

Linear Algebra

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Contents

I	General Information	7
1	Contact Information	7
2	Grades	7
II	Fields	8
1	Definition	8
1.1	Examples of Fields	8
1.2	Examples of Non-fields (Rings)	9
2	Examples	9
III	Matrices	10
1	Definition	10



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2	Addition of Matrices	10
2.0.1	Properties	10
3	Multiplication of a matrix by a scalar	10
4	Multiplication of matrices	11
5	Zero Divisor	11
6	Theorem ('Good properties of matrix multiplication')	12
7	Square Matrices	13
7.1	Diagonal Matrices	13
7.1.1	Proof	13
7.2	Upper-triangular Matrices	13
7.3	Lower-triangular Matrices	13
7.4	Theorem	14
7.4.1	Proof	14
7.5	Identity Matrix	14
7.6	Theorem	14
7.6.1	Proof	14
7.7	Inverse of Matrix	15
7.7.1	If $AB = I_n$ and $CA = I_n$, then $B = C$	15
7.7.2	Inverse of a Matrix	15
7.7.3	If $AB = I$, then $BA = I$	16
7.7.4	If A is invertible, then A <u>cannot</u> be a zero divisor.	16
7.7.5	If A and B are invertible, then $A + B$ may or may not be invertible.	16
7.7.6	If A and B are invertible, then AB must be invertible.	17
8	Transpose of a Matrix	17
8.1	Properties of A^t	17
9	Adjoint Matrix	17
9.0.1	Properties of Adjoint Matrices	18
10	Row Operations on Matrices	18
10.1	Elementary Row Operations	18
10.2	Theorems	18
10.2.1	$E_1A =$ the matrix obtained from A by an elementary row operation I	18

10.2.2	$E_{II}A$ = the matrix obtained from A by an elementary row operation II	19
10.2.3	$E_{III}A$ = the matrix obtained from A by an elementary row operation III	19
10.2.4	All elementary matrices are invertible, moreover, the inverses of E_I, E_{II}, E_{III} are also elementary matrices of the same type.	20
10.3	Row-equivalent of a Matrix	22
11	Row Echelon Form of a Matrix	22
11.1	Definition	22
11.2	Notation	23
12	Row Rank of a Matrix	23
13	Gauss Theorem	23
13.1	Elimination Algorithm	23
13.1.1	Example	24
13.2	Row Spaces of Matrices	24
13.3	Column Equivalence	25
IV	Linear Systems	27
1	Definition	27
2	Equivalent Systems	27
3	Solution of a System of Equations	28
4	Homogeneous Systems	29
4.1	Definition	29
4.2	Solutions of Homogeneous Systems	29
4.2.1	Example	30
4.2.2	General Solution	31
4.3	Properties	31
4.3.1	For a homogeneous system $Ax = 0$, if c and d are solutions, then $c + d$ is also a solution.	31
4.3.2	For a homogeneous system $Ax = 0$, if c is a solution and $\alpha \in \mathbb{F}$, then, αc is a solution too.	31
4.4	Fundamental Solutions	32

4.4.1	Theorem: Any solution d of the system $Ax = \mathbb{O}$ can be obtained from the basic solutions v_1, \dots, v_t as a linear combination of the basic solutions, $d = \alpha_1 v_1 + \dots + \alpha_t v_t$.	32
4.5	32
5	Non-Homogeneous Systems	33
5.1	Definition	33
5.2	Solutions of Non-Homogeneous Systems	33
5.2.1	Case I: $\tilde{r} = r$	33
5.2.2	Case II: $\tilde{r} > r$	33
5.3	General Solution	34
V	Vector Spaces	35
1	Definition	35
1.1	Examples	35
1.1.1	Geometric Vectors in Plane	35
1.1.2	Arithmetic Vector Space	35
1.1.3	36
2	Properties	36
2.0.1	Proof of 1	36
3	Subspaces	36
3.1	Examples	37
3.2	Operations on Subspaces	38
4	Spans	38
5	Linear Dependence	41
5.1	Properties of Linearly Dependent and Independent Sets	42
5.2	Changing a Basis	46
5.3	Representation of Vectors in a Basis	49
5.3.1	Properties of Representations	49
6	Determinants	50
6.1	Definition	50
6.2	Properties	50
6.3	Practical Methods for Computing Determinants	53
6.4	Expansion along a row/ column	53
6.5	Determinant Rank	53

7	Linear Maps	54
7.1	Definition	54
7.2	Properties	54
7.3	Matrix of a Linear Map	54
7.4	Change of Bases	56
7.5	Operations on Linear Maps	57
7.6	Kernel and Image	58
7.6.1	Dimensions of Kernel and Image	60
VI	Linear Operators	62
1	Definition	62
2	Similar Matrices	62
2.1	Properties of Similar Matrices	63
3	Diagonalization	63
4	Eigenvalues and Eigenvectors	63
5	Characteristic Polynomial	65
5.1	Properties	66
VII	Inner Product Spaces	71
1	Definition	71
2	Computation of Inner Products	72
2.1	Change of Basis	74
3	Norms	75
3.1	Definition	75
3.2	Properties	75
4	Orthogonality	75
4.1	Definition	75
4.2	Properties	76
5	Orthogonal and Orthonormal Bases	76
6	Unitary Matrices	78

7	Projections	80
7.1	Definition	80
7.2	Properties	82
7.3	Gram - Schmidt Process	82
7.4	Inequalities	84
8	Angle	87
9	Triangle Inequality	87
10	Orthogonal Decomposition	87
11	Distance	89
12	Adjoint Map	90
12.1	Construction	92
12.2	Properties	92
13	Special Linear Operators	93

Part I

General Information

1 Contact Information

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2 Grades

Final Exam: 80%

Midterm Exam: 10%

Homework: 10%

Passing Criteria: 60%

Part II

Fields

1 Definition

Definition 1 (Field). The set \mathbb{F} is a field if there are operations $+$, \cdot satisfying the following properties:

$$(A1) \quad \forall a, b \in \mathbb{F}; a + b = b + a$$

$$(A2) \quad \forall a, b \in \mathbb{F}; (a + b) + c = a + (b + c)$$

$$(A3) \quad \text{There is an element } 0 \in \mathbb{F} \text{ s.t. } a + 0 = 0 + a = a$$

$$(A4) \quad \forall a \in \mathbb{F}, \exists b \in \mathbb{F} \text{ s.t. } a + b = 0$$

$$(M1) \quad \forall a, b \in \mathbb{F}, a \cdot b = b \cdot a$$

$$(M2) \quad \forall a, b \in \mathbb{F}, (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

$$(M3) \quad \text{There is an element } 1 \in \mathbb{F} \text{ s.t. } a \cdot 1 = 1 \cdot a = a (1 \neq 0)$$

$$(M4) \quad \forall a \in \mathbb{F}, (a \neq 0), \exists b \in \mathbb{F} \text{ s.t. } a \cdot b = 1$$

$$(AM) \quad (a + b) \cdot c = (a \cdot c) + (b \cdot c)$$

If \mathbb{F} is a field, one can define subtraction and division as follows.

$$a - b \doteq a + (-b)$$

$$\frac{a}{b} \doteq a \cdot \frac{1}{b}$$

1.1 Examples of Fields

1. \mathbb{R}
2. \mathbb{C}
3. \mathbb{F}_p

1.2 Examples of Non-fields (Rings)

1. \mathbb{Z} , as M4 is not satisfied.

If we define $\mathbb{F}_2 = 0, 1; 0 + 0 = 0; 0 + 1 = 1 + 0 = 1; 1 + 1 = 0$, otherwise, 1 will have no additive inverse.

2 Examples

Example 1. Let p be a prime number. \mathbb{F}_p is defined as follows.

$$\forall m \in \mathbb{Z}, m = a \cdot p + \bar{m}$$

The operations $+$ and \cdot are defined as

$$\begin{aligned}\bar{a} + \bar{b} &= \overline{(a + b)} \\ \bar{a} \cdot \bar{b} &= \overline{(a \cdot b)}\end{aligned}$$

1. \mathbb{F}_p is a field.
2. If \mathbb{F} is a set of q elements, we can define on \mathbb{F} a structure of a field iff $q = p^t$, where p is prime, $t \geq 1$.

Example 2. For a field of 4 elements $\{0, 1, \alpha, \beta\}$, the addition and multiplication tables are as follows.

+	0	1	α	β
0	0	1	α	β
1	1	0	β	α
α	α	β	0	1
β	β	α	0	1

Part III

Matrices

1 Definition

Definition 2 (Matrix). Let \mathbb{F} be a field, $m, n \geq 1$. Then, $A(m \times n)$ is a table consisting of m rows and n columns, filled by elements of \mathbb{F} .

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

2 Addition of Matrices

Definition 3 (Addition of matrices). Let A, B be $m \times n$ matrices over \mathbb{F} . Then, $C = A + B$ is defined as follows.

$$c_{ij} = a_{ij} + b_{ij}$$

2.0.1 Properties

1. $A + B = B + A, \forall A, B$ s.t. the sum is defined
2. $(A + B) + C = A + (B + C), \forall A, B, C$ s.t. the sums are defined
3. There is a matrix \mathbb{O} , s.t. $A + \mathbb{O} = \mathbb{O} + A = A$
4. For any $A, \exists B$ s.t. $B = -A$

3 Multiplication of a matrix by a scalar

Definition 4 (Multiplication of a matrix by a scalar). Let A be a $m \times n$ matrix over \mathbb{F} . Let $\alpha \in \mathbb{F}$ be a scalar. Then, $C = \alpha A$ is defined as follows.

$$c_{ij} = \alpha a_{ij}$$

4 Multiplication of matrices

Definition 5 (Multiplication of matrices). Let A be a $m \times n$ matrix over \mathbb{F} . Let B be a $n \times p$ matrix over \mathbb{F} . Then, $C = AB$ is defined as follows.

$$c_{ik} = \sum_{j=1}^n a_{ij}b_{jk}$$

Example 3. For matrices A, B , of same size, is $AB = BA$?

Solution. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
 $\therefore AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
 $\therefore AB \neq BA$

Remark 1. $A \neq \mathbb{O}, B \neq \mathbb{O}$, but $AB = \mathbb{O}$.

5 Zero Divisor

Definition 6 (Zero divisor). We say that a square matrix $A \neq \mathbb{O}$ is a zero divisor if either there is a square matrix B s.t. $AB = \mathbb{O}$, or there is a square matrix C , s.t. $CA = \mathbb{O}$.

Remark 2. $\mathbb{O}B = C\mathbb{O} = \mathbb{O}$.

Remark 3. $AC = BC \not\Rightarrow A = B$. In general, we cannot cancel matrices on either side of an equation.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, C = \mathbb{O}$$

$$\therefore AB = CB = \mathbb{O} \& B \neq \mathbb{O}$$

But, we cannot cancel B , as $A \neq C$.

6 Theorem ('Good properties of matrix multiplication')

Theorem 1.

$$(AB)C = A(BC) \quad (1.1)$$

$$A(B + C) = AB + AC \quad (1.2)$$

$$(A + B)C = AC + BC \quad (1.3)$$

$$(\alpha A) = \alpha(AB) \quad (1.4)$$

Proof. Denote $AB = D, BC = G, (AB)C = F, A(BC) = H$

We need to prove $F = H$

Let the dimensions of the matrices be as follows.

$A_{m \times n}, B_{n \times p}, C_{p \times q}$

$\therefore F_{m \times q}, H_{m \times q}$

$$d_{ik} = \sum_j a_{ij}b_{jk}$$

$$\therefore g_{jl} = \sum_k b_{jk}b_{kl}$$

$$f_{il} = \sum_k d_{ik}c_{kl} = \sum_k \left(\sum_j a_{ij}b_{jk} \right) c_{kl} = \sum_k \sum_j a_{ij}b_{jk}c_{kl}$$

$$h_{il} = \sum_j a_{ij}g_{jl} = \sum_j a_{ij} \left(\sum_k b_{jk}c_{kl} \right) = \sum_k \sum_j a_{ij}b_{jk}c_{kl}$$

$$f_{il} = h_{il}$$

$$F = H$$

□

7 Square Matrices

Let A be a square matrix of size $n \times n, n \geq 1$

7.1 Diagonal Matrices

Definition 7 (Diagonal matrix). We say that A is a diagonal matrix if $a_{ij} = 0$, whenever $i \neq j$.

Theorem 2. Let A and B be diagonal $n \times n$ matrices.

$$a_{rr} = \alpha_r, b_{rr} = \beta_r$$

Then, $AB = BA = C, C$ is a diagonal matrix with $c_{rr} = a_{rr}b_{rr}$.

7.1.1 Proof

$$a_{ij} = \begin{cases} 0, & i \neq j \\ \alpha_i, & i = j \end{cases}$$

$$b_{ij} = \begin{cases} 0, & i \neq j \\ \beta_i, & i = j \end{cases}$$

$$c_{ik} = \sum_{j=1}^n a_{ij}b_{jk} = a_{ii}b_{ik} = \alpha_i b_{ik} = \begin{cases} 0, & i \neq k \\ \alpha_i \beta_i, & i = k \end{cases}$$

Similarly for BA .

7.2 Upper-triangular Matrices

We say that A is an upper-triangular matrix if $a_{ij} = 0$, whenever $i > j$.

7.3 Lower-triangular Matrices

We say that A is a lower-triangular matrix if $a_{ij} = 0$, whenever $i < j$.

Remark

Diagonal matrices are upper-triangular and lower-triangular. Conversely, if a matrix is both upper-triangular and lower-triangular, it is a diagonal matrix.

7.4 Theorem

If A and B are both upper-triangular, then AB and BA are upper-triangular too.

7.4.1 Proof

Denote $C = AB$.

$$\therefore c_{ik} = \sum_{j=1}^n a_{ij}b_{jk}$$

Suppose $i > k$, then, either $i > j$ or $j > k$. So, in each case, atleast one of a_{ij} or b_{jk} is 0.

7.5 Identity Matrix

Let $n \geq 1$. We call I_n the $n \times n$ identity matrix.

$$I_n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

7.6 Theorem

Let I_n be the identity $n \times n$ matrix. Then, for any $n \times n$ matrix B , we have

$$I_n B = B I_n = B$$

7.6.1 Proof

$$I_n = (e_{ij}); e_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Denote $C = I_n B$. We have

$$c_{ik} = \sum_{j=1}^n e_{ij}b_{jk} = e_{ii}b_{ik} = 1 \cdot b_{ik} = b_{ik}$$

$$\therefore C = B \Rightarrow I_n B = B$$

Similarly for $B I_n = B$.

7.7 Inverse of Matrix

Let A be an $n \times n$ matrix. We say that A is invertible if there exist B, C , s.t. $AB = I_n$ and $CA = I_n$

Remark

$A = \mathbb{O}$ is not invertible because $\mathbb{O}B = C\mathbb{O} = \mathbb{O} \neq I_n$

Remark

There are non-zero matrices which are not invertible.

$$\text{Let } A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

If possible, let there be C s.t. $CA = I_2$.

$$\text{Let } B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

We have $CA = I$.

$$\therefore (CA)B = IB$$

$$\therefore C(AB) = B$$

$$\therefore C\mathbb{O} = B$$

$$\therefore \mathbb{O} = B$$

But, $B \neq 0$. Therefore, C does not exist.

7.7.1 If $AB = I_n$ and $CA = I_n$, then $B = C$

$$\begin{aligned} C &= CI \\ &= C(AB) \\ &= (CA)B \\ &= IB \\ &= B \end{aligned}$$

7.7.2 Inverse of a Matrix

If A is invertible, i.e. if there exists B , s.t. $AB = BA = I$, then, B is called the inverse of A , and is denoted by A^{-1} .

7.7.3 If $AB = I$, then $BA = I$.

7.7.4 If A is invertible, then A cannot be a zero divisor.

If possible, let A be a zero divisor.

