

WELL-TOTAL DOMINATION OF K_3 -FREE, INDUCED- P_5 -FREE GRAPHS

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ABSTRACT. We show that finite, simple, K_3 -free, induced- P_5 -free graphs are well-totally-dominated. This answers Conjecture 314 from Written on the Wall II.

Let G be a finite, simple, connected, K_3 -free, induced- P_5 -free graph.

Claim 1. Each minimal total dominating set of G has size at least two.

Proof. Let M be a minimal total dominating set of G , and v be a vertex in M . Then, there exists a distinct vertex v' in M . $\square$

Lemma. Let M be a minimal total dominating set of G , and v be a vertex in M . Then, v has a *private neighbour* adjacent to no vertex in M but v .

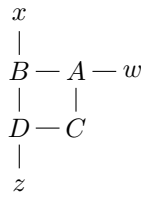
Proof. Suppose not. Then, vertex set $M \setminus \{v\}$ totally dominates G . This contradicts the minimality of M . $\square$

Lemma. Let M be a minimal total dominating set of G of size at least four, and subgraph $G[M]$ be connected. Then, exactly four vertices in M induce a C_4 , P_4 , or $K_{1,3}$.

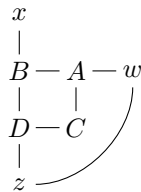
Proof. If some vertex in M has at least three neighbours, since any three of them are pairwise nonadjacent, exactly four vertices induce a $K_{1,3}$. Otherwise, since $G[M]$ is connected, exactly four vertices induce a C_4 or P_4 . $\square$

Claim 2. Each minimal total dominating set has size at most three.

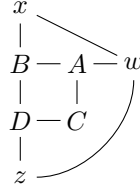
Proof. Let M be a minimal total dominating set of size at least four. Then, $G[M]$ is either connected or disconnected. Suppose $G[M]$ is connected. Let $S = \{A, B, C, D\}$ be a subset of M . Suppose S induces a C_4 . Let w be the private neighbour of A , x be the private neighbour of B , and z be the private neighbour of D . Then, we have:



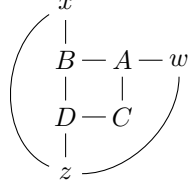
Clearly, w is either adjacent or not adjacent to z . If w is not adjacent to z , then $w - A - B - D - z$ is an induced P_5 . Otherwise, we have:



Since G is K_3 -free, x is not adjacent to w and z . Thus, x is either adjacent to w , z , or neither. If x is adjacent to neither w nor z , then $x - B - A - w - z$ is an induced P_5 . Otherwise, if x is adjacent to w , we have



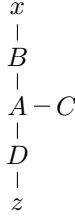
in which $x - w - z - D - C$ is an induced P_5 . Otherwise, we have



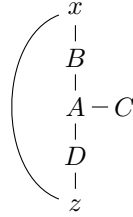
in which $x - z - w - A - C$ is an induced P_5 . Suppose S induces a P_4 . Without loss of generality, let z be the private neighbour of D . Then, we have

$$A - B - C - D - z$$

which is an induced P_5 . Suppose S induces a $K_{1,3}$. Let x be the private neighbour of B and z be the private neighbour of D . Then, we have:



Clearly, x is either adjacent or not adjacent to z . If x is not adjacent to z , then $x - B - A - D - z$ is an induced P_5 . Otherwise, we have



in which $x - z - D - A - C$ is an induced P_5 .

Suppose $G[M]$ is disconnected. Then, $G[M]$ has two or more connected components and each component has at least one edge. Let $A - B, C - D$ be edges of distinct components of $G[M]$. Then, we have:

$$B - A \quad C - D$$

Without loss of generality, consider the distance between A and C . Since G is connected and induced- P_5 -free, the distance is at most three. Since $A \neq C$, the distance is not zero. Since A is not adjacent to C , the distance is not one. Thus, the distance is either two or three. Suppose the distance is two. Then, there exists a vertex adjacent to A and C . Let α be such a vertex. Then, we have

$$B - A - \alpha - C - D$$

which is an induced P_5 . Otherwise, there exists a path of length three between A and C . Let $A - \alpha - \beta - C$ be such a path. Then, we have:

$$B - A - \alpha - \beta - C - D$$

Clearly, B is either adjacent or not adjacent to β . If B is not adjacent to β , then $B - A - \alpha - \beta - C$ is an induced P_5 . Otherwise, we have

$$B - A - \alpha - \beta - C - D$$

in which $A - B - \beta - C - D$ is an induced P_5 . □

Claim 3. Minimal total dominating sets of sizes two and three cannot coexist.

Proof. Let $U - V$ be a minimal total dominating set of G of size two, $A - B - C$ be a minimal total dominating set of G of size three, w be the private neighbour of A , and y be the private neighbour of C . Then, we have:

$$w - A - B - C - y$$

Without loss of generality, suppose A is adjacent to U . Then, we have:

$$\begin{array}{c} w - A - B - C - y \\ | \\ U \end{array}$$

Since $U - V$ totally dominates w and w must be adjacent to V , we have:

$$\begin{array}{c} w - A - B - C - y \\ | \quad | \\ V - U \end{array}$$

Since $U - V$ totally dominates B and B must be adjacent to V , we have:

$$\begin{array}{c} w - A - B - C - y \\ | \quad | \\ V - U \end{array}$$

Since $U - V$ totally dominates C and C must be adjacent to U , we have:

$$\begin{array}{c} w - A - B - C - y \\ | \quad | \\ V - U \end{array}$$

Since $U - V$ totally dominates y and y must be adjacent to V , we have

$$\begin{array}{c} w - A - B - C - y \\ | \quad | \\ V - U \end{array}$$

in which $w - A - B - C - y$ is an induced P_5 . □