

Ordinary Differential Equations

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1 First Order ODEs

1.1 Linear Differential Equations with Coefficients Independent of y : $y' + p(x)y = q(x)$

1: Calculate the integrating factor

$$\mu(x) = e^{\int p(x) dx}$$

2: Solve

$$\begin{aligned}\mu(x)y' + \mu(x)p(x)y &= \mu(x)q(x) \\ \therefore (\mu(x)y)' &= \mu(x)q(x)\end{aligned}$$

3:

$$y = \frac{1}{\mu(x)} \int \mu(x)q(x) dx$$

1.2 Exact Differential Equations : $M(x, y) + N(x, y)y' = 0$

and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

1: Solve

$$\begin{aligned}\psi &= \int M(x, y) dx \\ &= a(x, y) + h(y) \\ \therefore \frac{\partial \psi}{\partial y} &= \frac{\partial a}{\partial y} + h'(y)\end{aligned}$$

2: Compare $\frac{\partial \psi}{\partial y}$ and N to find $h'(y)$ and hence $h(y)$.

3:

$$\psi(x, y) = c$$

1.3 Bernoulli Differential Equations : $y' + p(x)y = q(x)y^n$, $y \neq 0, 1$

1: Divide the equation by y^n .

$$y^{-n}y' + p(x)y^{1-n} = q(x)$$

2: Substitute

$$\nu = y^{1-n}$$

3: Differentiate ν

$$\nu' = (1-n)y^{-n}y'$$

4: Substitute

$$y^{1-n} = \nu$$

and

$$y^{-n}y' = \frac{1}{1-n}\nu'$$

5: Solve the linear DE in ν

$$\frac{1}{1-n}\nu' + p(x)\nu = q(x)$$

1.4 Separable Differential Equations : $N(y)y' = M(x)$

1: Separate the variables and integrate

$$\begin{aligned} N(y) dy &= M(x) dx \\ \therefore \int N(y) dy &= \int M(x) dx \end{aligned}$$

1.5 Homogeneous Differential Equations : $y' = f(x, y) = F\left(\frac{y}{x}\right)$

1: Write the function as a function of $\frac{x}{y}$ or $\frac{y}{x}$.

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

2: Let

$$\begin{aligned} \frac{y}{x} &= z \\ \therefore y &= xz \\ \therefore \frac{dy}{dx} &= z + x \frac{dz}{dx} \end{aligned}$$

3: Substitute z and $\frac{dz}{dx}$

$$z + x \frac{dz}{dx} = F(z)$$

4: Solve the differential equation in z and x

1.6 Riccati Equations : $y' = f_0(t) + f_1(t)y + f_2(t)y^2$

This method is applicable only if at least one solution is known.

- 1: Let y_1 be a known solution.
- 2: Let

$$y = y_1 + \frac{1}{u(t)}$$

- 3: Differentiate y

$$y' = y_1' - \frac{u'}{u^2}$$

- 4: Substitute $y = y_1 + \frac{1}{u(t)}$ and $y' = y_1' - \frac{u'}{u^2}$ in original equation and simplify.

$$\begin{aligned}y_1' - \frac{u'}{u^2} &= f_0(x) + f_1(x) \left(y_1 + \frac{1}{u} \right) + f_2(x) \left(y_1 + \frac{1}{u} \right)^2 \\ \therefore y_1' - \frac{u'}{u^2} &= \cancel{(f_0(x) + f_1(x)y_1 + f_2(x)y_1^2)} + f_1(x) \frac{1}{u} + f_2(x) \left(\frac{2y_1}{u} + \frac{1}{u^2} \right) \\ \therefore -\frac{u'}{u^2} &= \frac{f_1(x)}{u} + \frac{2f_2(x)y_1u + f_2(x)}{u^2} \\ \therefore u' &= (-f_1(x) - 2f_2y_1)u - f_2(x)\end{aligned}$$