Therefore, either $AB = \mathbb{O}$, for some $B \neq \mathbb{O}$; or $CA = \mathbb{O}$, for some $C \neq \mathbb{O}$

Case I: $AB = \mathbb{O}$

$$\begin{aligned}AB &= \mathbb{O} \\ \therefore A^{-1}(AB) &= A^{-1}\mathbb{O} \\ \therefore (A^{-1}A)B &= \mathbb{O} \\ \therefore IB &= \mathbb{O} \\ \therefore B &= \mathbb{O}\end{aligned}$$

This contradicts the assumption $B \neq \mathbb{O}$

Case II: $CA = \mathbb{O}$

$$\begin{aligned}CA &= \mathbb{O} \\ \therefore (CA)A^{-1} &= \mathbb{O}A^{-1} \\ \therefore C(A^{-1}A) &= \mathbb{O} \\ \therefore CI &= \mathbb{O} \\ \therefore C &= \mathbb{O}\end{aligned}$$

This contradicts the assumption $C \neq \mathbb{O}$

7.7.5 If A and B are invertible, then $A + B$ may or may not be invertible.

If $A = B$, then $A + B = 2A$ is invertible.

If $A = -B$, then $A + B = \mathbb{O}$ is not invertible.

7.7.6 If A and B are invertible, then AB must be invertible.

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= AIA^{-1} \\ &= AA^{-1} \\ &= I\end{aligned}$$

$$\begin{aligned}\text{Similarly, } (B^{-1}A^{-1})(AB) &= I \\ \therefore (AB)^{-1} &= B^{-1}A^{-1}\end{aligned}$$

8 Transpose of a Matrix

Let A be a $m \times n$ matrix, $A = (a_{ij})_{1 \leq i \leq m; 1 \leq j \leq n}$

$B = A^t$ is defined as follows.

$$b_{ji} = a_{ij}$$

8.1 Properties of A^t

1. $(A + B)^t = A^t + B^t$
2. $(\alpha A)^t = \alpha A^t$
3. $(AB)^t = B^t A^t$
4. If A is invertible, then, A^t must be invertible, and $(A^t)^{-1} = (A^{-1})^t$

9 Adjoint Matrix

$$A^* \doteq \overline{A}^t$$

For example,

$$A = \begin{pmatrix} 1 & 1+i & 2-1 \\ i & -5i & 3 \end{pmatrix}$$
$$B = \begin{pmatrix} 1 & -i \\ 1-i & 5i \\ 2+i & 3 \end{pmatrix}$$

9.0.1 Properties of Adjoint Matrices

1. $(A + B)^* = A^* + B^*$
2. $(\alpha A)^* = \bar{\alpha}A^*$
3. $(AB)^* = B^*A^*$
4. If A is invertible, then A^* is invertible, and $(A^*)^{-1} = (A^{-1})^*$

10 Row Operations on Matrices

10.1 Elementary Row Operations

Let A be a $m \times n$ matrix with rows a_1, \dots, a_m . We define 3 types of elementary row operations.

- I $a_i \leftrightarrow a_j$ (Switch of the i^{th} and j^{th} rows.)
- II $a_i \rightarrow \alpha a_i (\alpha \neq 0)$ (Multiplication of a row by a non-zero scalar.)
- III $a_i \rightarrow a_i + \alpha a_j (j \neq i)$ (Addition of a row multiplied by a scalar, and another row.)

E_I, E_{II}, E_{III} are matrices obtained from the identity matrix by applying elementary row operations I, II, III, respectively. These matrices are called elementary matrices.

10.2 Theorems

Let $e_i = (0 \ \dots \ 0 \ 1 \ 0 \ \dots \ 0)$ be a $1 \times m$ matrix.

Let A be any $m \times n$ matrix.

Then, $e_i A =$ the i^{th} row of A .

10.2.1 $E_I A =$ the matrix obtained from A by an elementary row operation I

Proof

Let A be any $m \times n$ matrix.

$$\therefore E_I A = \begin{pmatrix} e_1 A \\ \vdots \\ e_j A \\ \vdots \\ e_i A \\ \vdots \\ e_m A \end{pmatrix}$$

10.2.2 $E_{II} A =$ the matrix obtained from A by an elementary row operation II

Proof

Let A be any $m \times n$ matrix.

$$\therefore E_{II} A = \begin{pmatrix} e_1 A \\ \vdots \\ \alpha e_i A \\ \vdots \\ e_m A \end{pmatrix}$$

10.2.3 $E_{III} A =$ the matrix obtained from A by an elementary row operation III

Proof

Let A be any $m \times n$ matrix.

$$\begin{aligned} \therefore E_1 A &= \begin{pmatrix} e_1 A \\ \vdots \\ a_{i1} + \alpha a_{j1} \cdots + a_{in} + \alpha a_{jn} \\ \vdots \\ e_j A \\ \vdots \\ e_m A \end{pmatrix} \\ &= \begin{pmatrix} 1^{\text{st}} \text{ row of } A \\ \vdots \\ i^{\text{th}} \text{ row of } A + \alpha(j^{\text{th}} \text{ row of } A) \\ \vdots \\ j^{\text{th}} \text{ row of } A \\ \vdots \\ m^{\text{th}} \text{ row of } A \end{pmatrix} \end{aligned}$$

10.2.4 All elementary matrices are invertible, moreover, the inverses of $E_{\text{I}}, E_{\text{II}}, E_{\text{III}}$ are also elementary matrices of the same type.

$$\begin{aligned} E_{\text{I}}^{-1} &= E_{\text{I}} \\ \Leftrightarrow E_{\text{I}}^2 &= I_m \end{aligned}$$

$$\begin{aligned}
E_1^2 &= E_1 E_1 \\
&= \begin{pmatrix} e_1 E_1 \\ \vdots \\ e_j E_1 \\ \vdots \\ e_i E_1 \\ \vdots \\ e_m E_1 \end{pmatrix} \\
&= \begin{pmatrix} 1^{\text{st}} \text{ row of } A \\ \vdots \\ j^{\text{th}} \text{ row of } A \\ \vdots \\ i^{\text{th}} \text{ row of } A \\ \vdots \\ m^{\text{th}} \text{ row of } A \end{pmatrix} \\
&= \begin{pmatrix} e_1 \\ \vdots \\ e_j \\ \vdots \\ e_i \\ \vdots \\ e_m \end{pmatrix} = I_m
\end{aligned}$$

Similarly for E_{II} , to get the inverse, α is replaced by $\frac{1}{\alpha}$

$$E_{\text{II}} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \alpha & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

$$\therefore E_{\text{II}}^{-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{\alpha} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Similarly for E_{III} , to get the inverse, α is replaced by $-\alpha$

$$E_{\text{III}} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & \alpha & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

$$\therefore E_{\text{III}}^{-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -\alpha & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

10.3 Row-equivalent of a Matrix

A matrix A' is a row-equivalent of A , if A' is obtained for A , by a finite sequence of elementary row operations.

11 Row Echelon Form of a Matrix

11.1 Definition

Let A be an $m \times n$ matrix.

Denote the i^{th} row of A by a_i .

The leading entry of a non-zero row a_i is its first non-zero entry.
Denote the column where the leading entry occurs by l_i .

$$a_{ij} = 0 \text{ if } j < l(i)$$

$$a_{ij} \neq 0 \text{ if } j = l(i)$$

We say that A is in row echelon form(REF) if the following conditions hold.

1. The non-zero rows are at the top of A . (r = the number of non-zero rows)
2. The leading entries go right as we go down, i.e. $l(1) < l_2 < \dots < l(r)$
3. All leading entries equal 1, i.e. if $j = l(i)$, then, $a_{ij} = 1$
4. Any column which contains a leading entry must have all other entries equal to 0, i.e. if $j = l(i)$, then, $a_{kj} = 0; \forall k \neq i$

11.2 Notation

The REF of A will be denoted by A_R .

12 Row Rank of a Matrix

The number of non-zero rows in A_R is called the row rank of A . It is denoted by r .

$$r \leq n$$

13 Gauss Theorem

Any $m \times n$ matrix A can be brought to REF by a sequence of elementary row operations.

13.1 Elimination Algorithm

Step 1 Find the first non-zero column C_p of A .

Step 2 Denote by a_{ip} the first non-zero entry of C_p .

Step 3 Switch the 1st and i^{th} rows.

Step 4 Multiply the 1st row by $\frac{1}{a_{ip}}$.

Step 5 Using row operations of type III, make all other entries of the p^{th} column zeros.

Step 6 Ignoring the top row and C_p , repeat steps Step 1 to Step 5.

13.1.1 Example

$$\begin{aligned}
 & \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 4 & 7 \\ 0 & -1 & 7 & 6 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 0 & -1 & 4 & 7 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 7 & 6 \end{pmatrix} \xrightarrow{R_1 \rightarrow -R_1} \begin{pmatrix} 0 & 1 & -4 & -7 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 7 & 6 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 + R_1} \\
 & \begin{pmatrix} 0 & 1 & -4 & -7 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 3 & -1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 0 & 1 & -4 & -7 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \xrightarrow{R_2 \rightarrow \frac{R_2}{3}} \begin{pmatrix} 0 & 1 & -4 & -7 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & -1 \end{pmatrix} \xrightarrow{R_1 \rightarrow R_1 + 4R_2} \\
 & \begin{pmatrix} 0 & 1 & 0 & -\frac{25}{3} \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & -1 \end{pmatrix} \xrightarrow{R_3 \rightarrow -R_3} \begin{pmatrix} 0 & 1 & 0 & -\frac{25}{3} \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \rightarrow R_1 + \frac{25}{3}R_3} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 + \frac{1}{3}R_3} \\
 & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
 \end{aligned}$$

13.2 Row Spaces of Matrices

Definition 8 (Row space of a matrix). Let A be a $m \times n$ matrix over \mathbb{F} . $R(A)$ is defined as

$$R(A) = \text{span } v_1, \dots, v_m$$

where v_1, \dots, v_m are rows of A .

$R(A)$ a subspace of the vector space of all rows of length n , is called the row space of A .

Definition 9 (Row rank of a matrix). $\dim R(A)$ is called the row-rank of A , and is denoted by $\text{rr}(A)$.

Theorem 3. Let P be a $l \times m$ matrix. Then

1. $R(PA) \subseteq R(A)$
2. If P is an invertible $m \times m$ matrix, then $R(PA) = R(A)$

Corollary 3.1.

$$A' \stackrel{R}{\sim} A \implies R(A') = R(A)$$

Theorem 4. If A is in REF, and if r is the number of non-zero rows in A , then

$$\text{rr}(A) = r$$

Corollary 4.1. The following are equivalent

1. $A \stackrel{R}{\sim} A'$
2. There is an invertible matrix P , s.t. $A' = PA$
3. $R(A) = R(A')$
4. A and A' have the same REF

13.3 Column Equivalence

Definition 10 (Elementary column operations, column equivalence, column echelon form, column space and column rank). If A is a $m \times n$ matrix, we can define elementary column operations, column equivalence ($A \stackrel{C}{\sim}$) and column echelon form (CEF), the column space of A ($C(A)$), and the column rank of A ($\text{cr}(A)$).

Theorem 5.

$$\text{cr}(A) = \text{rr}(A) = r$$

Proof. Let $r = \text{rr}(A) = \dim R(A)$.

Choose r rows of A which form a basis of $R(A)$, WLG, say v_1, \dots, v_r .

Let

$$X_{r \times n} = \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix}$$

$$\text{span}(X) = \text{R}(A)$$

Hence, any row of A can be expressed as a linear combination of v_1, \dots, v_r

$$v_i = \sum_{j=1}^r y_{ij} v_j$$

Let

$$Y_{m \times r} = (y_{ij})$$

Therefore,

$$A = YX$$

Considering each column of A as a linear combination of columns of Y ,

$$\begin{aligned} \text{C}(A) &\subseteq \text{C}(Y) \\ \therefore \text{cr}(A) &\leq \text{cr}(Y) \leq r = \text{rr}(A) \\ \therefore \text{cr}(A) &\leq \text{rr}(A) \end{aligned}$$

Similarly,

$$\text{rr}(A) \leq \text{cr}(A) \therefore \text{cr}(A) = \text{rr}(A)$$

□

Corollary 5.1. *The following are equivalent*

1. $A \stackrel{\mathcal{C}}{\sim} A'$
2. *There is an invertible matrix Q , s.t. $A' = QA$*
3. $\text{C}(A) = \text{C}(A')$
4. *A and A' have the same CEF*

Part IV

Linear Systems

1 Definition

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m\end{aligned}$$

Here, all x_i are taken to be unknowns, and all a_{ij}, b_i are given.

A solution to such a system is a collection d_1, \dots, d_n , s.t. after replacing x_i by d_i , we get equalities.

We assume that all a_{ij}, b_i belong to \mathbb{F} , and we are looking for solutions $d_i \in \mathbb{F}$.

Given such a system, we define $A_{m \times n} = (a_{ij}), b_{m \times 1} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, x_{n \times 1} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Then, we can write the system as

$$Ax = b$$

A solution to this system is $d_n = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$, s.t. $Ad = b$

Let D be the set of all $d = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$

D may be empty, infinite, or a singleton set.

2 Equivalent Systems

Two systems $Ax = b$ and $A'x = b'$ are called equivalent, if every solution of the first system is also a solution of the second system, and vice versa.

3 Solution of a System of Equations

We want to bring a given system

$$Ax = b$$

to the form

$$A_R x = b_R$$

using elementary row operations.

We denote the augmented or extended matrix of the system as follows.

$$\overline{A}_{m \times (n+1)} = (A_{m \times n} | b_{m \times 1})$$

Then apply Gaussian elimination method to \overline{A} , in order to get the matrix

$$(A_R | b_R)$$

As A_R is obtained from A using elementary row operations,

$$A_R = E_n \dots E_2 E_1 A$$

where every E_i is an elementary matrix.

Let $P = E_n \dots E_2 E_1$. P is invertible, as it is a product of elementary matrices.

$$\begin{aligned} A_R &= PA \\ \therefore A_R d &= PAd \\ &= Pb \\ &= b_R \end{aligned}$$

Conversely, let d be a solution to

$$\begin{aligned} A_R d &= b_R \\ \therefore PAd &= b_R \\ \therefore P^{-1}(PAd) &= P^{-1}b_R \\ \therefore Ad &= b \end{aligned}$$

If we have a system $Ax = b$, we may and will assume that A is in REF, i.e. $A = A_R, b = b_R$.

Let $l(1), \dots, l(r)$ denote the numbers of the columns containing leading entries.

$$\text{Let } b = \begin{pmatrix} b_1 \\ \vdots \\ b_r \\ b_{r+1} \\ \vdots \\ b_m \end{pmatrix}$$

Therefore,

$$\begin{aligned} 1 \cdot x_{l(1)} + \dots &= b_1 \\ 1 \cdot x_{l(2)} + \dots &= b_2 \\ &\vdots \\ 1 \cdot x_{l(r)} &= b_r \\ 0 &= b_{r+1} \\ &\vdots \\ 0 &= b_m \end{aligned}$$

4 Homogeneous Systems

4.1 Definition

A system of the form

$$Ax = \mathbb{O}$$

is called a homogeneous system.

Remark

Any homogeneous system is consistent and has a trivial solution $x = \mathbb{O}$

4.2 Solutions of Homogeneous Systems

If $r =$ number of non-zero rows, let $t = n - r =$ number of free variables. If $t > 0$, denote the numbers of the columns that do not contain leading entries by $z(1), \dots, z(t)$

4.2.1 Example

$$A = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore,

$$m = 4$$

$$n = 6$$

$$r = 3$$

$$t = 3$$

$$l(1) = 2$$

$$l(2) = 4$$

$$l(3) = 5$$

$$z(1) = 1$$

$$z(2) = 3$$

$$z(3) = 6$$

Therefore,

$$x_2 + 2x_3 - 3x_6 = 0$$

$$x_4 - x_6 = 0$$

$$x_5 + 7x_6 = 0$$

Therefore,

$$x_2 = -2x_3 + 3x_6$$

$$x_4 = x_6$$

$$x_5 = -7x_6$$

$$\begin{pmatrix} x_2 \\ x_4 \\ x_5 \end{pmatrix} = C_{3 \times 3} \begin{pmatrix} x_1 \\ x_3 \\ x_6 \end{pmatrix}$$

where $C_{3 \times 3} = \begin{pmatrix} 0 & -2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -7 \end{pmatrix}$

The free variables x_1, x_3, x_6 can be considered as parameters, $x_1 = \gamma_1, x_2 =$

$$\gamma_2, x_3 = \gamma_3.$$

Therefore,

$$x_2 = -2\gamma_3 + 3\gamma_6$$

$$x_4 = \gamma_6$$

$$x_5 = -7\gamma_6$$

4.2.2 General Solution

4.2.2.1 Case I: $t = 0$

If $t = 0$, there are no free variables, and the system has a unique trivial solution.

