mathfrak{a}}(ax) = \phi_{\delta_1}(x)$ for any $x \in \begin{cases} V(\mathbb{A}), & \text{sign } +1 \text{ case} \\ \mathbb{V}, & \text{sign } -1 \text{ case} \end{cases}$. For a different choice of a, $\phi_{\mathfrak{a},0}$ will differed by ± 1, we may fix one choice once for all.

For each $x \in V_{\delta_1\delta_2}$ with $(\delta_1, \delta_2) \in \mathfrak{X}_{\mathfrak{a}}$, we have choose $h_x := h_{\delta_1,\mathcal{K}}$ as before with $\mathcal{K} = \begin{cases} F(x), & \text{sign } +1 \text{ case} \\ \mathbb{A}(x), & \text{sign } -1 \text{ case} \end{cases}$ and we may further choose $h_x^{-1} \circ x$ such that for each $v \in \Sigma$ and $h_x^{-1} \circ x \in \mathcal{K}_v^o$. Here in $\mathcal{K}_v^o \subset \mathcal{K}_v^{\mathrm{tr}=0}$ is the oriented subset in 2.2 and in the case sign -1, $V \subset \mathbb{V}_{\mathrm{fin}}$ is in Section 4.1.

Proposition 5.4. *Let* $(f_{\delta_1,x}, \phi_{\delta_1,\mathfrak{a},x}) := (h_x(f_0 \otimes \eta_{\delta_1}), h_x \phi_{\delta_1,\mathfrak{a}})$*, then it follows that*

$$\varphi_{\delta_1,\mathfrak{a}} = \begin{cases} \theta_{\phi_{\delta_1,\mathfrak{a},x}}^{f_{\delta_1,x}}, & \textit{sign } +1 \textit{ case} \\ \vartheta_{\phi_{\delta_1,\mathfrak{a},x}}^{f_{\delta_1,x}}, & \textit{sign } -1 \textit{ case} \end{cases}$$

only depends on $\mathfrak{a}, \delta_1$*, does not depend on* δ_2 *here the theta lifting is with respect to* $\psi = \psi_{0,\delta_1^{-1}}$ *and* ψ_0 *is a fixed non-trivial additive character once for all.*

We have $\varphi_{\delta_1,\mathfrak{a}}$ is a test vector for $\psi_{\delta_1\delta_2}$ Whittaker functional for all $(\delta_1, \delta_2) \in \mathfrak{X}_{\mathfrak{a}}$. In the following, we will consider the uniform relation between Whittaker-Fourier coefficients and L-values.

5.2. **Explicit formulae for (arithmetic) Whittaker-Fourier periods and Rallis inner product.**

Type (I) and (I')

By the decomposition of (arithmetic) Whittaker-Fourier coefficients in Section 3 4.2 and formula for local Whittaker function 2.2, we have the following modularity of toric periods:

Define the normalized Whittaker-Fourier coefficient by $W^0_{\psi_\delta}(\varphi_{\delta_1,\mathfrak{a}}) = \frac{W_{\psi_\delta}(\varphi_{\mathfrak{a}})}{\prod_{v|\infty} C_{q(x),v}}$ whenever $\prod_{v|\infty} C_{q(x),v}$ is nonzero, where $C_{q(x),v}$ is defined behind Theorem 2.24. For each $(\delta_1,\delta_2)\in\mathfrak{X}_{\mathfrak{a}}$ and $x\in V_\delta$ with $\delta=\delta_1\delta_2$,

$$W^0_{\psi_\delta}(\varphi_{\delta_1,\mathfrak{a}}) = 2^{\epsilon}\cdot\prod_{v|\infty}|\delta|_v^{w_v/[F_v:\mathbb{R}]-3/4}\cdot\sqrt{\frac{|D_{\mathcal{K}_x}|}{|D_F|}}L^{(c_{1,2})}(1,\eta_{K_x})||c_{1,2}||P_{T_x}(f_x),$$

where $\epsilon=\sum\epsilon_v$ with $\epsilon_v\in\{1,2\}$ in Theorem 2.18, T_x be the stablizer of $x\in V$.

