# WELL-TOTAL DOMINATION OF $K_3$-FREE, INDUCED-$P_5$-FREE GRAPHS

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ABSTRACT. We show that finite, simple, $K_3$-free, induced-$P_5$-free graphs are well-totally-dominated. This answers Conjecture 314 from Written on the Wall II.

Let $G$ be a finite, simple, connected, $K_3$-free, induced-$P_5$-free graph.

**Claim 1.** Each minimal total dominating set of $G$ has size at least two.

*Proof.* Let $M$ be a minimal total dominating set of $G$, and $v$ be a vertex in $M$. Then, there exists a distinct vertex $v'$ in $M$. $\square$

**Lemma.** Let $M$ be a minimal total dominating set of $G$, and $v$ be a vertex in $M$. Then, $v$ has a *private neighbour* adjacent to no vertex in $M$ but $v$.

*Proof.* Suppose not. Then, vertex set $M \setminus \{v\}$ totally dominates $G$. This contradicts the minimality of $M$. $\square$

**Lemma.** Let $M$ be a minimal total dominating set of $G$ of size at least four, and subgraph $G[M]$ be connected. Then, exactly four vertices in $M$ induce a $C_4$, $P_4$, or $K_{1,3}$.

*Proof.* If some vertex in $M$ has at least three neighbours, since any three of them are pairwise nonadjacent, exactly four vertices induce a $K_{1,3}$. Otherwise, since $G[M]$ is connected, exactly four vertices induce a $C_4$ or $P_4$. $\square$

**Claim 2.** Each minimal total dominating set has size at most three.

*Proof.* Let $M$ be a minimal total dominating set of size at least four. Then, $G[M]$ is either connected or disconnected. Suppose $G[M]$ is connected. Let $S = \{A, B, C, D\}$ be a subset of $M$. Suppose $S$ induces a $C_4$. Let $w$ be the private neighbour of $A$, $x$ be the private neighbour of $B$, and $z$ be the private neighbour of $D$. Then, we have:

$$\begin{tikzcd}
x \arrow[d, dash] & & \\
B \arrow[r, dash] \arrow[d, dash] & A \arrow[r, dash] \arrow[d, dash] & w \\
D \arrow[r, dash] \arrow[d, dash] & C & \\
z & &
\end{tikzcd}$$

Clearly, $w$ is either adjacent or not adjacent to $z$. If $w$ is not adjacent to $z$, then $w - A - B - D - z$ is an induced $P_5$. Otherwise, we have:

$$\begin{tikzcd}
x \arrow[d, dash] & & \\
B \arrow[r, dash] \arrow[d, dash] & A \arrow[r, dash] \arrow[d, dash] & w \arrow[ddll, dash, bend left] \\
D \arrow[r, dash] \arrow[d, dash] & C & \\
z & &
\end{tikzcd}$$

Since $G$ is $K_3$-free, $x$ is not adjacent to $w$ and $z$. Thus, $x$ is either adjacent to $w$, $z$, or neither. If $x$ is adjacent to neither $w$ nor $z$, then $x - B - A - w - z$ is an induced $P_5$. Otherwise, if $x$ is adjacent to $w$, we have

$$\begin{tikzcd}
x \arrow[d, dash] \arrow[drr, dash] & & \\
B \arrow[r, dash] \arrow[d, dash] & A \arrow[r, dash] \arrow[d, dash] & w \arrow[ddll, dash, bend left] \\
D \arrow[r, dash] \arrow[d, dash] & C & \\
z & &
\end{tikzcd}$$

in which $x - w - z - D - C$ is an induced $P_5$. Otherwise, we have

$$\begin{tikzcd}
x \arrow[d, dash] \arrow[ddd, dash, bend right=60] & & \\
B \arrow[r, dash] \arrow[d, dash] & A \arrow[r, dash] \arrow[d, dash] & w \arrow[ddll, dash, bend left] \\
D \arrow[r, dash] \arrow[d, dash] & C & \\
z & &
\end{tikzcd}$$

in which $x - z - w - A - C$ is an induced $P_5$. Suppose $S$ induces a $P_4$. Without loss of generality, let $z$ be the private neighbour of $D$. Then, we have

$$A - B - C - D - z$$

which is an induced $P_5$. Suppose $S$ induces a $K_{1,3}$. Let $x$ be the private neighbour of $B$ and $z$ be the private neighbour of $D$. Then, we have:

$$\begin{tikzcd}
x \arrow[d, dash] & \\
B \arrow[d, dash] & \\
A \arrow[r, dash] \arrow[d, dash] & C \\
D \arrow[d, dash] & \\
z &
\end{tikzcd}$$