4.2.2.2 Case II: $t > 0$

$$\begin{pmatrix} x_{l(1)} \\ x_{l(2)} \\ \vdots \\ x_{l(r)} \end{pmatrix} = C_{r \times t} \begin{pmatrix} x_{z(1)} \\ x_{z(2)} \\ \vdots \\ x_{z(t)} \end{pmatrix}$$

C is filled by coefficients of the equations obtained after shifting the terms containing all z_i to the RHS.

4.3 Properties

4.3.1 For a homogeneous system $Ax = 0$, if c and d are solutions, then $c + d$ is also a solution.

$$\begin{aligned} Ac &= \mathbb{0} \\ Ad &= \mathbb{0} \\ \therefore A(c + d) &= Ac + Ad \\ &= \mathbb{0} + \mathbb{0} \\ &= \mathbb{0} \end{aligned}$$

4.3.2 For a homogeneous system $Ax = 0$, if c is a solution and $\alpha \in \mathbb{F}$, then, αc is a solution too.

$$\begin{aligned} Ac &= \mathbb{0} \\ \therefore A(\alpha c) &= \alpha(Ac) \\ &= \alpha\mathbb{0} \\ &= \mathbb{0} \end{aligned}$$

4.4 Fundamental Solutions

We define t fundamental solutions or basic solutions, v_1, \dots, v_t .

We define t columns, each of length n as follows.

For the i^{th} column v_i , we set

$$\begin{aligned}x_{z(1)} &= 0 \\x_{z(i)} &= 1 \\&\vdots \\x_{z(t)} &= 0\end{aligned}$$

and for $x_{l(1)}, \dots, x_{l(r)}$,

$$\begin{pmatrix} x_{l(1)} \\ \vdots \\ x_{l(r)} \end{pmatrix} = C \begin{pmatrix} x_{z(1)} \\ \vdots \\ x_{z(t)} \end{pmatrix} = i^{\text{th}} \text{column of } C$$

4.4.1 Theorem: Any solution d of the system $Ax = \mathbb{O}$ can be obtained from the basic solutions v_1, \dots, v_t as a linear combination of the basic solutions, $d = \alpha_1 v_1 + \dots + \alpha_t v_t$

One can choose another collection v'_1, \dots, v'_t s.t. any solution of $Ax = \mathbb{O}$ can be obtained as a linear combination of v'_1, \dots, v'_t . In such a case, we get another form of the general solution.

4.5

$$r \leq \min m, n$$

If $r = n$, i.e. $t = 0$, the system has a unique solution.

If $r < n$, i.e. $t > 0$, the system has more than one solutions. Its general solution can be expressed as in terms of t parameters, where each free variable serves as a parameter, whose value can be any element of \mathbb{F} .

If $m < n$, then $r < n$. Therefore, the system has more than one solution.

5 Non-Homogeneous Systems

5.1 Definition

Consider a system $Ax = b; b \neq \mathbb{O}$. The extended matrix is defined as

$$\tilde{A} = (A|b) = \begin{pmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{pmatrix}$$

5.2 Solutions of Non-Homogeneous Systems

Let \tilde{r} be the number of non-zero rows in the REF of \tilde{A} , i.e. \tilde{A}_R .

5.2.1 Case I: $\tilde{r} = r$

$$b'_{r+1} = \cdots = b'_m = 0$$

5.2.1.1 Case a: $r = n$, i.e. $t = 0$

Therefore,

$$\begin{aligned} x_1 &= b'_1 \\ \cdots \\ x_r &= b'_r \end{aligned}$$

Hence, the system has a unique solution.

5.2.1.2 Case b: $r < n$, i.e. $t > 0$

Therefore,

$$\begin{aligned} x_l(1) &= b'_1 + c_{11}x_{z(1)} + \cdots + c_{1t}x_{z(t)} \\ &\vdots \\ x_l(r) &= b'_1 + c_{r1}x_{z(1)} + \cdots + c_{rt}x_{z(t)} \end{aligned}$$

5.2.2 Case II: $\tilde{r} > r$

In this case, the $(r + 1)^{\text{th}}$ row represents an equation of the form $0 = 1$. Therefore, the system is inconsistent.

5.3 General Solution

The general solution of $Ax = b$ can be expressed by adding the general solution of $Ax = b$ and any particular solution of $Ax = b$.

If c is a solution of $Ax = \mathbb{O}$, and d is a solution of $Ax = b$, then $c + d$ is a solution of $Ax = b$.

Conversely, if d and d' are solutions of $Ax = b$, then, $c = d' - d$ is a solution of $Ax = \mathbb{O}$.

Part V

Vector Spaces

1 Definition

Let \mathbb{F} be a field. A vector space V , over \mathbb{F} , is a set on which there are two operations, denoted by $+$ and \cdot , where

$+$ is the addition of elements of V

\cdot is the multiplication of an element of V by an element of \mathbb{F}

s.t. the sum of elements of V lies in V , and the product of an element of V by an element of \mathbb{F} lies in V , and the following properties hold.

$$(A1) \quad x + y = y + x; \forall x, y \in V$$

$$(A2) \quad (x + y) + z = x + (y + z); \forall x, y, z \in V$$

$$(A3) \quad \exists \mathbb{O} \in V, \text{ s.t. } \mathbb{O} + x = x + \mathbb{O} = x; \forall x \in V$$

$$(A4) \quad \forall x \in V, \exists y \in V, \text{ s.t. } x + y = \mathbb{O}. \text{ (} y \text{ is denoted as } -x \text{.)}$$

$$(M1) \quad \alpha(x + y) = \alpha x + \alpha y; \forall \alpha \in \mathbb{F}, \forall x, y \in V$$

$$(M2) \quad (\alpha + \beta)x = \alpha x + \beta x; \forall \alpha, \beta \in \mathbb{F}, \forall x \in V$$

$$(M3) \quad (\alpha\beta)x = \alpha(\beta x) = \beta(\alpha x); \forall \alpha, \beta \in \mathbb{F}, \forall x \in V$$

$$(M4) \quad 1 \cdot x = x; \forall x \in V$$

Elements of V are called vectors, and elements of \mathbb{F} are called scalars.

1.1 Examples

1.1.1 Geometric Vectors in Plane

1.1.2 Arithmetic Vector Space

Let \mathbb{F} be a field, and $n \geq 1 \in \mathbb{Z}$.

Let $V = \mathbb{F}^n$ be a set of ordered n -tuples.

We define

$$\begin{aligned} (\alpha_1, \dots, \alpha_n) + (\beta_1, \dots, \beta_n) &= (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n) \\ \alpha(\alpha_1, \dots, \alpha_n) &= (\alpha\alpha_1, \dots, \alpha\alpha_n) \end{aligned}$$

1.1.3

Let \mathbb{F} be a field, and $m, n \geq 1 \in \mathbb{Z}$.

Let $V = \mathbb{F}^{mn}$ be the set of all $(m \times n)$ matrices over \mathbb{F} , i.e. a set of ordered mn -tuples. For $X, Y \in V$, we use the usual definitions of $X + Y$ and αX from algebra of matrices.

2 Properties

1. $\alpha\mathbb{O} = \mathbb{O}; \forall \alpha \in F$
2. $\alpha(-x) = -(\alpha x)$
3. $x - y \doteq x + (-y)$
4. $0x = \mathbb{O}; \forall x \in V$
5. $(-1)x = -x; \forall x \in V$
6. $(\alpha - \beta)x = \alpha x - \beta x; \forall \alpha, \beta \in F, \forall x \in V$

2.0.1 Proof of 1

$$\begin{aligned}\alpha\mathbb{O} &= \alpha(\mathbb{O} + \mathbb{O}) \\ &= \alpha\mathbb{O} + \alpha\mathbb{O}\end{aligned}$$

For $\alpha\mathbb{O} \exists y$ s.t. $\alpha\mathbb{O} + y = \mathbb{O}$.

Therefore,

$$\begin{aligned}\alpha\mathbb{O} + y &= (\alpha\mathbb{O} + \alpha\mathbb{O}) + y \\ \therefore \mathbb{O} &= \alpha\mathbb{O} + (\mathbb{O} + y) \\ &= \alpha\mathbb{O} + \mathbb{O} \\ &= \alpha\mathbb{O}\end{aligned}$$

3 Subspaces

Let V be a vector space over \mathbb{F} . Let $U \subseteq V$. U is called a subspace of V if the following properties hold.

Axiom 1 $\mathbb{O} \in U$

Axiom 2 If $x, y \in U$, then, $(x + y) \in U$

Axiom 3 If $x \in U, \alpha \in \mathbb{F}$, then, $\alpha x \in U$

3.1 Examples

Example 4. Let V be the set of all geometric vectors in plane.

If U_1 is the set of all vectors along the x -axis, U_2 is the singleton set of a specific vector along the x -axis, and U_3 is the set of all vectors along the x -axis and a specific vector not along the x -axis. Which of U_1, U_2, U_3 are subspaces of V ?

Solution. U_1 is a subspace of V as it satisfies all three axioms.

U_2 is not a subspace of V as it does not satisfy any of the three axioms.

U_3 is not a subspace of V as it does not satisfy Axiom 3

Example 5.

$$\mathbb{F} = \mathbb{R}$$

$$V = \mathbb{C} = \{\alpha + \beta i; \alpha, \beta \in \mathbb{R}\}$$

where $+$ is addition in \mathbb{C} and \cdot is multiplication by real scalars.

$$U_1 = \{\alpha + 0i\}$$

$$U_2 = \{0 + \beta i\}$$

Which of U_1, U_2, U_3 are subspaces of V ?

Solution. Both U_1 and U_2 are subspaces of V , as they satisfy all three axioms.

Example 6. Let $V = \mathbb{F}$, where $+$ is addition in \mathbb{F} , and \cdot is multiplication in \mathbb{F} .

$$U_1 = \{\alpha + 0i\}$$

$$U_2 = \{0 + \beta i\}$$

Which of U_1, U_2 are subspaces of V ?

Solution. Neither U_1 nor U_2 are subspaces of V .

Example 7. Let $V = \{f : [0, 1] \rightarrow \mathbb{R}\}$, where $+$ and \cdot is defined as follows.

$$(f + g)(x) = f(x) + g(x)$$

$$(\alpha f)(x) = \alpha f(x)$$

\mathbb{O} is the function with graph $x = 0$.

$$U = \{\text{all continuous functions } [0, 1] \rightarrow \mathbb{R}\}$$

Is U is subspace?

Solution. $\mathbb{O} \in \mathbb{R}$. Therefore, Axiom 1 is satisfied. Similarly, Axiom 2 and Axiom 3 are also satisfied.

3.2 Operations on Subspaces

Let V/F be a vector space, and U_1, U_2 be subspaces of V .

$$U_1 \cap U_2 = \{x \in V : x \in U_1 \text{ and } x \in U_2\}$$

$$U_1 \cup U_2 = \{x \in V : x \in U_1 \text{ or } x \in U_2\}$$

$$U_1 + U_2 = \{x \in V : x = x_1 + x_2, x_1 \in U_1, x_2 \in U_2\}$$

Example 8. Let V be a set of geometric vectors in 3D space. Let U_1 be the xy -plane, and U_2 be the yz -plane. Is $U_1 \cap U_2$ a subspace of V ?

Solution.

$$\mathbb{O} \in U_1, \mathbb{O} \in U_2 \Rightarrow \mathbb{O} \in U_1 \cap U_2$$

$$x, y \in U_1 \cap U_2 \Rightarrow x, y \in U_1, x, y \in U_2$$

$$\Rightarrow x + y \in U_1, x + y \in U_2$$

$$= x + y \in U_1 \cap U_2$$

Similarly, if $x \in U_1 \cap U_2, \alpha \in \mathbb{F}$, then, $\alpha x \in U_1 \cap U_2$. Therefore, $U_1 \cap U_2$ is a subspace of V .

4 Spans

Definition 11 (Span). Let V/\mathbb{F} be a vector space. Let $S \subset V$ be non-empty.

$$\text{span}(S) = \{x \in V : x = \alpha_1 v_1 + \dots + \alpha_m v_m, \alpha_1, \dots, \alpha_m \in \mathbb{F}, v_1, \dots, v_m \in S\}$$

$\text{span}(S)$ is the collection of all linear combinations of finite number of vectors of S with coefficients from \mathbb{F}

Theorem 1. $\text{span}(S)$ is a subspace of V

Proof.

$$\mathbb{0} = 0v \Rightarrow \mathbb{0} \in \text{span}(S)$$

$$\begin{aligned} x, y \in \text{span}(S) &\Rightarrow x = \alpha_1 v_1 + \cdots + \alpha_m v_m, \beta_1 w_1 + \cdots + \beta_m w_m \\ &\Rightarrow x + y = \alpha_1 v_1 + \cdots + \alpha_m v_m + \beta_1 w_1 + \cdots + \beta_m w_m \in \text{span}(S) \end{aligned}$$

$$\begin{aligned} x \in \text{span}(S), \alpha \in \mathbb{F} &\Rightarrow \alpha_1 v_1 + \cdots + \alpha_m v_m \\ &\Rightarrow \alpha x = \alpha(\alpha_1 v_1 + \cdots + \alpha_m v_m) \\ &\Rightarrow \alpha x = \alpha \alpha_1 v_1 + \cdots + \alpha \alpha_m v_m \in \text{span}(S) \end{aligned}$$

□

Definition 12 (Spanning sets and dimensionality). Let V/\mathbb{F} be a vector space. A set $S \subseteq V$ is said to be a spanning set, if $\text{span}(S) = V$. If V has at least one finite spanning set, V is said to be finite-dimensional. Otherwise, V is said to be infinite-dimensional.

Remark 4. V may have many finite spanning sets, of different sizes

Definition 13 (Basis of a vector space). Let V/\mathbb{F} be a vector space. We say that $B = \{v_1, \dots, v_n\} \subset V$ is a basis of V if every vector $v \in V$ can be expressed in a unique way

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n \quad ; \alpha_1, \dots, \alpha_n \in \mathbb{F}$$

that is, as a linear combination of elements of B .

Definition 14 (Isomorphic spaces). Let V/\mathbb{F} and W/\mathbb{F} be vector spaces. We say that V is isomorphic to W if there is a map $\varphi : V \rightarrow W$, s.t.

1. φ is one-to-one and onto
2. $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2); \forall v_1, v_2 \in V$
3. $\varphi(\alpha v) = \alpha \varphi(v); \forall v \in V, \forall \alpha \in \mathbb{F}$

Theorem 2. *If a vector space V/\mathbb{F} has a basis $B = \{v_1, \dots, v_n\}$ consisting of n elements, then it is isomorphic to the space*

$$W = \mathbb{F}^n = \left\{ \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \right\}$$

Proof. Let $B' = \{e_1, \dots, e_n\}$, where

$$e_1 = \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$$

B' is a basis of Q , as any $w = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in W$ can be expressed in a unique way

$$w = \alpha_1 e_1 + \dots + \alpha_n e_n$$

Let $\varphi : V \rightarrow W$,

$$\begin{aligned} \varphi(v_1) &= e_1 \\ &\vdots \\ \varphi(v_n) &= e_n \end{aligned}$$

For any $v = \alpha_1 v_1 + \dots + \alpha_n v_n \in V$,

$$\varphi(v) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Therefore,

$$\begin{aligned} \varphi(\alpha_1 v_1 + \dots + \alpha_n v_n) &= \alpha_1 e_1 + \dots + \alpha_n e_n \\ &= \alpha_1 \varphi(v_1) + \dots + \alpha_n \varphi(v_n) \end{aligned}$$

If $v \neq v'$,

$$\begin{aligned} v &= \alpha_1 v_1 + \dots + \alpha_n v_n \\ v' &= \alpha'_1 v_1 + \dots + \alpha'_n v_n \end{aligned}$$

Hence φ is one-to-one.

For any $w = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in W$.

Let $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$.

Therefore,

$$\varphi(v) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = w$$

Therefore, φ is onto. □

5 Linear Dependence

Definition 15 (Linearly dependent subsets). Let V/\mathbb{F} be a vector space. Let $S \subseteq V$ be a finite subset. S is said to be linearly dependent if there exist scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$, not all equal to zero, s.t.

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = \mathbb{O}$$

Otherwise, S is said to be linearly independent if all $\alpha_1 = \cdots = \alpha_n = 0$.

Example 9. Is $S = \{v_1, \dots, v_l, v, \alpha v\}$ linearly dependent?

