ORDINARY DIFFERENTIAL EQUATIONS

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1 First Order ODEs

1.1 Linear Differential Equations with Coefficients Independent of y : $y'+p(x)y=q(x)$

1: Calculate the integrating factor

$$\mu(x)=e^{\int p(x)\mathrm{d}x}$$

2: Solve

$$\mu(x)y'+\mu(x)p(x)y=\mu(x)q(x)$$
$$\therefore \big(\mu(x)y\big)'=\mu(x)q(x)$$

3:

$$y=\frac{1}{\mu(x)}\int \mu(x)q(x)\mathrm{d}t$$

1.2 Exact Differential Equations : $M(x,y)+N(x,y)y'=0$ and $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

1: Solve

$$\psi=\int M(x,y)\mathrm{d}x$$
$$=a(x,y)+h(y)$$
$$\therefore \frac{\partial \psi}{\partial y}=\frac{\partial a}{\partial y}+h'(y)$$

2: Compare $\frac{\partial \psi}{\partial y}$ and N to find $h'(y)$ and hence $h(y)$.

3:

$$\psi(x,y)=c$$

1.3 Bernoulli Differential Equations : $y'+p(x)y=q(x)y^n$, $y\neq 0,1$

1: Divide the equation by y^n.

$$y^{-n}y'+p(x)y^{1-n}=q(x)$$

2: Substitute

$$\nu=y^{1-n}$$

3: Differentiate ν

$$\nu'=(1-n)y^{-n}y'$$

4: Substitute

$$y^{1-n}=\nu$$

and

$$y^{-n}y'=\frac{1}{1-n}\nu'$$

5: Solve the linear DE in ν

$$\frac{1}{1-n}\nu'+p(x)\nu=q(x)$$

1.4 Separable Differential Equations : $N(y)y'=M(x)$

1: Separate the variables and integrate

$$N(y)\mathrm{d}y=M(x)\mathrm{d}x$$
$$\therefore \int N(y)\mathrm{d}y=\int M(x)\mathrm{d}x$$

1.5 Homogeneous Differential Equations : $y'=f(x,y)=F\left(\frac{y}{x}\right)$

1: Write the function as a function of $\frac{x}{y}$ or $\frac{y}{x}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=F\left(\frac{y}{x}\right)$$

2: Let

$$\frac{y}{x}=z$$
$$\therefore y=xz$$
$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=z+x\frac{\mathrm{d}z}{\mathrm{d}x}$$

3: Substitute z and $\frac{\mathrm{d}z}{\mathrm{d}x}$

$$z+x\frac{\mathrm{d}z}{\mathrm{d}x}=F(z)$$

4: Solve the differential equation in z and x

1.6 Riccati Equations : $y'=f_0(t)+f_1(t)y+f_2(t)y^2$

This method is applicable only if at least one solution is known.

1: Let y_1 be a known solution.

2: Let

$$y=y_1+\frac{1}{u(t)}$$

3: Differentiate y

$$y'={y_1}'-\frac{u'}{u^2}$$

4: Substitute $y=y_1+\frac{1}{u(t)}$ and $y'={y_1}'-\frac{u'}{u^2}$ in original equation and simplify.

$${y_1}'-\frac{u'}{u^2}=f_0(x)+f_1(x)\left(y_1+\frac{1}{u}\right)+f_2(x)\left(y_1+\frac{1}{u}\right)^2$$
$$\therefore \cancel{{y_1}'}-\frac{u'}{u^2}=\cancel{\left(f_0(x)+f_1(x)y_1+f_2(x){y_1}^2\right)}+f_1(x)\frac{1}{u}+f_2(x)\left(\frac{2y_1}{u}+\frac{1}{u^2}\right)$$
$$\therefore -\frac{u'}{u^2}=\frac{f_1(x)}{u}+\frac{2f_2(x)y_1u+f_2(x)}{u^2}$$
$$\therefore u'=\big(-f_1(x)-2f_2y_1\big)u-f_2(x)$$

5: Solve this differential equation in u.