By Theorem 5.3, we have the following explicit and uniform relation:

Theorem 5.5. *For each $(\delta_1,\delta_2)\in\mathfrak{X}_{\mathfrak{a}}$,*

$$\begin{cases}|W^0_{\psi_\delta}(\varphi_{\delta_1,\mathfrak{a}})|^2\\ |W^0_{\psi_\delta}(\varphi_{\delta_1,\mathfrak{a}})|^2_M\end{cases} = C'_{\mathfrak{X}_1,\mathfrak{X}_2}\cdot\frac{(f_0,f_0)}{(\phi^0,\phi^0)}\cdot\prod_{v|\infty}|\delta_1\delta_2|_v^{2w_v/[F_v:\mathbb{R}]-1}\cdot\begin{cases}L^{(\Sigma')}(1/2,\pi_{\delta_1,\mathcal{K}}), & \textit{sign } +1 \textit{ case}\\ L'^{(\Sigma')}(1/2,\pi_{\delta_1,\mathcal{K}}), & \textit{sign } -1 \textit{ case}\end{cases},$$

where

$$C'_{\mathfrak{X}_1,\mathfrak{X}_2} = C_1\cdot C_{\mathfrak{X}_1,\mathfrak{X}_2}$$

with C_1 only depends on $\sigma_0,\mathfrak{X}_1,\mathfrak{X}_2$ given by local factors contributed from places in Σ, $\mathfrak{a}$ and bad places of σ_0: $C_1 = 4^{\epsilon}||c'||^{1/2}||\mathfrak{a}||^{-1}$, $c' = \prod_{v|\Sigma_{fin}}\frac{D_{\mathcal{K}_v/F_v}}{D_{\mathcal{K}_{1,v}/F_v}D_{\mathcal{K}_{2,v}/F_v}}\prod_{v\in\Sigma,(II).(a).}\mathfrak{p}_v^{2c_v}$ with $\mathcal{K}_{i,v}=F_v(\sqrt{\delta_i})$ for $v\in\Sigma$ only depends on $\mathfrak{X}_i$, c_v the conductor of $\eta_{\delta_1,v}\circ N_{\mathcal{K}_v/F_v}$, $\epsilon=\sum\epsilon_v$ with $\epsilon_v\in\{1,2\}$ in Theorem 2.18.

Type (II) and (II')

Let notations be as before,

Theorem 5.6. *For each $(\delta_1,\delta_2)\in\mathfrak{X}_{\mathfrak{a}}$,*

$$\begin{cases}|W^0_{\psi_\delta}(\varphi_{\delta_1,\mathfrak{a}})|^2\\ |W^0_{\psi_\delta}(\varphi_{\delta_1,\mathfrak{a}})|^2_M\end{cases} = C''_{\mathfrak{X}_1,\mathfrak{X}_2}\cdot\frac{(\varphi_{\delta_1,\mathfrak{a}},\varphi_{\delta_1,\mathfrak{a}})}{(\phi^0,\phi^0)}\frac{\prod_{v\notin\Sigma}||D_{F_v(\delta_1\delta_0)/F_v}||_v}{||(\delta_1)^{(\Sigma)}||}\prod_{v|\infty}|\delta_2|_v^{2w_v/[F_v:\mathbb{R}]-1}L^{(\Sigma)}(1/2,\pi_0\otimes\eta_{\delta_2})$$

where $C''_{\mathfrak{X}_1,\mathfrak{X}_2}$ only depends on $\sigma_0,\mathfrak{X}_1,\mathfrak{X}_2$ given by local factors mainly contributed from places in Σ, $\mathfrak{a}$ and bad places of σ_0:

$$C''_{\mathfrak{X}_1,\mathfrak{X}_2} = ||\mathfrak{a}||^2\prod_{v|\infty}|c_0|_v^{2w_v/[F_v:\mathbb{R}]}|D_F|2^{n+\#\Sigma_\infty}\pi^b||(\delta_2)_\Sigma||$$
$$\prod_{\substack{v\notin\Sigma,\\ \pi_{0,v}\text{ is ramified}}}\frac{L(2,1_{F_v})}{L(1,\pi_{0,v},ad)L(1,1_{F_v})}\prod_{\substack{v\in\Sigma_{fin}\\ \pi_{0,v}\text{ is unramified}}}\frac{L(1,\pi_{0,v},ad)L(1,1_F)}{L(2,1_{F_v})}\prod_{v|\infty}\frac{(W_{0,v},W_{0,v})}{c_{\mathfrak{X}_1\cdot\mathfrak{X}_2,v}(W_v^o,W_v^o)_2}$$

$n=\sum_{v\in\Sigma}n_v$ with n_v defined in Theorem 2.31, c_0 is such that $\psi_{0,c_0^{-1}}$ is the standard additive character of $F\backslash\mathbb{A}$, $b=\{v|\infty \mid F_v\simeq\mathbb{C}\}$.