Solution.

$$(0)v_1 + \cdots + (0)v_l + (-\alpha)v + (1)\alpha v = \mathbb{O}$$

Therefore, as not all coefficients are zero, S is linearly dependent.

Example 10. Is $S = \{v_1, \dots, v_l, \mathbb{O}\}$ linearly dependent?

Solution.

$$(0)v_1 + \cdots + (0)v_l + (1)\mathbb{O} = \mathbb{O}$$

Therefore, as not all coefficients are zero, S is linearly dependent.

Theorem 3. Any basis $B = \{v_1, \dots, v_n\}$ of a vector space V is linearly independent.

Proof. Let

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = \mathbb{O}$$

Also,

$$(0)v_1 + \cdots + (0)v_n = \mathbb{O} \tag{3.1}$$

Therefore, there are two representations of $v = \mathbb{O}$ as linear combinations of elements of B . By the definition of basis, they must coincide.

Therefore,

$$\begin{aligned}\alpha_1 &= 0 \\ &\vdots \\ \alpha_n &= 0\end{aligned}$$

Hence, B is linearly independent. □

5.1 Properties of Linearly Dependent and Independent Sets

Theorem 4. *If $S \subseteq S'$ and S is linearly dependent, then S' is also linearly dependent.*

Theorem 5. *If $S \subseteq S'$ and S' is linearly independent, then S is also linearly independent.*

Theorem 6. *Let $S = \{v_1, \dots, v_n\}$. S is linearly dependent iff one of the v_i s is a linear combination of the others.*

Proof of statement. Suppose

$$\begin{aligned}v_n &= \alpha_1 v_1 + \dots + \alpha_{n-1} v_{n-1} \\ \therefore \alpha_1 v_1 + \dots + \alpha_{n-1} v_{n-1} + (-1)v_n &= \mathbb{O}\end{aligned}$$

Therefore, S is linearly dependent. □

Proof of converse. Suppose

$$\alpha_1 v_1 + \dots + \alpha_{n-1} v_{n-1} + \alpha_n v_n = \mathbb{O}$$

not all of α_i s are 0. WLG, let $\alpha_n \neq 0$

$$\therefore v_n = -\frac{\alpha_1}{\alpha_n} v_1 - \dots - \frac{\alpha_{n-1}}{\alpha_n} v_{n-1}$$

□

Theorem 7. *Let $S = \{v_1, \dots, v_m\}$. Let $w \in V$. Suppose w is a linear combination of v_i s*

$$w = \alpha_1 v_1 + \dots + \alpha_n v_n$$

Then, such an expression is unique iff S is linearly dependent.

Proof of statement. Let

$$w = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

be unique.

If possible, let

$$\beta_1 v_1 + \cdots + \beta_n v_n = \mathbb{O}$$

not all β_i s are zero.

Then,

$$(\alpha_1 + \beta_1)v_1 + \cdots + (\alpha_n + \beta_n)v_n = w$$

This is another expression for w , and contradicts the assumption. \square

Proof of converse. If possible, let S be linearly independent. Assume

$$w = \alpha'_1 v_1 + \cdots + \alpha'_n v_n$$

Therefore,

$$(\alpha_1 - \alpha'_1)v_1 + \cdots + (\alpha_n - \alpha'_n)v_n = \mathbb{O}$$

Therefore, S is linearly dependent, which contradicts the assumption. \square

Theorem 8 (Main Lemma on Linear Independence). *Suppose V is spanned by n vectors.*

Let $S = \{v_1, \dots, v_m\} \subset V$. Suppose $m > n$.

Then, S is linearly dependent.

Proof. Let $E = \{w_1, \dots, w_n\}$ be a spanning set for V , $V = \text{span}(E)$.

Therefore, all elements of S can be represented as linear combinations of elements of E .

$$\begin{aligned} v_1 &= \beta_{11}w_1 + \cdots + \beta_{1n}w_n \\ &\vdots \\ v_m &= \beta_{m1}w_1 + \cdots + \beta_{mn}w_n \end{aligned}$$

Let

$$\begin{aligned} &\alpha_1 v_1 + \cdots + \alpha_m v_m = \mathbb{O} \\ \therefore \alpha_1(\beta_{11}w_1 + \cdots + \beta_{1n}w_n) + \cdots + \alpha_m(\beta_{m1}w_1 + \cdots + \beta_{mn}w_n) &= \mathbb{O} \\ \therefore (\alpha_1\beta_{11} + \cdots + \alpha_m\beta_{m1})w_1 + \cdots + (\alpha_1\beta_{1n} + \cdots + \alpha_m\beta_{mn}) &= \mathbb{O} \end{aligned}$$

Therefore

$$\begin{aligned}\alpha_1\beta_{11} + \cdots + \alpha_m\beta_{m1} &= 0 \\ &\vdots \\ \alpha_1\beta_{1n} + \cdots + \alpha_m\beta_{mn} &= 0\end{aligned}$$

These equations form a homogeneous linear system with respect to $\alpha_1, \dots, \alpha_m$. As $m > n$, the system has a non-zero solution. Therefore not all α_i s are zero. Hence S is linearly dependent. \square

Definition 16 (Alternative definition of a basis). $B = \{v_1, \dots, v_n\}$ is said to be a basis of V if B is a spanning set and B is linearly independent.

Theorem 9. *If B and B' are bases of V , then they contain the same number of elements.*

Proof. If possible, let B contain n elements $\{v_1, \dots, v_n\}$, and B' contain m elements $\{w_1, \dots, w_m\}$, $m > n$.

Therefore, B is a spanning set and B' contains more elements than n , hence by Main Lemma on Linear Independence, B' is linearly dependent. Also, B' is a basis, so it is linearly independent.

This is a contradiction. \square

Definition 17 (Dimension of a vector space). Let V/\mathbb{F} be a finite-dimensional vector space. The number of elements in any basis B of V is called the dimension of V .

$$n = \dim V$$

Remark 5. If V and W are vector spaces over \mathbb{F} , s.t.

$$\dim V = \dim W$$

then, V is isomorphic to W

Theorem 10. *If $S = \{v_1, \dots, v_m\}$ is a spanning set of V , and if S is not a basis of V , a basis B of V can be obtained by removing some elements from S .*

Proof. If S is linearly independent, then it is a basis.

Otherwise, if S is linearly dependent, it has an element, WLG, say v_m , which is a linear combination of the others.

$$v_m = \alpha_1v_1 + \cdots + \alpha_{m-1}v_{m-1}$$

Let

$$S' = S - \{v_m\}$$

S' is a spanning set.

Therefore, $\forall v \in V$

$$\begin{aligned} v &= \beta_1 v_1 + \cdots + \beta_{m-1} v_{m-1} + \beta_m v_m \\ &= \beta_1 v_1 + \cdots + \beta_{m-1} v_{m-1} + \beta_m (\alpha_1 v_1 + \cdots + \alpha_{m-1} v_{m-1}) \\ &= \gamma_1 v_1 + \cdots + \gamma_{m-1} v_{m-1} \end{aligned}$$

If S' is linearly independent, then it is a basis, else the same process above can be repeated till we get a basis.

Therefore, a basis is a smallest spanning set. □

Theorem 11. *If $B_0 = \{v_1, \dots, v_n\}$ is a linearly independent set, and if B_0 is a basis of V , a basis of V can be obtained by adding elements to B_0 .*

Theorem 12. *Let V be a vector space, s.t. $\dim V = n$.*

If B satisfies 2 out of the 3 following conditions, then it is a basis.

1. B has n elements.
2. B is a spanning set.
3. B is linearly dependent.

Theorem 13 (Dimension Theorem).

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$$

Theorem 14.

$$U + W = \text{span}(U \cup W)$$

If

$$U = \text{span}(B)$$

$$W = \text{span}(B')$$

then,

$$U + W = \text{span}(B \cup B')$$

Proof. Let $v \in U + W$.

Then,

$$\begin{aligned}v &= u + w \quad ; u \in U, w \in W \\u &\in U \cup W \\w &\in U \cup W \\\therefore v &\in \text{span}(U \cup W)\end{aligned}$$

Let

$$v \in \text{span}(U \cup W) \therefore v = \alpha_1 v_1 + \cdots + \alpha_k v_k \quad ; v_i \in U \cup W$$

Let

$$\begin{aligned}v_1, \dots, v_l &\in U \\v_{l+1}, \dots, v_k &\in W\end{aligned}$$

Therefore,

$$\begin{aligned}v &= (\alpha_1 v_1 + \cdots + \alpha_l v_l) + (\alpha_{l+1} v_{l+1} + \cdots + \alpha_k v_k) \\\therefore v &\in U + W\end{aligned}$$

□

5.2 Changing a Basis

Let $B = \{v_1, \dots, v_n\}$ be a basis of V , s.t. $\dim V = n$. Let $B' = \{v'_1, \dots, v'_n\}$. As B is a spanning set, all of v'_1, \dots, v'_n can be expressed as a linear combination of v_1, \dots, v_n .

$$\begin{aligned}v'_1 &= \gamma_{11} v_1 + \cdots + \gamma_{n1} v_n \\&\vdots \\v'_n &= \gamma_{1n} v_1 + \cdots + \gamma_{nn} v_n\end{aligned}$$

Definition 18 (Transition matrix). The matrix

$$C = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{pmatrix}$$

is called the transition matrix from B to B' .

If B and B' are considered as row vectors of length n filled by vectors,

$$\begin{aligned}v'_1 &= \gamma_{11}v_1 + \cdots + \gamma_{n1}v_n \\ &\vdots \\ v'_n &= \gamma_{1n}v_1 + \cdots + \gamma_{nn}v_n\end{aligned}$$

can be written as

$$B'_{1 \times n} = B_{1 \times n}C_{n \times n}$$

Theorem 15. B' is a basis of V iff C is invertible.

Proof of statement. Let $B' = BC$ be a basis.

B' is a basis, and hence is a spanning set. Therefore, any vector from B can be expressed as a linear combination of elements of B' .

Therefore,

$$\begin{aligned}B &= B'Q \\ &= BCQ\end{aligned}$$

Also,

$$B = BI$$

Therefore,

$$I = CQ$$

Similarly,

$$\begin{aligned}B' &= BC \\ &= B'QC\end{aligned}$$

Also,

$$B' = B'I$$

Therefore,

$$I = QC$$

Therefore,

$$CQ = QC = I$$

Hence C is invertible. □

Proof of converse. Let $B' = BC$ and C be invertible. Therefore, B' is a basis iff B is a spanning set.

Let $z \in V$. As B is a spanning set,

$$z = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

Therefore,

$$z = Bg$$

where

$$g = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$$\begin{aligned} \therefore z &= Bg \\ &= B(Ig) \\ &= B(CC^{-1})g \\ &= (BC)(C^{-1}g) \end{aligned}$$

Let $C^{-1}g = f$

$$\therefore z = B'f$$

Therefore, z can be expressed as a linear combination of vectors from B' . \square

Remark 6. Let B be a basis of V . If

$$BP = BQ$$

where P and Q are $n \times n$ matrices, then

$$P = Q$$

Example 11. Let $B = \{e_1, e_2\}$ and $B' = \{e'_1, e'_2\}$, where

$$\begin{aligned} e'_1 &= e_1 + e_2 \\ e'_2 &= -e_1 + e_2 \end{aligned}$$

Solution.

$$\begin{aligned} e'_1 &= e_1 + e_2 \\ e'_2 &= -e_1 + e_2 \\ \therefore C &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
e_1 &= \frac{1}{2}e'_1 - \frac{1}{2}e'_2 \\
e_2 &= \frac{1}{2}e'_1 + \frac{1}{2}e'_2 \\
\therefore C^{-1} &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}
\end{aligned}$$

5.3 Representation of Vectors in a Basis

Let V be a vector space of dimension n . Let $B = \{v_1, \dots, v_n\}$ be a basis of V .

Let $z \in V$.

z can be written as a unique linear combination of elements of B .

$$z = \alpha_1 v_1 + \dots + \alpha_n v_n$$

The representation of z w.r.t B can be represented as

$$[z]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

5.3.1 Properties of Representations

1. $[z_1 + z_2]_B = [z_1]_B + [z_2]_B$
2. $[\alpha z]_B = \alpha [z]_B$
3. $[z_1]_B = [z_2]_B \iff z_1 = z_2$
4. $\forall \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{F}^n, \exists z \in V, \text{ s.t. } [z]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$

6 Determinants

6.1 Definition

Definition 19 (Determinants). Given an $n \times n$ matrix A , $n \geq 1$, $\det(A)$ is defined as follows.

$$\begin{aligned}
 n = 1 & \qquad \det(a) = a \\
 n = 2 & \qquad \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \\
 & \qquad \vdots \\
 n = n &
 \end{aligned}$$

The determinant of a $n \times n$ matrix is the summation of $n!$ summands. Each summand is the product of n elements, each from a different row and column.

Summand	Permutation	Number of Elementary Permutations ¹	Parity
$a_{11}a_{22}a_{33}$	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	0	even
$a_{12}a_{23}a_{31}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	2 ((1, 2, 3) \rightarrow (2, 1, 3) \rightarrow (2, 3, 1))	even
$a_{13}a_{21}a_{32}$	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	2 ((1, 2, 3) \rightarrow (1, 3, 2) \rightarrow (3, 1, 2))	even
$a_{13}a_{22}a_{31}$	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$	1 ((1, 2, 3) \rightarrow (3, 2, 1))	odd
$a_{12}a_{21}a_{33}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$	1 ((1, 2, 3) \rightarrow (2, 1, 3))	odd
$a_{11}a_{23}a_{32}$	$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$	1 ((1, 2, 3) \rightarrow (1, 3, 2))	odd

6.2 Properties

Theorem 16. *If A , A' are matrices s.t. all rows except the i^{th} row are identical, and A'' is obtained by addition of i^{th} row of A and i^{th} row of A' , then*

$$\det(A'') = \det(A) + \det(A')$$

¹Any permutation can be represented as a result of a series of elementary permutations, i.e. permutations of 2 elements only. The parity of a particular permutation depends of the parity of the number of elementary functions required for it.

Theorem 17. *If A' is obtained from A by switching two rows, then*

$$\det(A') = -\det(A)$$

Theorem 18. *If A' is obtained from A by multiplication of a row by a scalar α , then*

$$\det(A') = \alpha \det(A)$$

Theorem 19. *If A' is obtained from A by adding to the i^{th} row the j^{th} row multiplied by a scalar α , then*

$$\det(A') = \det(A)$$

Corollary 19.1 (Corollary of Property 2). *If A has two identical rows, then $\det(A) = 0$.*

Theorem 20. *The determinant of upper triangular and lower triangular matrices is the product of the elements on the principal diagonal.*

Theorem 21.

$$\det(A^t) = \det(A)$$

Corollary 21.1. *In all above theorems, the properties which are applicable to rows, are also applicable to columns.*

Theorem 22. *If A , B , C are some matrices, and \mathbb{O} is the zero matrix,*

$$\begin{pmatrix} A_{m \times m} & B \\ \mathbb{O} & C_{n \times n} \end{pmatrix} = \det(A) \cdot \det(C)$$

Theorem 23.

$$\det(AB) = \det(A) \det(B)$$

Corollary 23.1. *If A is invertible, then*

$$\det(A) \neq 0$$

Proof. A is invertible.

Therefore, $\exists P$, s.t.

$$\begin{aligned} PA &= I \\ \therefore \det(PA) &= \det(I) \\ \therefore \det(P) \det(A) &= 1 \\ \therefore \det(A) &\neq 0 \end{aligned}$$

□

Theorem 24. *If*

$$\det(A) \neq 0$$

then A is invertible.

Proof. If possible let A be non invertible.

Let the REF of A be A_R .

As A is non invertible, A_R has a zero row. Therefore,

$$\det(A_R) = 0$$

But

$$\det(A) = 0$$

This is not possible as elementary row operations cannot change a non-zero determinant to zero.

Therefore, A is invertible. □

Theorem 25.

$$\det(A) \neq 0$$

iff the rows of A are linearly independent iff the columns of A are linearly independent.

Proof. If possible, let the rows of A be linearly dependent.

Therefore, either all of them are zeros, or one row is the linear combination of the others.