6: Substitute u in

$$y=y_1+\frac{1}{u(t)}$$

1.7 Non-exact Differential Equations : $M(x,y)\,\mathrm{d}\,x+N(x,y)\mathrm{d}y=0$ and $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$

1: If $\frac{M_y-N_x}{N}$ is a function of x only,

$$\mu(x)=e^{\int \frac{M_y-N_x}{N}\mathrm{d}x}$$

2: If $\frac{N_x-M_y}{M}$ is a function of y only,

$$\mu(y)=e^{\int \frac{N_x-M_y}{M}\mathrm{d}y}$$

3: If $\frac{y^2(M_y-N_x)}{xM+yN}$ is a function of $\frac{x}{y}$ only,

$$\mu\left(\frac{x}{y}\right)=e^{\int \frac{y^2(M_y-N_x)}{xM+yN}\mathrm{d}\left(\frac{x}{y}\right)}$$

4: If $\frac{x^2(N_x-M_y)}{xM+yN}$ is a function of $\frac{y}{x}$ only,

$$\mu\left(\frac{y}{x}\right)=e^{\int \frac{x^2(N_x-M_y)}{xM+yN}\mathrm{d}\left(\frac{y}{x}\right)}$$

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5: If $\frac{N_x - M_y}{xM - yN}$ is a function of xy only,
$$\mu(xy) = e^{\int \frac{N_x - M_y}{xM - yN} \mathrm{d}(xy)}$$
6: If $\frac{M_y - N_x}{z_x N - z_y M}$ is a function of $z(x,y)$ only,
$$\mu(z) = e^{\int \frac{M_y - N_x}{z_x N - z_y M} \mathrm{d}z}$$
7: Multiply the equation by μ and solve the exact differential equation
$$\mu M(x,y)\mathrm{d}x + \mu N(x,y)\mathrm{d}y = 0$$

1.8 Existence and Uniqueness

Definition 1 (Lipschitz function)**.** A function if said to be Lipschitz in y if
$$\left|f(x) - f(y)\right| \le C|x-y|$$
for all x and y in the interval, and where C is independent of x and y.

Theorem 1 (Existence and Uniqueness Theorem)**.** *Let $f(x,y)$ be a continuous function of x, y in an open rectangle D, i.e. not including its boundaries, and Lipschitz in y. Then there exists an interval I such that $x_0 \in I$ and the solution for the initial value problem $y' = f(x,y)$, $y(x_0) = y_0$, exists and is unique in I.*

1.8.1 Showing that a IVP has a unique solution in a particular interval

1: Let the IVP be
$$y' = f(x,y)$$
$$y(x_0) = y_0$$
2: Show that $f(x,y)$ is continuous in the interval.
3: Show that $f(x,y)$ is Lipschitz in y in the interval, i.e.
$$\left|f(x) - f(y)\right| \le C|x-y|$$
for all x and y in the interval, with C independent of x and y.

2 Second Order ODEs

2.1 Linear Homogeneous Differential Equations with Constant Coefficients : $ay'' + by' + cy = 0$

1: Let
$$y = e^{\lambda t}$$
$$y' = \lambda e^{\lambda t}$$
$$y'' = \lambda^2 e^{\lambda t}$$
2: Substitute into the equation
$$a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} = 0$$
$$\therefore a\lambda^2 + b\lambda + c = 0$$
3: Solve the quadratic equation in y
$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
4: If λ_1 and λ_2 are real and distinct,
$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
5: If $\lambda_1 = \lambda_2$,
$$y = c_1 e^{\lambda_1 t} + tc_2 e^{\lambda_1 t}$$
6: If $\lambda_1 = \overline{\lambda}_2 = \alpha + i\beta$,
$$y = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t$$

2.2 Linear Non-homogeneous Differential Equations : $y'' + p(t)y' + q(t)y = g(t)$

1: Solve the corresponding homogeneous differential equation $y'' + p(t)y' + q(t)y = 0$.
2: Let the solution of the corresponding homogeneous differential equation be y_h.
3: Guess a particular solution, $y_p(t)$, using the method of undetermined coefficients or the method of variation of parameters.
4: The solution to the ODE is
$$y = y_h + y_p$$

2.2.1 Method of Undetermined Coefficients

1: Guess a particular solution to the equation.