Type (III) and (III')

Let notations be as before,

Theorem 5.7. *For each $(\delta_1,\delta_2)\in\mathfrak{X}_{\mathfrak{a}}$,*

$$\frac{(\varphi_{\delta_1,\mathfrak{a}},\varphi_{\delta_1,\mathfrak{a}})}{(f_0,f_0)} = C'''_{\mathfrak{X}_1,\mathfrak{X}_2}\cdot\frac{||(\delta_1)^{(\Sigma)}||}{\prod_{v\notin\Sigma}||D_{F_v(\delta_1\delta_0)/F_v}||_v}\prod_{v|\infty}|\delta_1|_v^{2w_v/[F_v:\mathbb{R}]-1}\cdot\begin{cases}L^{(\Sigma')}(1/2,\pi_0\otimes\eta_{\delta_1}), & \textit{sign } +1 \textit{ case}\\ L'^{(\Sigma')}(1/2,\pi_0\otimes\eta_{\delta_1}), & \textit{sign } -1 \textit{ case}\end{cases}$$

where $C'''_{\mathfrak{X}_1,\mathfrak{X}_2}$ only depends on $\mathfrak{X}_1$, $\mathfrak{X}_2$ given by local factors contributed from places in Σ, $\mathfrak{a}$ and bad places of σ_0: $C'''_{\mathfrak{X}_1,\mathfrak{X}_2} = C'_{\mathfrak{X}_1,\mathfrak{X}_2}/C''_{\mathfrak{X}_1,\mathfrak{X}_2}L_{\Sigma\backslash\Sigma'}(1/2,\pi_0\otimes\eta_{\delta_2})$

6. Arithmetic application: Tunnell-Gross Type formula

Let notations be as in sign $+1$ case of Theorem 5.5 in Section 5. We consider the case $\pi_{0,\infty}$ is trivial. Equivalently, F is totally real, $\sigma_{0,\infty}$ is parallel of weight 2 and $\mathfrak{X}_1\mathfrak{X}_2$ is totally negative.

We have known the explicit and uniform relation between Fourier coefficients of theta series and L-values, in the following we consider relation between Fourier coefficients and arithmetic of ternary quadratic lattices.

Recall for $v \in \Sigma$, we say $(\pi_v, \mathcal{K}_v)$ has no local obstruction if $(*)$ in Proposition 2.17 holds. It is equivalent to that the involution define in the proposition acts on local test vector in π_v by $\epsilon(B_v)\epsilon(\pi_v) = +1$. Let

$$L_v^\circ = \begin{cases} L_v, & \pi_v \text{ is unramified } (\mathrm{ord}_v(\delta_{0,v}\delta_{1,v}) \text{ is even) and } v \notin \Sigma \\ \mathcal{R}_v^\times \circ \begin{pmatrix} & \mathfrak{p}_v \\ 1 & \end{pmatrix}, & \pi_v \text{ is ramified } (\mathrm{ord}_v(\delta_{0,v}\delta_{1,v}) \text{ is odd) and } v \notin \Sigma \\ L_v, & v \in \Sigma \text{ finite and without local obstruction} \\ \mathcal{R}_v^\times \circ \mathcal{K}_v^\circ, & v \in \Sigma \text{ finite, other case.} \end{cases},$$

where $\mathfrak{p}_v$ is the prime ideal of F associated to v and in the last case, L_v° is the oriented subset defined in 2.2.

And let $L_{\mathfrak{a}}^\circ = (\prod_{v<\infty} a_v L_v^\circ) \cap V(F)$, here recall $a \in \mathbb{A}_{\mathrm{fin}}$ is a generator of $\mathfrak{a}$. For each $\delta = \delta_1\delta_2$ with $(\delta_1, \delta_2) \in \mathfrak{X}_{\mathfrak{a}}$, let

$$w_{\delta_1} = \prod_{\substack{v\notin\Sigma \\ \pi_v \text{ is ramified}}} w_{\delta_1,v} \prod_{v\in\Sigma, Case(II),(a).}$$

be the weight function on $L_{\mathfrak{a}}$ defined by

$$w_{\delta_1,v}\left(r \circ \left(a_v \begin{pmatrix} & b \\ 1 & \end{pmatrix}\right)\right) = \eta_{\delta_{1,v}\delta_{0,v}} \circ \det(r), \quad r \in \mathcal{R}_v^\times, \quad b \in \mathfrak{p}_v$$

in the case $v \notin \Sigma$ and π_v is ramified and

$$w_{\delta_1,v}(r \circ k) = \eta_{\delta_{1,v}} \circ \det(r), \quad r \in \mathcal{R}_v^\times, \quad k \in \mathcal{K}_v^\circ.$$

in the Case (II).(a). for $v \in \Sigma$.