Case 1 (All rows are zeros).

$$\therefore \det(A) = 0$$

Case 2 (One row is a linear combination of the others). Let

$$v_n = \alpha_1 v_1 + \cdots + \alpha_{n-1} v_{n-1}$$
$$\therefore A = \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix}$$

$$v_n \rightarrow v_n - \alpha_1 v_1 + \cdots + \alpha_{n-1} v_{n-1}$$

$$\therefore A' = \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ \mathbb{O} \end{pmatrix}$$

$$\therefore \det(A') = 0$$

$$\therefore \det(A) = 0$$

This contradicts $\det(A) \neq 0$. Therefore, the rows of A must be linearly independent.

If v_1, \dots, v_n are linearly independent,

$$\dim R(A) = n$$

$$\therefore r = n$$

Therefore, there are no zero rows in REF of A . Hence A is invertible.

$$\therefore \det(A) \neq 0$$

□

6.3 Practical Methods for Computing Determinants

6.4 Expansion along a row/ column

Let A be a $m \times n$ matrix, and let A_{ij} be the matrix obtained by removing the i^{th} row and j^{th} column from A .

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij})$$

6.5 Determinant Rank

Definition 20 (Determinant rank). Let A be any $m \times n$ matrix. Consider all square sub-matrices of A and compute their determinants. If there is an $r \times r$

sub-matrix of A s.t. its determinant is non-zero, but the determinants of all $(r + 1) \times (r + 1)$ sub-matrices of A are zero, then, r is called the determinant rank of A .

Theorem 26. *The determinant rank of A is equal to the rank of A .*

7 Linear Maps

7.1 Definition

Definition 21 (Linear map). Let V and W be vector spaces over the same field \mathbb{F} .

$$\varphi : V \rightarrow W$$

is said to be a linear map if

1. $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2); \forall v_1, v_2 \in V$
2. $\varphi(\alpha v) = \alpha\varphi(v); \forall v \in V, \forall \alpha \in \mathbb{F}$

7.2 Properties

1. $\varphi(\mathbb{0}) = \mathbb{0}$
2. $\varphi(-v) = -\varphi(v)$

7.3 Matrix of a Linear Map

Definition 22 (Matrix of a linear map). Let $\varphi : V \rightarrow W$ be a linear map. Let

$$\begin{aligned} n &= \dim V \\ m &= \dim W \end{aligned}$$

Let

$$\begin{aligned} B &= \{v_1, \dots, v_n\} \\ B' &= \{w_1, \dots, w_m\} \end{aligned}$$

be bases of V and W respectively.

Let

$$\begin{aligned}\varphi(v_1) &= \alpha_{11}w_1 + \cdots + \alpha_{m1}w_m \\ &\vdots \\ \varphi(v_n) &= \alpha_{1n}w_1 + \cdots + \alpha_{mn}w_m\end{aligned}$$

The matrix

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix}$$

is called the matrix of φ with respect to the bases B and B' . It is denoted as

$$A = [\varphi]_{B,B'}$$

Theorem 27. *Let*

$$\varphi : V \rightarrow W$$

be a linear map.

Let B and B' be bases of V and W respectively, and let

$$A = [\varphi]_{B,B'}$$

be the matrix of φ with respect to B and B' . Then, $\forall x \in V$,

$$[\varphi(z)]_{B'} = A[z]_B$$

Proof. Let

$$\begin{aligned}B &= \{v_1, \dots, v_n\} \\ B' &= \{w_1, \dots, w_m\}\end{aligned}$$

Case 3 ($z \in B$). WLG, let $z = v_i$. Then,

$$[z]_B = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

i.e. all rows except the i^{th} row are 0.

Let this vector be e_i .

Therefore,

$$A[z]_B = Ae_i$$

is the i^{th} column of A .

$$[\varphi(z)]_{B'} = [\varphi(v_i)]_{B'}$$

is the i^{th} row in the formulae of $\varphi(v_1), \dots, \varphi(v_n)$.

Therefore, it is the i^{th} column of A .

Case 4 ($z \in V$ is an arbitrary vector). Let

$$z = \alpha_1 v_1 + \dots + \alpha_n v_n$$

Therefore,

$$\begin{aligned} [\varphi(z)]_{B'} &= [\varphi(\alpha_1 v_1 + \dots + \alpha_n v_n)]_{B'} \\ &= [\alpha_1 \varphi(v_1) + \dots + \alpha_n \varphi(v_n)]_{B'} \\ &= \alpha_1 [\varphi(v_1)]_{B'} + \dots + \alpha_n [\varphi(v_n)]_{B'} \\ &= \alpha_1 \cdot (1^{\text{st}} \text{column of } A) + \dots + \alpha_n c_n \cdot (n^{\text{th}} \text{column of } A) \\ &= A[z]_B \end{aligned}$$

□

7.4 Change of Bases

Theorem 28. *Let V, W be vector spaces over \mathbb{F} , $\dim(V) = n$, $\dim(W) = m$. Let $\varphi : V \rightarrow W$ be a linear map. Let B, \tilde{B} be bases of V and let B' and \tilde{B}' be bases of W . Let $A = [\varphi]_{B, B'}$ and $\tilde{A} = [\varphi]_{\tilde{B}, \tilde{B}'}$ be the matrices of φ w.r.t. the pairs B, B' and \tilde{B}, \tilde{B}' . Let P denote the transition matrix from B to \tilde{B} , and let Q denote the transition matrix from B' to \tilde{B}' . Then,*

$$\tilde{A}_{m \times n} = Q_{m \times m}^{-1} A_{m \times n} P_{n \times n}$$

Proof. $\forall z \in V$,

$$[\varphi(z)]_{B'} = A[z]_B \tag{28.1}$$

$$[\varphi(z)]_{\tilde{B}'} = A[z]_{\tilde{B}} \tag{28.2}$$

We have

$$[z]_B = P[z]_{\tilde{B}} \quad (28.3)$$

$$[\varphi(z)]_{B'} = Q[\varphi(z)]_{\tilde{B}'} \quad (28.4)$$

Therefore,

(28.1) in (28.4) \implies

$$A[z]_B = Q[\varphi(z)]_{\tilde{B}'} \quad (28.5)$$

(28.3) in (28.5) \implies

$$AP[z]_{\tilde{B}} = Q[\varphi(z)]_{\tilde{B}'} \quad (28.6)$$

Multiplying on the left by Q^{-1} ,

$$\begin{aligned} Q^{-1}AP[z]_{\tilde{B}} &= [\varphi(z)]_{\tilde{B}'} \\ \therefore [\varphi(z)]_{\tilde{B}'} &= Q^{-1}AP[z]_{\tilde{B}} \end{aligned}$$

Comparing with (28.2),

$$\tilde{A} = Q^{-1}AP$$

□

7.5 Operations on Linear Maps

Definition 23 (Operations on linear maps). Let

$$\begin{aligned} \varphi &: V \rightarrow W \\ \varphi' &: V \rightarrow W \end{aligned}$$

be linear maps.

$$\varphi + \varphi' : V \rightarrow W$$

is defined as

$$(\varphi + \varphi')(v) = \varphi(v) + \varphi'(v)$$

and

$$\alpha\varphi : V \rightarrow W$$

is defined as

$$(\alpha\varphi)(v) = \alpha\varphi(v)$$

Definition 24 (Composed map). Let

$$\begin{aligned}\varphi &: V \rightarrow W \\ \varphi' &: W \rightarrow U\end{aligned}$$

be linear maps.

$$(\varphi' \circ \varphi) : V \rightarrow U$$

is defined as

$$(\varphi' \circ \varphi)(v) = \varphi'(\varphi(v))$$

Theorem 29 (Matrix of composed map). *Let $\varphi : V \rightarrow W$, $\varphi' : W \rightarrow U$ be linear maps. Let $(\varphi' \circ \varphi) : V \rightarrow U$ be the composed map. Let $\dim V = n$, $\dim W = m$, $\dim U = l$. Let B, B', B'' be bases of V, W, U respectively. Let $A = [\varphi]_{B, B'}$, $A' = [\varphi']_{B', B''}$ be the matrices of φ, φ' . Let $A'' = [(\varphi' \circ \varphi)]_{B, B''}$ be the matrix of the composed map. Then,*

$$A'' = A'A$$

Proof. Let $z \in V$.

$$\begin{aligned}[(\varphi' \circ \varphi)(z)]_{B''} &= [\varphi'(\varphi(z))]_{B''} \\ &= A'[\varphi(z)]_{B'} \\ &= A'A[z]_B\end{aligned}$$

By definition,

$$[(\varphi' \circ \varphi)(z)]_{B''} = A''[z]_B$$

Therefore,

$$A'' = A'A$$

□

7.6 Kernel and Image

Definition 25 (Kernel and image). Let $\varphi : V \rightarrow W$ be a linear map.

$$\begin{aligned}\ker \varphi &\doteq \{v \in V : \varphi(v) = \mathbb{0}\} \\ \operatorname{im} \varphi &\doteq \{\varphi(v) : v \in V\}\end{aligned}$$

Theorem 30. $\ker \varphi$ is a subspace of V and $\operatorname{im} \varphi$ is a subspace of W .

Proof.

$$\begin{aligned}\varphi(\mathbb{0}) &= \mathbb{0} \\ \therefore \mathbb{0} &\in \ker \varphi\end{aligned}$$

If $v_1, v_2 \in \ker \varphi$, then

$$\begin{aligned}\varphi(v_1 + v_2) &= \varphi(v_1) + \varphi(v_2) \\ &= \mathbb{0} + \mathbb{0} \\ &= \mathbb{0} \\ \therefore v_1 + v_2 &\in \ker V\end{aligned}$$

If $v \in \ker \varphi$, $\alpha \in \mathbb{F}$, then

$$\begin{aligned}\varphi(\alpha v) &= \alpha \varphi(v) \\ &= \alpha \mathbb{0} \\ &= \mathbb{0} \therefore \alpha v \in \ker \varphi\end{aligned}$$

Therefore, $\ker \varphi$ is a subspace of W .

$$\begin{aligned}\varphi(\mathbb{0}) &= \mathbb{0} \\ \therefore \mathbb{0} &\in \operatorname{im} \varphi\end{aligned}$$

If $w_1, w_2 \in \operatorname{im} \varphi$, then

$$\begin{aligned}w_1 &= \varphi(v_1) \\ w_2 &= \varphi(v_2) \\ \therefore w_1 + w_2 &= \varphi(v_1) + \varphi(v_2) \\ &= \varphi(v_1 + v_2) \\ \therefore w_1 + w_2 &\in \operatorname{im} \varphi\end{aligned}$$

If $w \in W$, $\alpha \in \mathbb{F}$, then

$$\begin{aligned}\alpha w &= \alpha \varphi(v) \\ &= \varphi(\alpha v) \\ \therefore \alpha w &\in \operatorname{im} \varphi\end{aligned}$$

Therefore, $\operatorname{im} \varphi$ is a subspace of W . □

7.6.1 Dimensions of Kernel and Image

Theorem 31. *Let $\varphi : V \rightarrow W$ be a linear map. Then*

$$\dim(\ker(\varphi)) + \dim(\text{im}(\varphi))$$

Proof. Let $\ker \varphi = U$, $U \subseteq V$.

Let $B_0 = \{v_1, \dots, v_k\}$ be a basis of U .

Completing B_0 to a basis B of V ,

$$B = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$$

Let

$$w_{k+1} = \varphi(v_{k+1})$$

\vdots

$$w_n = \varphi(v_n)$$

Therefore, we need to prove that B' is a basis of $W' = \text{im}(\varphi)$, by proving that B' is a spanning set and that B' is linearly independent.

Take $w \in \text{im}(\varphi)$, so that there is $v \in V$ s.t. $\varphi(v) = w$.

Representing v as a linear combination of elements of B ,

$$\begin{aligned} v &= \alpha_1 v_1 + \dots + \alpha_k v_k + \alpha_{k+1} v_{k+1} + \dots + \alpha_n v_n \\ \therefore w &= \varphi(v) \\ &= \varphi(\alpha_1 v_1 + \dots + \alpha_k v_k + \alpha_{k+1} v_{k+1} + \dots + \alpha_n v_n) \\ &= \alpha_1 \varphi(v_1) + \dots + \alpha_k \varphi(v_k) + \alpha_{k+1} \varphi(v_{k+1}) + \dots + \alpha_n \varphi(v_n) \\ &= \alpha_{k+1} \varphi(v_{k+1}) + \dots + \alpha_n \varphi(v_n) \\ &= \alpha_{k+1} w_{k+1} + \dots + \alpha_n w_n \\ &\in \text{span}(B') \end{aligned}$$

Therefore, B' is a spanning set for W' .

Let

$$\beta_{k+1} w_{k+1} + \dots + \beta_n w_n = \mathbb{0}$$

Therefore, B' is linearly independent iff

$$\beta_{k+1} = \dots = \beta_n = 0$$

As φ is a linear map,

$$\begin{aligned} \varphi(\beta_{k+1} v_{k+1} + \dots + \beta_n v_n) &= \mathbb{0} \\ \therefore \beta_{k+1} v_{k+1} + \dots + \beta_n v_n &\in \ker \varphi \end{aligned}$$

Therefore, it can be expressed as a linear combination of vectors of B_0 , which is a basis of $\ker \varphi$.

Let

$$\begin{aligned} \beta_{k+1}v_{k+1} + \cdots + \beta_nv_n &= \alpha_{k+1}v_{k+1} + \cdots + \alpha_nv_n \\ \therefore \alpha_{k+1}v_{k+1} + \cdots + \alpha_nv_n - \beta_{k+1}v_{k+1} - \cdots - \beta_nv_n &= \mathbb{O} \end{aligned}$$

As $\{v_1, \dots, v_n\}$ is a basis of V , all coefficients must be 0

Therefore,

$$\beta_{k+1}v_{k+1} = \cdots = \beta_nv_n = 0$$

Hence, as B' is a spanning set of $\text{im } \varphi$ and also linearly independent, B' is a basis of $\text{im } \varphi$.

Therefore,

$$\begin{aligned} \dim(\text{im } \varphi) &= \text{size of } B' \\ &= n - k \\ &= n - \dim(\ker \varphi) \\ \therefore \dim(\text{im } \varphi) + \dim(\ker \varphi) &= \dim V \end{aligned}$$

□

Corollary 31.1.

$$\dim(\text{im } \varphi) = r$$

where r is the rank of A

Corollary 31.2. Let $A_{m \times n}$ be a matrix of rank r . Let $C(A)$ be the column space of A , and let $\dim C(A)$ be the column rank of A . Then

$$\dim C(A) = r$$

Proof. Define

$$\varphi : \mathbb{F}^n \rightarrow \mathbb{F}^m$$

s.t. $A = [\varphi]_{B, B'}$, where B is the standard basis of \mathbb{F}^n .

$$\begin{aligned} B &= \left\{ \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \right\} \\ &= \{e_1, \dots, e_n\} \end{aligned}$$

$\forall v \in \mathbb{F}^n$, we have

$$[\varphi(v)]_{B'} = A[v]_B$$

If $v = e_i$,

$$[\varphi(e_i)] = Ae_i$$

which is the i^{th} column of A . So, the space spanned by $\{\varphi(e_1), \dots, \varphi(e_n)\}$ is equal to $C(A)$. But it is also in $\text{im } \varphi$.

Therefore,

$$\text{im } \varphi = C(A)$$

and

$$\begin{aligned} \dim(\text{im } \varphi) &= \dim(C(A)) \\ \therefore r &= \dim(C(A)) \end{aligned}$$

□

Remark 7. Let $\varphi : V \rightarrow W$ be a linear map. Let $w \in \text{im } (\varphi)$, so that there is $v \in V$ s.t. $\varphi(v) = w$. Then any v' s.t. $\varphi(v') = w$ can be written down as $v' = v + v_0$ where $v_0 \in \ker \varphi$.

Part VI

Linear Operators

1 Definition

Definition 26 (Linear operator). A linear operator or transformation

$$T : V \rightarrow V$$

is a linear map from a vector space V to itself.

2 Similar Matrices

Let B and \tilde{B} be bases of V . Let A and \tilde{A} be the representing matrices

$$\begin{aligned} A &= [T]_B \\ \tilde{A} &= [T]_{\tilde{B}} \end{aligned}$$

Both these are $n \times n$ matrices, where $n = \dim V$. Let P denote the transition matrix from B to \tilde{B} . Then,

$$\tilde{A} = P^{-1}AP$$

Definition 27 (Similarity of matrices). Let A, \tilde{A} be $n \times n$ matrices. A is said to be similar to \tilde{A} , denoted as $A \sim \tilde{A}$, if there exists an invertible $n \times n$ matrix P , s.t. $\tilde{A} = P^{-1}AP$.