Clearly, $x$ is either adjacent or not adjacent to $z$. If $x$ is not adjacent to $z$, then $x - B - A - D - z$ is an induced $P_5$. Otherwise, we have

$$\begin{tikzcd}
x \arrow[d, dash] \arrow[dddd, dash, bend right=60] & \\
B \arrow[d, dash] & \\
A \arrow[r, dash] \arrow[d, dash] & C \\
D \arrow[d, dash] & \\
z &
\end{tikzcd}$$

in which $x - z - D - A - C$ is an induced $P_5$.

Suppose $G[M]$ is disconnected. Then, $G[M]$ has two or more connected components and each component has at least one edge. Let $A - B, C - D$ be edges of distinct components of $G[M]$. Then, we have:

$$B - A \qquad C - D$$

Without loss of generality, consider the distance between $A$ and $C$. Since $G$ is connected and induced-$P_5$-free, the distance is at most three. Since $A \neq C$, the distance is not zero. Since $A$ is not adjacent to $C$, the distance is not one. Thus, the distance is either two or three. Suppose the distance is two. Then, there exists a vertex adjacent to $A$ and $C$. Let $\alpha$ be such a vertex. Then, we have

$$B - A - \alpha - C - D$$

which is an induced $P_5$. Otherwise, there exists a path of length three between $A$ and $C$. Let $A - \alpha - \beta - C$ be such a path. Then, we have:

$$B - A - \alpha - \beta - C - D$$

Clearly, $B$ is either adjacent or not adjacent to $\beta$. If $B$ is not adjacent to $\beta$, then $B - A - \alpha - \beta - C$ is an induced $P_5$. Otherwise, we have

$$B - A - \alpha - \beta - C - D$$

in which $A - B - \beta - C - D$ is an induced $P_5$. $\square$

**Claim 3.** Minimal total dominating sets of sizes two and three cannot coexist.

*Proof.* Let $U - V$ be a minimal total dominating set of $G$ of size two, $A - B - C$ be a minimal total dominating set of $G$ of size three, $w$ be the private neighbour of $A$, and $y$ be the private neighbour of $C$. Then, we have:

$$w - A - B - C - y$$

Without loss of generality, suppose $A$ is adjacent to $U$. Then, we have:

$$\begin{array}{ccccccccc} w & - & A & - & B & - & C & - & y \\ & & | & & & & & & \\ & & U & & & & & & \end{array}$$

Since $U - V$ totally dominates $w$ and $w$ must be adjacent to $V$, we have:

$$\begin{array}{ccccccccc} w & - & A & - & B & - & C & - & y \\ | & & | & & & & & & \\ V & - & U & & & & & & \end{array}$$

Since $U - V$ totally dominates $B$ and $B$ must be adjacent to $V$, we have:

$$\begin{array}{ccccccccc} w & - & A & - & B & - & C & - & y \\ | & & | & & & & & & \\ V & - & U & & & & & & \end{array}$$

Since $U - V$ totally dominates $C$ and $C$ must be adjacent to $U$, we have:

$$\begin{array}{ccccccccc} w & - & A & - & B & - & C & - & y \\ | & & | & & & & & & \\ V & - & U & & & & & & \end{array}$$

Since $U - V$ totally dominates $y$ and $y$ must be adjacent to $V$, we have

$$\begin{array}{ccccccccc} w & - & A & - & B & - & C & - & y \\ | & & | & & & & & & \\ V & - & U & & & & & & \end{array}$$

in which $w - A - B - C - y$ is an induced $P_5$. $\square$