Denoted $(L_{\mathfrak{a},h}^\circ, w_{\delta_1,h})$ by the conjugation of $(L_{\mathfrak{a}}^\circ, w_{\delta_1})$ by h, more precisely,

$$L_{\mathfrak{a},h}^\circ = (\prod_{v<\infty} h_v \circ a_v L_v^\circ) \cap V(F)$$

$$w_{\delta_1,h,v}(\cdot) = w_{\delta_{1,v}}(h_v^{-1} \circ \cdot).$$

The following lemma follows directly from our choice of vector and explicit description of Weil representation.

Lemma 6.1. *For each $(\delta_1, \delta_2) \in \mathfrak{X}_{\mathfrak{a}}$,*

$$W^0_{\psi_\delta}(\theta_{\mathfrak{a}}) = 2^{[F:\mathbb{Q}]} \sum_{[h]\in X_{\mathcal{R}^\times}} \frac{f_0(h)\eta_{\delta_1} \circ \det(h)}{w_h} \sum_{x \in L_{\mathfrak{a},h}^\circ \cap V_\delta(F)} w_{\delta_1}(x)$$

where $\delta = \delta_1\delta_2$, $w_h = \#(h\mathcal{R}^\times h^{-1} \cap H(F))$, $X_{\mathcal{R}^\times} = H(F)\backslash H(\mathbb{A}_{fin})/\mathcal{R}^\times$.

Remark 6.2. The local weight function has the following concrete description in the case $v \notin \Sigma$ and π_v is ramified: Fix $x_0 \in \begin{pmatrix} & \mathfrak{p}_v \\ 1 & \end{pmatrix}$. Then it is the function whose support is contained in $a_v^{-1}M_2(\mathcal{O}_{F,v})$ given by

$$w_{\delta_1,v}(a_v^{-1}x) = \begin{cases} 0, & \mathrm{ord}_v(q(x)) = 0 \\ \eta_{\delta_{1,v}\delta_{0,v}}(-\langle x, x_0\rangle), & \mathrm{ord}_v(\langle x, x_0\rangle) = 0 \\ \eta_{\delta_{1,v}\delta_{0,v}}(u), & \mathrm{ord}_v(\langle x, x_0\rangle) > 0 \end{cases},$$

here $\langle x, y\rangle = q(x+y) - q(x) - q(y)$, $u \in \mathcal{O}_{F,v}^\times$ is such that $\mathrm{ord}_v(x - ux_0) > 0$.

Together with Theorem 5.5, we have the following generalization of Tunnell type theorem:

Theorem 6.3. *For each* $(\delta_1,\delta_2)\in\mathfrak{X}_\mathfrak{a}$,

$$\frac{|\delta_1\delta_2|^{1/2}\cdot L^{(\Sigma')}(1/2,\sigma_0\otimes\eta_{\delta_1})L^{(\Sigma')}(1/2,\sigma_0\otimes\eta_{\delta_2})}{\pi^3(\phi^0,\phi^0)}\cdot C^{\circ}_{\mathfrak{X}_1,\mathfrak{X}_2}=\frac{1}{(f_0,f_0)}\Big|\sum_{[h]\in X_{\mathcal{R}^\times}}\frac{f_0(h)\eta_{\delta_1}\circ\det(h)}{w_h}\sum_{x\in L^{\circ}_{\mathfrak{a},h}\cap V_\delta(F)}w_{\delta_1}(x)\Big|^2\in\overline{\mathbb{Q}}$$

where

$$C^{\circ}_{\mathfrak{X}_1,\mathfrak{X}_2}=\pi^3 4^{-[F:\mathbb{Q}]}\cdot C'_{\mathfrak{X}_1,\mathfrak{X}_2}\in\overline{\mathbb{Q}^\times}$$

with $C'_{\mathfrak{X}_1,\mathfrak{X}_2}$ *be the same as in Theorem 5.5 only depends on* σ_0, $\mathfrak{a}_0$ *and* $\mathfrak{X}_i$ *given by local factors contributed from places in* Σ, $\mathfrak{a}$ *and bad places of* σ_0.

If π is unramified outside Σ and $\delta_1=1$, the formulae in $|\cdot|$ is simply

$$\sum_{[h]\in X_{\mathcal{R}^\times}}\frac{f_0(h)}{w_h}\#(L^{\circ}_{\mathfrak{a},h}\cap V_\delta(F)).$$

Similar for general F and B, the weight function will involve archimedean places and one should count the lattice points by modulo suitable automorphism so that the sum appeared in the formulae is still finite.

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