2.1 Properties of Similar Matrices

1. $A \sim A$
2. If $A \sim \tilde{A}$, then $\tilde{A} \sim A$
3. If $A \sim \tilde{A}$ and $\tilde{A} \sim \tilde{\tilde{A}}$, then $A \sim \tilde{\tilde{A}}$
4. If $A \sim \tilde{A}$, then $\det(A) = \det(\tilde{A})$
5. If $A \sim I$, then $A = I$

3 Diagonalization

Given a square matrix $A_{n \times n}$, decide whether or not A is similar to some diagonal matrix D . If it is, find D , and P s.t. $P^{-1}AP = D$.

Alternatively,

Given an operator $T : V \rightarrow V$, decide whether or not there exists a basis B of V , s.t. $[T]_B$ is a diagonal matrix D . If it exists, find D , and B , s.t. $[T]_B = D$.

Definition 28 (Diagonalizability). If A is similar to a diagonal matrix, A is said to be diagonalizable. P , s.t. $P^{-1}AP = D$ is called a diagonalizing matrix for A . D is called a diagonal form of A .

4 Eigenvalues and Eigenvectors

Definition 29 (Eigenvalue and eigenvector). Let A be a $n \times n$ matrix over \mathbb{F} . $\lambda \in \mathbb{F}$ is said to be an eigenvalue of A , if $\exists v \in \mathbb{F}, v \neq 0$, such that

$$Av = \lambda v$$

v is called an eigenvector corresponding to λ .

Definition 30 (Alternate definition of eigenvalue and eigenvector). Let $T : V \rightarrow V$ be a linear operator, where V is a vector space over \mathbb{F} . $\lambda \in \mathbb{F}$ is said to be an eigenvalue of A , if $\exists v \in V, v \neq 0$, such that

$$T(v) = \lambda v$$

v is called an eigenvector corresponding to λ .

Definition 31 (Spectrum). The collection of all eigenvalues of a matrix, or a linear operator is called the spectrum.

Theorem 1. Let A be a $n \times n$ matrix. $\lambda \in \mathbb{F}$ is an eigenvalue of A iff

$$\det(\lambda I_n - A) = 0$$

Proof. λ is an eigenvalue of A

$$\iff \exists v \in \mathbb{F}^n, v \neq 0, \text{ s.t. } Av = \lambda v$$

$$\iff \exists v \in \mathbb{F}^n, v \neq 0, \text{ s.t. } (\lambda I - A)v = \mathbf{0}$$

$$\iff v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\iff (\lambda I - A) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{0} \text{ has a non-zero solution}$$

$$\iff \text{there are free variables}$$

$$\iff \det(\lambda I - A) = 0$$

□

Theorem 2 (General criterion for diagonalization). Let A be a $n \times n$ matrix. A is diagonalizable if and only if there exists a basis $B = \{v_1, \dots, v_n\}$ of \mathbb{F}^n consisting of eigenvectors of A . In such a case, the diagonal entries of D are eigenvalues of A , and B can be chosen as consisting of the columns of P , where $P^{-1}AP = D$.

Corollary 2.1. If A has no eigenvalues, then it is not diagonalizable.

Theorem 3. Let $\lambda_1, \dots, \lambda_s$ be pairwise distinct eigenvalues of an $n \times n$ matrix A , i.e. $\forall i \neq j, \lambda_i \neq \lambda_j$. Let v_1, \dots, v_s be eigenvectors of A corresponding to $\lambda_1, \dots, \lambda_s$. Then the set $S = \{v_1, \dots, v_s\}$ is linearly independent.

Proof. If possible, let S be linearly dependent. Let S' denote a linearly dependent subset of S of smallest possible size, say l . WLG, let $S' = \{v_1, \dots, v_l\}$.

Hence, $\exists \alpha_1, \dots, \alpha_l \in \mathbb{F}$, s.t.

$$\alpha_1 v_1 + \dots + \alpha_l v_l = \mathbb{0} \quad (3.1)$$

Multiplying (3.1) on both sides by A ,

$$\alpha_1 A v_1 + \dots + \alpha_l A v_l = \mathbb{0} \quad (3.2)$$

$$\alpha_1 \lambda_1 v_1 + \dots + \alpha_l \lambda_l v_l = \mathbb{0} \quad (3.3)$$

Multiplying (3.1) on both sides by λ_l

$$\alpha_1 \lambda_l v_1 + \dots + \alpha_l \lambda_l v_l = \mathbb{0} \quad (3.4)$$

Subtracting (3.4) from (3.3)

$$\alpha_1 (\lambda_1 - \lambda_l) v_1 + \dots + \alpha_{l-1} (\lambda_{l-1} - \lambda_l) v_{l-1} = \mathbb{0} \quad (3.5)$$

Solving,

$$\alpha_1 = \alpha_l = 0$$

This is a contradiction. □

Corollary 3.1. *Let $A_{n \times n}$ have n distinct eigenvalues. Then, A is diagonalizable.*

Proof. Let v_1, \dots, v_n be eigenvectors of A , corresponding to $\lambda_1, \dots, \lambda_n$. As they are distinct, by the above theorem, they are linearly independent. The number of elements in the set $\{v_1, \dots, v_n\}$ is n . Therefore, the set is a basis. Hence, according to General criterion for diagonalization, A is diagonalizable. □

5 Characteristic Polynomial

Definition 32 (Characteristic Polynomial). Let A be any $n \times n$ matrix.

$$p_A(x) = \det(xI_n - A)$$

is called the characteristic polynomial.

5.1 Properties

1. The roots of $p_A(x)$ are the eigenvalues of A .
2. $\deg p_A(x) = n$
3. The coefficient of x^n is 1.
4. The constant term is $\alpha_0 = (-1)^n \det(A)$.
5. The coefficient of x^{n-1} is $\alpha_{n-1} = -(a_{11} + \cdots + a_{nn})$.

Theorem 4. *If $A \sim A'$, then $p_A(x) = p_{A'}(x)$.*

Proof.

$$\begin{aligned}
 A' &= P^{-1}AP \\
 \therefore p_{A'}(x) &= \det(xI - A') \\
 &= \det(xI - P^{-1}AP) \\
 &= \det(P^{-1}(xI)P - P^{-1}AP) \\
 &= \det(P^{-1}(xI - A)P) \\
 &= \cancel{\det(P^{-1})} \det(xI - A) \cancel{\det(P)} \\
 &= \det(xI - A) \\
 &= p_A(x)
 \end{aligned}$$

□

Definition 33 (Alternative definition of characteristic polynomial). Let $T : V \rightarrow V$ be a linear operator. The characteristic polynomial of T is defined as the characteristic polynomial of any representing matrix of T .

Theorem 5. *Let $f(x), g(x)$ be polynomials. Then $\exists q(x), r(x)$, s.t.*

$$f(x) = g(x)q(x) + r(x)$$

and $\deg r(x) < \deg g(x)$.

Definition 34 (Remainder). If

$$f(x) = g(x)q(x) + r(x)$$

$r(x)$ is called the remainder after division of $f(x)$ by $g(x)$. If $r(x) = \mathbb{O}$, $f(x)$ is said to be divisible by $g(x)$.

Corollary 5.1. *Let $f(x)$ be a polynomial and let α be a root of f . Then $f(x)$ is divisible by $(x - \alpha)$.*

Definition 35 (Algebraic multiplicity of eigenvalue). Let A be a $n \times n$ matrix, and let $p_A(x)$ be the characteristic polynomial of A , and let λ be an eigenvalue of A . The algebraic multiplicity of λ is defined as the largest possible integer value of k such that $p_A(x)$ is divisible by $(x - \lambda)^k$.

Definition 36 (Eigenspace). Let A be a $n \times n$ matrix, and let λ be an eigenvalue of A . The eigenspace of A corresponding to λ is defined as

$$V_\lambda = \{v \in \mathbb{F}^n; Av = \lambda v\}$$

Theorem 6. *An eigenspace of a matrix is a subspace of the field over which the matrix is defined.*

Definition 37 (Geometric multiplicity of eigenvalue). $m = \dim V_\lambda$ is called the geometric multiplicity of λ .

Theorem 7. *Let λ be an eigenvalue of $A_{n \times n}$. Let k be the algebraic multiplicity of λ and let m be the geometric multiplicity of λ . Then*

$$m \leq k$$

Proof.

$$m = \dim V_\lambda$$

Therefore, let $B_0 = \{v_1, \dots, v_m\}$ be a basis of V_λ .

Completing B_0 to $B = \{v_1, \dots, v_m, v_{m+1}, \dots, v_n\}$, a basis of \mathbb{F}^n .

Let $P_{n \times n}$ be a matrix with columns v_1, \dots, v_n .

$$P = \begin{pmatrix} v_1 & \dots & v_m & v_{m+1} & \dots & v_n \end{pmatrix}$$

P is invertible as v_1, \dots, v_n are linearly independent.

Consider $A' = P^{-1}AP$.

$$\begin{aligned} \therefore P^{-1}AP &= P^{-1}A \begin{pmatrix} v_1 & \dots & v_m & v_{m+1} & \dots & v_n \end{pmatrix} \\ &= P^{-1} \begin{pmatrix} Av_1 & \dots & Av_m & Av_{m+1} & \dots & Av_n \end{pmatrix} \\ &= P^{-1} \begin{pmatrix} \lambda v_1 & \dots & \lambda v_m & \star & \dots & \star \end{pmatrix} \\ &= \begin{pmatrix} P^{-1}(\lambda v_1) & \dots & P^{-1}(\lambda v_m) & \star & \dots & \star \end{pmatrix} \\ &= \begin{pmatrix} \lambda e_1 & \dots & \lambda e_m & \star & \dots & \star \end{pmatrix} \\ &= \begin{pmatrix} \lambda I_m & \star \\ 0 & \tilde{A} \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
p_{A'}(x) &= \det(xI_n + A') \\
&= \det\left(\begin{pmatrix} xI_m & 0 \\ 0 & xI_{n-m} \end{pmatrix} - \begin{pmatrix} \lambda I_m & \star \\ 0 & \tilde{A} \end{pmatrix}\right) \\
&= \det\begin{pmatrix} (x-\lambda)I_m & \star \\ 0 & xI_{n-m} - \tilde{A} \end{pmatrix} \\
&= \det((x-\lambda)I_m) \cdot \det(xI_{n-m} - \tilde{A}) \\
&= (x-\lambda)^m \cdot p_{\tilde{A}}(x)
\end{aligned}$$

As $A \sim \tilde{A}$,

$$p_A(x) = p_{A'} = (x-\lambda)^m p_{\tilde{A}}(x)$$

By the definition of Algebraic multiplicity of eigenvalue, $k \geq m$. \square

Theorem 8. *If a matrix $A_{n \times n}$ is diagonalizable, then its characteristic polynomial $p_A(x)$ can be represented as a product of linear factors.*

$$p_A(x) = (x - \lambda_1)^{k_1} \dots (x - \lambda_s)^{k_s}$$

where k_i is the algebraic multiplicity of λ_i and $\lambda_1, \dots, \lambda_s$ are pairwise distinct.

Proof. As A is diagonalizable, let $A \sim D$,

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_s \end{pmatrix}$$

Then,

$$\begin{aligned}
p_A(x) &= p_D(x) \\
&= \det \begin{pmatrix} x - \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x - \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x - \lambda_s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x - \lambda_s \end{pmatrix} \\
&= (x - \lambda_1)^{k_1} \dots (x - \lambda_s)^{k_s}
\end{aligned}$$

□

Theorem 9 (Explicit criterion for diagonalization). *Let A be an $n \times n$ matrix, s.t. $p_A(x)$ splits completely. Then A is diagonalizable if and only if $\forall \lambda_i$ of A , the algebraic multiplicity coincides with the geometric multiplicity.*

Proof of statement. If p_A splits completely, then $k_1 + \dots + k_n = n$.

If A is diagonalizable, then by the General criterion for diagonalization, there is $B = \{v_1, \dots, v_n\}$, a basis of \mathbb{F}^n , s.t. each v_i is an eigenvector of A .

Dividing v_1, \dots, v_n into s groups corresponding to $\lambda_1, \dots, \lambda_s$, to each λ_i , there correspond at most $m_i = \dim V_{\lambda_i}$ eigenvectors, as they are a part of a basis and hence linearly independent.

Therefore,

$$n \leq m_1 + \dots + m_s$$

As p_A splits completely,

$$n = k_1 + \dots + k_s$$

Also, $k_i \geq m_i$

$$\therefore k_1 + \dots + k_s = m_1 + \dots + m_s$$

moreover, $\forall i$, s.t. $1 \leq i \leq s$,

$$k_i = m_i$$

□

Proof of converse.

$\forall i$, s.t. $1 \leq i \leq s$

$$\therefore k_i = m_i$$

As $k_1 + \dots + k_s = n$,

$$m_1 + \dots + m_s = n$$

Let the bases of the eigenspaces $V_{\lambda_1}, \dots, V_{\lambda_s}$ be B_1, \dots, B_s .

$$\begin{aligned} |B_1| &= m_1 \\ &\vdots \\ |B_s| &= m_s \end{aligned}$$

Let $B = B_1 \cup \dots \cup B_s$. $|B| = n$.

It is enough to prove that B is linearly independent.

Let

$$B = \{v_1, v_2, \dots, w_1, w_2, \dots, u_1, u_2, \dots\}$$

Suppose

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \beta_1 w_1 + \beta_2 w_2 + \dots + \gamma_1 u_1 + \gamma_2 u_2 + \dots = \mathbb{O}$$

If possible, let at least one coefficient be non-zero. WLG, let $\alpha_1 \neq 0$.

Hence, as v_1, v_2, \dots form B_1 which is a basis of V_{λ_1} ,

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots \neq \mathbb{O}$$

Let

$$w = \beta_1 w_1 + \beta_2 w_2 + \dots$$

...

$$u = \gamma_1 u_1 + \gamma_2 u_2 + \dots$$

Therefore,

$$v + w + \dots + u = \mathbb{O}$$

where $v \neq \mathbb{O}$ and $v \in V_{\lambda_1}, w \in V_{\lambda_2}, \dots, u \in V_{\lambda_s}$.

But as $\lambda_1, \dots, \lambda_s$ are pairwise distinct, v, w, \dots, u are linearly independent.

This is a contradiction. Therefore, B is a basis. Hence, as B consists of eigenvectors of A , by the General criterion for diagonalization, A is diagonalizable. \square

Theorem 10 (Criterion for triangularization). *An operator $T : V \rightarrow V$ is triangularizable, i.e. there is a basis B of V such that $[T]_B$ is upper triangular, if and only if $p_T(x)$ splits completely.*

Theorem 11 (Jordan Theorem). *Let $T : V \rightarrow V$ be a linear operator such that $p_T(x)$ splits completely. Then there exists a basis B of V such that $[T]_B$ is of the form*

$$[T]_B = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_l \end{pmatrix}$$

where each J_i is of the form

$$\begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

where λ is some eigenvalue of T .

Part VII

Inner Product Spaces

1 Definition

Definition 38 (Inner product). Let \mathbb{F} be \mathbb{R} or \mathbb{C} . Let V be a vector space over \mathbb{F} . An inner product on V is a function in two vector arguments with scalar values which associates to two given vectors $v, w \in V$ their product $\langle v, w \rangle \in \mathbb{F}$ so that the following properties are satisfied.

1. $\langle \alpha_1 v_1 + \alpha_2 v_2, w \rangle = \alpha_1 \langle v_1, w \rangle + \alpha_2 \langle v_2, w \rangle, \forall v_1, v_2, w \in V, \forall \alpha_1, \alpha_2 \in \mathbb{F}$
2. $\langle v, w \rangle = \overline{\langle w, v \rangle}, \forall v, w \in V$
3. $\langle v, v \rangle$ is a real non-negative number, $\forall v \in V$

Example 12. The dot product of two vectors is defined as follows. Is it an inner product?

$$V = \mathbb{F}^n$$

$$\left\langle \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \right\rangle = \alpha_1 \overline{\beta_1} + \cdots + \alpha_n \overline{\beta_n}$$

Solution. All three axioms are satisfied by this product. Hence, it is an inner product.