- 5: Solve this differential equation in u .
- 6: Substitute u in

$$y = y_1 + \frac{1}{u(t)}$$

1.7 Non-exact Differential Equations : $M(x, y) dx + N(x, y) dy =$

$$0 \text{ and } \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

- 1: If $\frac{M_y - N_x}{N}$ is a function of x only,

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

- 2: If $\frac{N_x - M_y}{M}$ is a function of y only,

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

3: If $\frac{y^2(M_y - N_x)}{xM + yN}$ is a function of $\frac{x}{y}$ only,

$$\mu\left(\frac{x}{y}\right) = e^{\int \frac{y^2(M_y - N_x)}{xM + yN} d\left(\frac{x}{y}\right)}$$

4: If $\frac{x^2(N_x - M_y)}{xM + yN}$ is a function of $\frac{y}{x}$ only,

$$\mu\left(\frac{y}{x}\right) = e^{\int \frac{x^2(N_x - M_y)}{xM + yN} d\left(\frac{y}{x}\right)}$$

5: If $\frac{N_x - M_y}{xM - yN}$ is a function of xy only,

$$\mu(xy) = e^{\int \frac{N_x - M_y}{xM - yN} d(xy)}$$

6: If $\frac{M_y - N_x}{z_x N - z_y M}$ is a function of $z(x, y)$ only,

$$\mu(z) = e^{\int \frac{M_y - N_x}{z_x N - z_y M} dz}$$

7: Multiply the equation by μ and solve the exact differential equation

$$\mu M(x, y) dx + \mu N(x, y) dy = 0$$

1.8 Existence and Uniqueness

Definition 1 (Lipschitz function). A function is said to be Lipschitz in y if

$$|f(x) - f(y)| \leq C|x - y|$$

for all x and y in the interval, and where C is independent of x and y .

Theorem 1 (Existence and Uniqueness Theorem). *Let $f(x, y)$ be a continuous function of x, y in an open rectangle D , i.e. not including its boundaries, and Lipschitz in y . Then there exists an interval I such that $x_0 \in I$ and the solution for the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$, exists and is unique in I .*

1.8.1 Showing that a IVP has a unique solution in a particular interval

1: Let the IVP be

$$\begin{aligned} y' &= f(x, y) \\ y(x_0) &= y_0 \end{aligned}$$

2: Show that $f(x, y)$ is continuous in the interval.

3: Show that $f(x, y)$ is Lipschitz in y in the interval, i.e.

$$|f(x) - f(y)| \leq C|x - y|$$

for all x and y in the interval, with C independent of x and y .

2 Second Order ODEs

2.1 Linear Homogeneous Differential Equations with Constant Coefficients : $ay'' + by' + cy = 0$

1: Let

$$\begin{aligned}y &= e^{\lambda t} \\y' &= \lambda e^{\lambda t} \\y'' &= \lambda^2 e^{\lambda t}\end{aligned}$$

2: Substitute into the equation

$$\begin{aligned}a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} &= 0 \\ \therefore a\lambda^2 + b\lambda + c &= 0\end{aligned}$$

3: Solve the quadratic equation in λ

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

4: If λ_1 and λ_2 are real and distinct,

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

5: If $\lambda_1 = \lambda_2$,

$$y = c_1 e^{\lambda_1 t} + t c_2 e^{\lambda_1 t}$$

6: If $\lambda_1 = \bar{\lambda}_2 = \alpha + i\beta$,

$$y = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t$$

2.2 Linear Non-homogeneous Differential Equations :

$$y'' + p(t)y' + q(t)y = g(t)$$

- 1: Solve the corresponding homogeneous differential equation $y'' + p(t)y' + q(t)y = 0$.
- 2: Let the solution of the corresponding homogeneous differential equation be y_h .
- 3: Guess a particular solution, $y_p(t)$, using the method of undetermined coefficients or the method of variation of parameters.
- 4: The solution to the ODE is

$$y = y_h + y_p$$

2.2.1 Method of Undetermined Coefficients

- 1: Guess a particular solution to the equation.

	$g(t)$	$y_p(t)$
	$\sum_{i=0}^n a_i t^i$	$\sum_{i=0}^n A_i t^i$
2:	$a e^{\beta t}$	$A e^{\beta t}$
	$a \cos(\beta t)$	$A \cos(\beta t) + B \sin(\beta t)$
	$a \sin(\beta t)$	$A \cos(\beta t) + B \sin(\beta t)$
	$a \cos(\beta t) + b \sin(\beta t)$	$A \cos(\beta t) + B \sin(\beta t)$

- 3: The general solution to the equation is

$$y = y_h + y_p$$

2.2.2 Method of Variation of Parameters

- 1: Let $y_1(t)$ and $y_2(t)$ be two solutions to the corresponding homogeneous equation.
 2: Solve the equation

$$\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

for $u_1'(t)$ and $u_2'(t)$.