2:

$g(t)$	$y_p(t)$
$\sum_{i=0}^{n} a_i t^i$	$\sum_{i=0}^{n} A_i t^i$
$ae^{\beta t}$	$Ae^{\beta t}$
$a\cos(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$a\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$a\cos(\beta t) + b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$

3: The general solution to the equation is
$$y = y_h + y_p$$

2.2.2 Method of Variation of Parameters

1: Let $y_1(t)$ and $y_2(t)$ be two solutions to the corresponding homogeneous equation.
2: Solve the equation
$$\begin{pmatrix} y_1(t) & y_2(t) \\ {y_1}'(t) & {y_2}'(t) \end{pmatrix} \begin{pmatrix} {u_1}'(t) \\ {u_2}'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$
for ${u_1}'(t)$ and ${u_2}'(t)$.
3:
$$y_p = u_1(t)y_1(t) + u_2(t)y_2(t)$$

2.3 Fundamental Set of Solutions of Linear Second Order Homogeneous ODEs

1: Find the Wronskian
$$W(y_1,y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ {y_1}'(x) & {y_2}'(x) \end{vmatrix}$$
$$= y_1(x){y_2}'(x) - {y_1}'(x)y_2(x)$$
2: If $W(y_1,y_2)(x) \ne 0$, then $\{y_1,y_2\}$ is a fundamental set of solutions.

2.4 Abel's Theorem

Theorem 2 (Abel's Theorem)**.**
$$W(y_1,y_2)(x) = y_1(x)y_2(x)' - y_1(x)'y_2(x) = Ce^{-\int p(x)\mathrm{d}x}$$
Therefore, as $Ce^{-\int p(x)\mathrm{d}x}$ can either be always zero or never zero, the Wronskian can also be always zero or never zero. Hence, a set of solutions y_1 and y_2, for which the Wronskian is zero for finite values of x cannot be a fundamental set of solutions.

2.5 Euler's Equations : $ax^2y'' + bxy' + cy = 0$

1: Let
$$y = x^r$$
$$y' = rx^{r-1}$$
$$y'' = r(r-1)x^{r-2}$$
2: Substitute into the equation,
$$ax^2r(r-1)x^{r-2} + bxrx^{r-1} + cx^r = 0$$
$$\therefore x^r\left(ar(r-1) + br + c\right) = 0$$
$$\therefore ar(r-1) + br + c = 0$$
3: Solve the equation in r,
$$ar^2 - ar + br + c = 0$$
$$\therefore r^2(a) - r(b-a) + c = 0$$
$$\therefore r_{1,2} = \frac{(a-b) \pm \sqrt{(b-a)^2 - 4ac}}{2a}$$
4: If r_1 and r_2 are real and distinct,
$$y = c_1x^{r_1} + c_2x^{r_2}$$
5: If $r_1 = r_2$,
$$y = c_1x^{r_1} + c_2x^{r_1}\ln x$$
6: If $r_1 = \overline{r}_2 = \alpha + i\beta$,
$$y = c_1x^\alpha\cos(\beta\ln x) + c_2x^\alpha\sin(\beta\ln x)$$

2.6 Existence and Uniqueness

Theorem 3 (Existence and Uniqueness Theorem)**.** *The IVP*
$$y'' + p(t)y' + q(t) = g(t)$$
$$y(t_0) = y_0$$
$$y'(t_0) = y'_0$$
has a unique solution in an interval I if and only if the functions $p(t)$, $q(t)$, $g(t)$ are continuous in an interval I, and $t_0 \in I$.

2.7 Reduction of Order : $y'' + p(t)y' + q(t) = 0$, $y_1(t)$

1: Let
$$y_2(t) = y_1(t)\nu(t)$$
$$\therefore {y_2}' = {y_1}'(t)\nu(t) + y_1(t)\nu'(t)$$
$$\therefore {y_2}'' = {y_1}''(t)\nu(t) + 2{y_1}'(t)\nu'(t) + y_1(t)\nu''(t)$$
2: Substitute into the equation to get an ODE with $\nu''(t)$ and $\nu'(t)$.
$$0 = {y_1}''(t)\nu(t) + 2{y_1}'(t)\nu'(t) + y_1(t)\nu''(t)$$
$$+ \left({y_1}'(t)\nu(t) + y_1(t)\nu'(t)\right)p(t)$$
$$+ y_1(t)\nu(t)q(t)$$
3: Let
$$k(t) = \nu'(t)$$
$$\therefore k'(t) = \nu''(t)$$
4: Substitute and solve the first order ODE in k.