Theorem 1 (Sesquilinearity).

$$\langle v, \beta_1 w_1 + \beta_2 w_2 \rangle = \overline{\beta_1} \langle v, w_1 \rangle + \overline{\beta_2} \langle v, w_2 \rangle$$

$$\forall v, w_1, w_2 \in V, \beta_1, \beta_2 \in \mathbb{F}$$

Definition 39 (Length). The length of a vector

$$v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

is defined to be

$$\|v\| = \sqrt{\alpha_1^2 + \cdots + \alpha_n^2}$$

Example 13. Let V be the vector space consisting of all continuous functions $f : [a, b] \rightarrow \mathbb{R}$.

$$\langle f, g \rangle = \int_a^b f(x)g(x) \, dx$$

Solution. All three axioms are satisfied by this product. Hence, it is an inner product.

2 Computation of Inner Products

Definition 40 (Gram matrix). Let V be an inner product space. Let

$$B = \{v_1, \dots, v_n\}$$

be a basis of V .

$$G_B = \begin{pmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_n \rangle \\ \vdots & & \vdots \\ \langle v_n, v_1 \rangle & \cdots & \langle v_n, v_n \rangle \end{pmatrix}$$

is called the Gram matrix of the inner product with respect to B .

Example 14. Find the Gram matrix of $V = \mathbb{F}^n$ with standard dot product with respect to

$$B = \left\{ \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

Solution.

$$\begin{aligned} G_B &= \begin{pmatrix} \langle e_1, e_1 \rangle & \dots & \langle e_1, e_n \rangle \\ \vdots & & \vdots \\ \langle e_n, e_1 \rangle & \dots & \langle e_n, e_n \rangle \end{pmatrix} \\ &= \begin{pmatrix} 1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1 \end{pmatrix} \end{aligned}$$

Example 15. Find the Gram matrix of $V = \mathbb{F}^n$ with standard dot product with respect to

$$B = \left\{ \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 7 \end{pmatrix} \right\}$$

Solution.

$$\begin{aligned} G_B &= \begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle \end{pmatrix} \\ &= \begin{pmatrix} 25 & 46 \\ 46 & 85 \end{pmatrix} \end{aligned}$$

Theorem 2.

$$\langle v, w \rangle = [v]_B^t G_B \overline{[w]_B}$$

Proof. Let

$$B = \{v_1, \dots, v_n\}$$

be a basis of V .

The Gram matrix is

$$G_B = (\langle v_i, v_j \rangle) = (g_{ij})$$

To compute $\langle v, w \rangle$, find

$$[v]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$$[w]_B = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\begin{aligned} \langle v, w \rangle &= \langle \alpha_1 v_1 + \cdots + \alpha_n v_n, \beta_1 v_1 + \cdots + \beta_n v_n \rangle \\ &= \alpha_1 \overline{\beta_1} \langle v_1, v_1 \rangle + \cdots + \alpha_1 \overline{\beta_n} \langle v_1, v_n \rangle \\ &\quad + \alpha_2 \overline{\beta_1} \langle v_2, v_1 \rangle + \cdots + \alpha_2 \overline{\beta_n} \langle v_2, v_n \rangle \\ &\quad + \cdots \\ &\quad + \alpha_n \overline{\beta_1} \langle v_n, v_1 \rangle + \cdots + \alpha_n \overline{\beta_n} \langle v_n, v_n \rangle \\ &= \alpha_1 g_{11} \overline{\beta_1} + \cdots + \alpha_1 g_{1n} \overline{\beta_n} \\ &\quad + \alpha_2 g_{21} \overline{\beta_1} + \cdots + \alpha_2 g_{2n} \overline{\beta_n} \\ &\quad + \cdots \\ &\quad + \alpha_n g_{n1} \overline{\beta_1} + \cdots + \alpha_n g_{nn} \overline{\beta_n} \\ &= [v]_B^t G_B \overline{[w]_B} \end{aligned}$$

□

2.1 Change of Basis

Theorem 3. Let B, \tilde{B} be bases of V . Let P be the transition matrix from B to \tilde{B} . Then

$$G_{\tilde{B}} = P^t G_B \overline{P}$$

where \overline{P} is the matrix obtained by replacing all elements of P by their complex conjugates.

Proof.

$$[v]_B = P[v]_{\tilde{B}}$$

$$\begin{aligned}
\langle v, w \rangle &= [v]_B^t G_B \overline{[w]_B} \\
&= (P[v]_{\tilde{B}})^t G_B \overline{(P[w]_{\tilde{B}})} \\
&= [v]_{\tilde{B}}^t (P^t G_B \overline{P}) \overline{[w]_{\tilde{B}}}
\end{aligned}$$

Also,

$$\langle v, w \rangle = [v]_{\tilde{B}}^t G_{\tilde{B}} \overline{[w]_{\tilde{B}}}$$

Therefore,

$$G_{\tilde{B}} = P^t G_B \overline{P}$$

□

3 Norms

3.1 Definition

Definition 41 (Norm). Let V be a vector space over \mathbb{F} with inner product. $\forall v \in V$,

$$\|v\| \doteq \sqrt{\langle v, v \rangle}$$

$\|v\|$ is called the norm of v .

3.2 Properties

1. Positivity

$$\|v\| \geq 0, \forall v \in V$$

$$\|v\| = 0 \iff v = \mathbb{0}$$

2. Homogeneity

$$\|\alpha v\| = |\alpha| \|v\|, \forall v \in V, \forall \alpha \in \mathbb{F}$$

3. Triangle Inequality

$$\|u + v\| \leq \|u\| + \|v\|, \forall u, v \in V$$

4 Orthogonality

4.1 Definition

Definition 42 (Orthogonality). A vector $u \in V$ is said to be orthogonal to $v \in V$ if

$$\langle u, v \rangle = 0$$

It is denoted as $u \perp v$.

4.2 Properties

1. If $u \perp v$, then $v \perp u$.
2. If $u \perp v$, $\alpha, \beta \in \mathbb{F}$, then $\alpha u \perp \beta v$.
3. $\mathbb{0} \perp v, \forall v \in V$.

5 Orthogonal and Orthonormal Bases

Let V be a vector space over \mathbb{F} with an inner product. Let $S \subset V$.

Definition 43 (Orthogonal set). S is said to be orthogonal if any two distinct vectors from S are orthogonal.

Definition 44 (Orthonormal set). S is said to be orthonormal if it is orthogonal and the norm of every vector is 1.

Definition 45 (Orthogonal basis). S is said to be an orthogonal basis of V if it is orthogonal and a basis of V .

Definition 46 (Orthonormal basis). S is said to be an orthonormal basis of V if it is orthonormal and a basis of V .

Theorem 4. *Let S be an orthogonal set such that $\mathbb{0} \notin S$. Then S is linearly independent.*

Proof. Let

$$\begin{aligned}\alpha_1, \dots, \alpha_m &\in \mathbb{F} \\ v_1, \dots, v_m &\in S\end{aligned}$$

Let

$$\alpha_1 v_1 + \dots + \alpha_m v_m = \mathbb{0}$$

S is linearly independent if and only if

$$\alpha_1 = \dots = \alpha_m = 0$$

$$\alpha_1 v_1 + \dots + \alpha_m v_m = \mathbb{0}$$

Multiplying both sides by v_1 ,

$$\begin{aligned}\langle \alpha_1 v_1 + \cdots + \alpha_m v_m, v_1 \rangle &= \langle \mathbb{O}, v_1 \rangle \\ \therefore \alpha_1 \langle v_1, v_1 \rangle + \cdots + \alpha_m \langle v_m, v_1 \rangle &= 0\end{aligned}$$

As v_1, \dots, v_m are orthogonal,

$$\langle v_2, v_1 \rangle = \cdots = \langle v_m, v_1 \rangle$$

$$\therefore \alpha_1 \langle v_1, v_1 \rangle = 0$$

As $v_1 \neq \mathbb{O}$

$$\begin{aligned}\langle v_1, v_1 \rangle &\neq 0 \\ \therefore \alpha_1 &= 0\end{aligned}$$

Similarly,

$$\alpha_2 = \cdots = \alpha_m = 0$$

□

Corollary 4.1. *Any orthonormal set is linearly independent.*

Corollary 4.2. *Any orthonormal set consisting of $n = \dim V$ vectors is an orthonormal basis of V .*

Example 16. Is the set

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

orthonormal?

Solution. The norm of the elements of S is not 1. Hence S is not orthonormal.

Theorem 5. *Let $B = \{v_1, \dots, v_n\}$ be an orthonormal basis of V . Let $v \in V$.*

Let $[v]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$. Then,

$$\alpha_1 = \langle v, v_1 \rangle$$

\vdots

$$\alpha_n = \langle v, v_n \rangle$$

Proof.

$$\begin{aligned}v &= \alpha_1 v_1 + \cdots + \alpha_n v_n \\ \therefore \langle v, v_1 \rangle &= \langle \alpha_1 v_1 + \cdots + \alpha_n v_n, v_1 \rangle \\ &= \alpha_1 \langle v_1, v_1 \rangle + \cdots + \alpha_n \langle v_n, v_1 \rangle \\ &= \alpha_1\end{aligned}$$

Similarly, in general, $\forall 1 \leq i \leq n$,

$$\langle v, v_i \rangle = \alpha_i$$

□

Theorem 6 (Pythagoras Theorem). *Let $B = \{v_1, \dots, v_n\}$ be an orthonormal basis of V . Let $v \in V$. Let $[v]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$. Then,*

$$\|v\|^2 = |\alpha_1|^2 + \cdots + |\alpha_n|^2$$

Proof.

$$\begin{aligned}\|v\|^2 &= \langle v, v \rangle \\ &= \langle \alpha_1 v_1 + \cdots + \alpha_n v_n, \alpha_1 v_1 + \cdots + \alpha_n v_n \rangle \\ &= \alpha_1 \overline{\alpha_1} + \cdots + \alpha_n \overline{\alpha_n} \\ &= |\alpha_1|^2 + \cdots + |\alpha_n|^2\end{aligned}$$

□

6 Unitary Matrices

Definition 47. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let A be an $n \times n$ matrix. A is said to be a unitary matrix if

$$A^* = \overline{A}^t = \overline{A^t} = A^{-1}$$

If $\mathbb{F} = \mathbb{R}$, unitary matrices are called orthogonal matrices.

1. I is a unitary matrix.
2. If A_1 and A_2 are unitary matrices, then $(A_1 A_2)^* = A_2^* A_1^*$.

3. If A is unitary, A^{-1} is also unitary.

Theorem 7. Let A be an $n \times n$ matrix. Let v_1, \dots, v_n be the columns of A . Let r_1, \dots, r_n be the columns of A^t . Then the following are equivalent.

1. A is unitary.
2. $\{v_1, \dots, v_n\}$ is an orthonormal basis of \mathbb{F}^n , with respect to standard dot product.
3. $\{r_1, \dots, r_n\}$ is an orthonormal basis of \mathbb{F}^n , with respect to standard dot product.

Proof. As A is unitary, A^t is also unitary.

$$\begin{aligned} (A^t)^* &= (\overline{A^t})^t \\ &= (A^*)^t \\ &= (A^{-1})^t \\ &= (A^t)^{-1} \end{aligned}$$

$$\begin{aligned} &A \text{ is unitary} \\ \iff &A^* = A^{-1} \\ \iff &AA^* = I \\ \iff &A\overline{A^t} = I \\ \iff &(A\overline{A^t})_{ik} = I_{ik} \\ &= \sum_{j=1}^n a_{ij} \overline{a_{jk}} \\ &= r_i \cdot \overline{r_k} \iff \{r_1, \dots, r_n\} \text{ is an orthonormal basis} \end{aligned}$$

□

Theorem 8. Let V be an inner product space. Let B be an orthonormal basis of V . Let B' be another basis of V . Let P be the transition matrix from B to B' . Then B' is orthonormal if and only if P is unitary.

Proof of statement.

$$G_{B'} = P^t G_B \overline{P}$$

If B' is orthonormal,

$$\begin{aligned}\therefore I &= P^t I \bar{P} \\ &= P^t \bar{P}\end{aligned}$$

Therefore, P is unitary. □

Proof of converse. If P is unitary,

$$G_{B'} = P^t G_B \bar{P}$$

As B is orthonormal,

$$\begin{aligned}G_B &= I \\ \therefore G_{B'} &= P^t \bar{P}\end{aligned}$$

As P is unitary,

$$\begin{aligned}P^t \bar{P} &= I \\ \therefore G_{B'} &= I\end{aligned}$$

Therefore, B' is orthonormal. □

7 Projections

7.1 Definition

Definition 48. Let $S \subset V$ be a set of vectors.

$$S^\perp \doteq \{v \in V \mid \langle u, v \rangle = 0 \forall u \in S\}$$

Theorem 9. S^\perp is a subspace of V .

Proof.

$$\langle u, \mathbb{O} \rangle = 0 \therefore \mathbb{O} \in S^\perp$$

If $v_1, v_2 \in S^\perp$,

$$\begin{aligned}\langle u, v_1 + v_2 \rangle &= \langle u, v_1 \rangle + \langle u, v_2 \rangle \\ &= 0 + 0 \\ &= 0\end{aligned}$$

If $v \in S^\perp$,

$$\begin{aligned}\langle u, \alpha v \rangle &= \bar{\alpha} \langle u, v \rangle \\ &= 0\end{aligned}$$

□

Theorem 10.

$$S^\perp = \text{span}(S)^\perp$$

Proof. Let $v \in S^\perp$, $u \in \text{span}(S)$.

Let $\alpha_1, \dots, \alpha_m \in \mathbb{F}$, $u_1, \dots, u_m \in S$.

Therefore,

$$\begin{aligned}u &= \alpha_1 u_1 + \dots + \alpha_m u_m \\ \therefore \langle u, v \rangle &= \langle \alpha_1 u_1 + \dots + \alpha_m u_m, v \rangle \\ &= \alpha_1 \langle u_1, v \rangle + \dots + \alpha_m \langle u_m, v \rangle \\ &= \alpha_1 \cdot 0 + \dots + \alpha_m \cdot 0 \\ &= 0\end{aligned}$$

Therefore, $v \in S^\perp$.

Therefore, $S^\perp \subset \text{span}(S)^\perp$.

$S \subset \text{span}(S)$. Therefore, let $v \in \text{span}(S)^\perp$. Then,

$$\langle u, v \rangle = 0$$

for all $u \in \text{span}(S)$.

Hence for all $u \in S$,

$$\langle u, v \rangle = 0$$

Therefore, $\text{span}(S)^\perp \subset S^\perp$. □

Definition 49 (Projection). Let V be an inner product space. Let W be a subspace of V . Let $v \in V$. Let $B = \{w_1, \dots, w_m\}$ be a basis of W . The projection of v onto W is defined as follows.

$$\pi_B(v) = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \dots + \frac{\langle v, w_m \rangle}{\langle w_m, w_m \rangle} w_m$$

7.2 Properties

1. $\pi_B(v) \in W$
2. $\pi_B(v) = v \iff v \in W$
3. $v - \pi_B(v) \in W^\perp$

7.3 Gram - Schmidt Process

Input	Any basis $B = \{v_1, \dots, v_n\}$ of V .
Intermediate Output	Orthogonal basis $\widetilde{B} = \{\widetilde{v}_1, \dots, \widetilde{v}_n\}$ of V
Final Output	Orthonormal basis $B^0 = \{v_1^0, \dots, v_n^0\}$ of V

Step 1 $\widetilde{v}_1 = v_1$, denote $w_1 = \text{span}\{\widetilde{v}_1\} = \text{span}\{v_1\}$, $B_1 = \{\widetilde{v}_1\}$

Step 2 $\widetilde{v}_2 = v_2 - \pi_{B_1}(v_2) = v_2 - \frac{\langle v_2, \widetilde{v}_1 \rangle}{\langle \widetilde{v}_1, \widetilde{v}_1 \rangle} \widetilde{v}_1$

As $\widetilde{v}_2 \perp \widetilde{v}_1$, $B_2 = \{\widetilde{v}_1, \widetilde{v}_2\}$ is an orthogonal set. Denote $W_2 = \text{span}\{\widetilde{v}_1, \widetilde{v}_2\} = \text{span}\{v_1, v_2\}$.