- 3:

$$y_p = u_1(t)y_1(t) + u_2(t)y_2(t)$$

2.3 Fundamental Set of Solutions of Linear Second Order Homogeneous ODEs

- 1: Find the Wronskian

$$\begin{aligned} W(y_1, y_2)(x) &= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \\ &= y_1(x)y_2'(x) - y_1'(x)y_2(x) \end{aligned}$$

- 2: If $W(y_1, y_2)(x) \neq 0$, then $\{y_1, y_2\}$ is a fundamental set of solutions.

2.4 Abel's Theorem

Theorem 2 (Abel's Theorem).

$$W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = C e^{-\int p(x) dx}$$

Therefore, as $Ce^{-\int p(x) dx}$ can either be always zero or never zero, the Wronskian can also be always zero or never zero. Hence, a set of solutions y_1 and y_2 , for which the Wronskian is zero for finite values of x cannot be a fundamental set of solutions.

2.5 Euler's Equations : $ax^2y'' + bxy' + cy = 0$

1: Let

$$\begin{aligned}y &= x^r \\y' &= rx^{r-1} \\y'' &= r(r-1)x^{r-2}\end{aligned}$$

2: Substitute into the equation,

$$\begin{aligned}ax^2r(r-1)x^{r-2} + bxx^{r-1} + cx^r &= 0 \\ \therefore x^r (ar(r-1) + br + c) &= 0 \\ \therefore ar(r-1) + br + c &= 0\end{aligned}$$

3: Solve the equation in r ,

$$\begin{aligned}ar^2 - ar + br + c &= 0 \\ \therefore r^2(a) - r(b-a) + c &= 0 \\ \therefore r_{1,2} &= \frac{(a-b) \pm \sqrt{(b-a)^2 - 4ac}}{2a}\end{aligned}$$

4: If r_1 and r_2 are real and distinct,

$$y = c_1x^{r_1} + c_2x^{r_2}$$

5: If $r_1 = r_2$,

$$y = c_1x^{r_1} + c_2x^{r_1} \ln x$$

6: If $r_1 = \bar{r}_2 = \alpha + i\beta$,

$$y = c_1x^\alpha \cos(\beta \ln x) + c_2x^\alpha \sin(\beta \ln x)$$

2.6 Existence and Uniqueness

Theorem 3 (Existence and Uniqueness Theorem). *The IVP*

$$\begin{aligned}y'' + p(t)y' + q(t) &= g(t) \\ y(t_0) &= y_0 \\ y'(t_0) &= y'_0\end{aligned}$$

has a unique solution in an interval I if and only if the functions $p(t)$, $q(t)$, $g(t)$ are continuous in an interval I , and $t_0 \in I$.

2.7 Reduction of Order : $y'' + p(t)y' + q(t) = 0$, $y_1(t)$

1: Let

$$\begin{aligned}y_2(t) &= y_1(t)\nu(t) \\ \therefore y_2' &= y_1'(t)\nu(t) + y_1(t)\nu'(t) \\ \therefore y_2'' &= y_1''(t)\nu(t) + 2y_1'(t)\nu'(t) + y_1(t)\nu''(t)\end{aligned}$$

2: Substitute into the equation to get an ODE with $\nu''(t)$ and $\nu'(t)$.

$$\begin{aligned}0 &= y_1''(t)\nu(t) + 2y_1'(t)\nu'(t) + y_1(t)\nu''(t) \\ &\quad + (y_1'(t)\nu(t) + y_1(t)\nu'(t))p(t) \\ &\quad + y_1(t)\nu(t)q(t)\end{aligned}$$

3: Let

$$\begin{aligned}k(t) &= \nu'(t) \\ \therefore k'(t) &= \nu''(t)\end{aligned}$$

4: Substitute and solve the first order ODE in k .