Step 3 $\widetilde{v}_3 = v_3 - \pi_{B_2}(v_3) = v_3 - \frac{\langle v_3, \widetilde{v}_1 \rangle}{\langle \widetilde{v}_1, \widetilde{v}_1 \rangle} \widetilde{v}_1 - \frac{\langle v_3, \widetilde{v}_2 \rangle}{\langle \widetilde{v}_2, \widetilde{v}_2 \rangle} \widetilde{v}_2$

As $\widetilde{v}_3 \in W_2^\perp$, $B_3 = \{\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3\}$ is an orthogonal set. Denote $W_3 = \text{span}\{\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3\} = \text{span}\{v_1, v_2, v_3\}$.

⋮

Step n The n^{th} step gives $\widetilde{B}_n = \{\widetilde{v}_1, \dots, \widetilde{v}_n\}$ which is an orthogonal basis of V .

B^0 is obtained by normalization of \widetilde{B}_n .

$$v_1^0 = \frac{1}{\|\widetilde{v}_1\|}$$

$$\vdots$$

$$v_n^0 = \frac{1}{\|\widetilde{v}_n\|}$$

Example 17.

$$\begin{aligned} B &= \{v_1, v_2, v_3\} \\ &= \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \end{aligned}$$

Solution.

$$\begin{aligned} \widetilde{v}_1 &= v_1 \\ &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ \widetilde{v}_2 &= v_2 - \frac{\langle v_2, \widetilde{v}_1 \rangle}{\langle \widetilde{v}_1, \widetilde{v}_1 \rangle} \widetilde{v}_1 \\ &= \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \widetilde{v}_3 &= v_3 - \frac{\langle v_3, \widetilde{v}_1 \rangle}{\langle \widetilde{v}_1, \widetilde{v}_1 \rangle} \widetilde{v}_1 - \frac{\langle v_3, \widetilde{v}_2 \rangle}{\langle \widetilde{v}_2, \widetilde{v}_2 \rangle} \widetilde{v}_2 \\ &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - 0 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \therefore \widetilde{B}_3 &= \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \end{aligned}$$

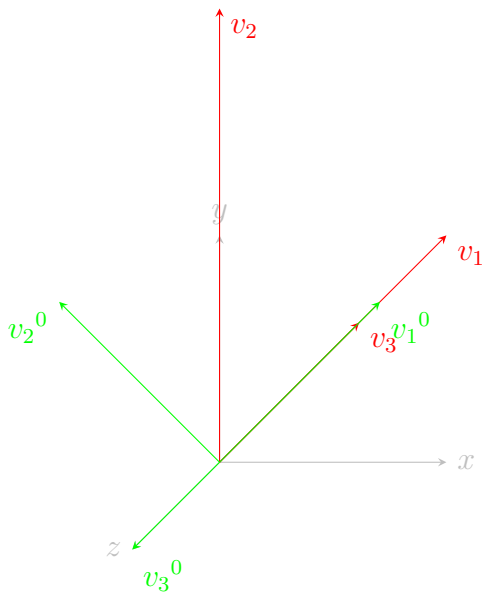
Therefore, normalizing \widetilde{B}_3 ,

$$v_1^0 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$v_2^0 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$v_3^0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore B^0 = \left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$



7.4 Inequalities

Theorem 11 (Bessel's Inequality). *Let $\{v_1, \dots, v_m\}$ be an orthonormal set. Let $v \in V$ be any vector. Then*

$$\|v\|^2 \geq |\langle v, v_1 \rangle|^2 + \dots + |\langle v, v_m \rangle|^2$$

and the equality holds if and only if $v \in \text{span}\{v_1, \dots, v_m\}$.

Proof. $\{v_1, \dots, v_m\}$ can be completed to an orthonormal basis

$$B = \{v_1, \dots, v_m, v_{m+1}, \dots, v_n\}$$

Using Pythagoras Theorem,

$$\begin{aligned} \|v\|^2 &= |\langle v, v_1 \rangle|^2 + \dots + |\langle v, v_m \rangle|^2 + |\langle v, v_{m+1} \rangle|^2 + \dots + |\langle v, v_n \rangle|^2 \\ \therefore \|v\|^2 &\geq |\langle v, v_1 \rangle|^2 + \dots + |\langle v, v_m \rangle|^2 \end{aligned}$$

The equality holds if and only if

$$|\langle v, v_{m+1} \rangle|^2 + \dots + |\langle v, v_n \rangle|^2 = 0$$

if and only if

$$\begin{aligned} |\langle v, v_{m+1} \rangle|^2 &= 0 \\ &\vdots \\ |\langle v, v_n \rangle|^2 &= 0 \end{aligned}$$

If $v \in \text{span}\{v_1, \dots, v_m\}$,

$$v = \alpha_1 v_1 + \dots + \alpha_m v_m$$

Therefore,

$$\begin{aligned} \langle v, v_{m+1} \rangle &= \langle \alpha_1 v_1 + \dots + \alpha_m v_m, v_{m+1} \rangle \\ &= \alpha_1 \langle v_1, v_{m+1} \rangle + \dots + \alpha_m \langle v_m, v_{m+1} \rangle \end{aligned}$$

as the basis is orthonormal, $\langle v_i, v_{m+1} \rangle$

$$\therefore \langle v, v_{m+1} \rangle = 0$$

Similarly,

$$\begin{aligned} |\langle v, v_{m+2} \rangle|^2 &= 0 \\ &\vdots \\ |\langle v, v_n \rangle|^2 &= 0 \end{aligned}$$

Conversely, if

$$\begin{aligned} |\langle v, v_{m+1} \rangle|^2 &= 0 \\ &\vdots \\ |\langle v, v_n \rangle|^2 &= 0 \end{aligned}$$

let

$$\begin{aligned}v &= \alpha_1 v_1 + \cdots + \alpha_m v_m + \alpha_{m+1} v_{m+1} + \alpha_n v_n \\ \therefore 0 &= \langle v, v_{m+1} \rangle \\ \therefore 0 &= \langle \alpha_1 v_1 + \cdots + \alpha_m v_m + \alpha_{m+1} v_{m+1} + \alpha_n v_n, v_{m+1} \rangle\end{aligned}$$

All $\langle v_i, v_{m+1} \rangle$ except $\langle v_{m+1}, v_{m+1} \rangle$ are 0.

Therefore,

$$|\langle v, v_{m+1} \rangle|^2 + \cdots + |\langle v, v_n \rangle|^2 = 0$$

□

Theorem 12 (Cauchy - Schwarz Inequality). *Let $u, v \in V$ be any vectors. Then*

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$$

and the equality holds if and only if $\{u, v\}$ is linearly dependent.

Proof. If $u = \mathbb{O}$, the equality holds.

Let $u \neq \mathbb{O}$.

Let

$$\begin{aligned}u^0 &= \frac{1}{\|u\|} \\ \|u^0\| &= 1\end{aligned}$$

Applying Bessel's Inequality to the orthonormal set $\{u^0\}$,

$$\begin{aligned}\|v\|^2 &\geq |\langle v, u^0 \rangle|^2 \\ |\langle v, u^0 \rangle|^2 &= \left| \left\langle v, \frac{1}{\|u\|} u \right\rangle \right|^2 \\ &= \left| \frac{1}{\|u\|} \langle v, u \rangle \right|^2 \\ &= \left(\frac{1}{\|u\|} |\langle v, u \rangle| \right)^2 \\ &= \frac{1}{\|u\|^2} |\langle v, u \rangle|^2 \\ \therefore \|v\|^2 &\geq \frac{1}{\|u\|^2} |\langle v, u \rangle|^2\end{aligned}$$

By Bessel's Inequality, the equality holds if and only if

$$v \in \text{span}\{u^0\} = \text{span}\{u\}$$

Therefore, v and u are linearly independent. □

8 Angle

Definition 50 (Angle). Let V be a vector space over \mathbb{R} with inner product $\langle \cdot, \cdot \rangle$. Let $u, v \in V$, $u \neq \mathbb{O}$, $v \neq \mathbb{O}$. The angle between u and v is defined as

$$\cos \varphi \doteq \frac{\langle u, v \rangle}{\|u\| \cdot \|v\|}$$

9 Triangle Inequality

Theorem 13 (Triangle Inequality Theorem). Let $u, v \in V$. Then

$$\|u + v\| \leq \|u\| + \|v\|$$

Proof.

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ &= \|u\|^2 + \langle u, v \rangle + \overline{\langle u, v \rangle} + \|v\|^2 \\ &= \|u\|^2 + 2\Re(\langle u, v \rangle) + \|v\|^2 \end{aligned}$$

As $\Re(z) \leq |z|$,

$$\|u + v\|^2 \leq \|u\|^2 + 2|\langle u, v \rangle| + \|v\|^2$$

Hence, by Cauchy - Schwarz Inequality,

$$\begin{aligned} \|u + v\|^2 &\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \\ \therefore \|u + v\|^2 &\leq (\|u\| + \|v\|)^2 \\ \therefore \|u + v\| &\leq \|u\| + \|v\| \end{aligned}$$

□

10 Orthogonal Decomposition

Theorem 14. Let W be a subspace of V . Then

$$V = W \oplus W^\perp$$

Proof. Let B be an orthogonal basis of V . Consider a projection $\pi_B(v)$. Therefore,

$$v = \pi_B(v) + (v - \pi_B(v))$$

$$\begin{aligned}\pi_B(v) &\in W \\ v - \pi_B(v) &\in W^\perp\end{aligned}$$

Therefore,

$$V = W + W^\perp$$

If possible, let $u \in W \cap W^\perp$. Therefore, $u \in W$ and $u \in W^\perp$. By the definition of orthogonality,

$$\begin{aligned}\langle u \in W, u \in W^\perp \rangle &= 0 \\ \therefore u &= 0\end{aligned}$$

Therefore,

$$V = W \oplus W^\perp$$

□

Corollary 14.1. *Let B be an orthogonal basis of W . Then $\pi_B(v)$ does not depend on the choice of B .*

Proof. As B is an orthogonal basis of W ,

$$v = \pi_B(v) + (v - \pi_B(v))$$

Let B' be another orthogonal basis of W . Therefore,

$$v = \pi_{B'}(v) + (v - \pi_{B'}(v))$$

Therefore,

$$\begin{aligned}\pi_B(v) &\in W \\ \pi_{B'}(v) &\in W\end{aligned}$$

and

$$\begin{aligned}v - \pi_B(v) &\in W^\perp \\ v - \pi_{B'}(v) &\in W^\perp\end{aligned}$$

As

$$V = W \oplus W^\perp$$

such a representation is unique. Therefore,

$$\pi_B(v) = \pi_{B'}(v)$$

□

Theorem 15. *Let $u, v \in V$, s.t. $u \perp v$. Then*

$$\|u \pm v\|^2 = \|u\|^2 + \|v\|^2$$

Proof.

$$\begin{aligned} \|u \pm v\|^2 &= \|u\|^2 + \|v\|^2 \\ &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ &= \langle u, u \rangle + \langle v, v \rangle \\ &= \|u\|^2 + \|v\|^2 \end{aligned}$$

□

11 Distance

Definition 51 (Distance). Let $u, v \in V$. The distance $d(u, v)$ from u to v is defined as

$$d(u, v) \doteq \|u - v\|$$

Theorem 16. *Let $u, v \in V$. Then*

$$d(u, v) \geq 0$$

and the equality holds if and only if $u = v$.

Theorem 17. *Let $u, v \in V$. Then*

$$d(u, v) = d(v, u)$$

Theorem 18. *Let $u, v \in V$. Then*

$$d(u, v) + d(v, w) \geq d(u, w)$$

Theorem 19. *The projection $\pi_W(v)$ is the vector in W closest to v , i.e.*

$$d(v, \pi_W(v)) = \min_{w \in W} d(v, w)$$

Proof. Let $v \in V$. For any vector $w \in W$,

$$\begin{aligned} (d(v, u))^2 &= \|v - w\|^2 \\ &= \|(v - \pi_W(v)) + (\pi_W(v) - w)\|^2 \\ &= \|v - \pi_W(v)\|^2 + \|\pi_W(v) - w\|^2 \\ &\geq \|v - \pi_W(v)\|^2 \\ \therefore (d(v, u))^2 &\geq d(v, \pi_W(v))^2 \end{aligned}$$

□

12 Adjoint Map

Definition 52 (Linear functional). A linear functional $\varphi : V \rightarrow \mathbb{F}$ is a linear map, with \mathbb{F} considered as a 1 dimensional vector space over itself.

Theorem 20 (Riesz's Representation Theorem). *Let V be an inner product space, s.t. $n = \dim V$. Let $\varphi : V \rightarrow \mathbb{F}$ be any linear functional. Then there exists a unique vector $u \in V$, dependent on φ , s.t. $\forall v \in V$,*

$$\varphi(v) = \langle v, u \rangle$$

Proof. If possible, let $u_1, u_2 \in V$, s.t. $\forall v \in V$,

$$\varphi(v) = \langle v, u_1 \rangle = \langle v, u_2 \rangle$$

Therefore,

$$\langle v, u_1 - u_2 \rangle = 0$$

Let $v = u_1 - u_2$. Therefore,

$$\begin{aligned} \langle v, u_1 - u_2 \rangle &= \langle u_1 - u_2, u_1 - u_2 \rangle \\ \therefore \langle u_1 - u_2, u_1 - u_2 \rangle &= 0 \\ \therefore u_1 - u_2 &= 0 \\ \therefore u_1 &= u_2 \end{aligned}$$

Therefore, u , if it exists, is unique.

Let

$$\begin{aligned} B &= \{v_1, \dots, v_n\} \\ \tilde{B} &= \{1\} \end{aligned}$$

be orthonormal bases of V and \mathbb{F} respectively.

Let

$$\begin{aligned} A &= [\varphi]_{B, \tilde{B}} \\ &= (\alpha_1 \quad \dots \quad \alpha_n) \end{aligned}$$

be the representation matrix.

Therefore,

$$[\varphi(v)]_{\tilde{B}} = A[v]_B$$

Let

$$\begin{aligned} v &= \beta_1 v_1 + \dots + \beta_n v_n \\ \therefore [v]_B &= \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \end{aligned}$$

Therefore,

$$\begin{aligned} [\varphi(v)]_{\tilde{B}} &= (\alpha_1 \quad \dots \quad \alpha_n) \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \\ &= \alpha_1 \beta_1 + \dots + \alpha_n \beta_n \\ &= \beta_1 \alpha_1 + \dots + \beta_n \alpha_n \\ &= \beta_1 \overline{\overline{\alpha_1}} + \dots + \beta_n \overline{\overline{\alpha_n}} \\ &= (\beta_1 \quad \dots \quad \beta_n) \overline{\begin{pmatrix} \overline{\alpha_1} \\ \vdots \\ \overline{\alpha_n} \end{pmatrix}} \\ &= \left\langle \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}, \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \right\rangle \end{aligned}$$

Let $u \in V$, s.t.

$$[u]_B = \begin{pmatrix} \overline{\alpha_1} \\ \vdots \\ \overline{\alpha_n} \end{pmatrix}$$

$$[\varphi(v)]_{\tilde{B}} = \langle v, u \rangle$$

and

$$\begin{aligned} [\varphi(v)]_{\tilde{B}} &= \varphi(v) \cdot 1 \\ \therefore \varphi(v) &= \langle v, u \rangle \end{aligned}$$

□

12.1 Construction

1. Let $T : V \rightarrow W$ be a linear map.
2. Fix $w \in W$.
3. Let $\varphi_w : V \rightarrow \mathbb{F}$ be a linear functional, s.t. $\varphi_w(v) = \langle T(v), w \rangle$.
 $\varphi_w(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 \varphi_w(v_1) + \alpha_2 \varphi_w(v_2)$.
4. By Riesz's Representation Theorem, $\exists! u \in V$, s.t. $\varphi_w(v) = \langle v, u \rangle$.
5. Define $T^*(w) = u$.
 Therefore, it can be expressed as

$$\langle T(v), w \rangle = \langle v, T^*(w) \rangle$$

12.2 Properties

Theorem 21. *Let B be an orthonormal basis of V and let \tilde{B} be an orthonormal basis of W . Let $A = [T]_{B, \tilde{B}}$ be the representing matrix of $T : V \rightarrow W$ with respect to B, \tilde{B} . Let $\tilde{A} = [T^*]_{B, \tilde{B}}$ be the representing matrix of $T^* : W \rightarrow V$ with respect to B, \tilde{B} . Then*

$$\tilde{A} = \overline{A}^t = A^*$$

Theorem 22. *If $T_1, T_2 : V \rightarrow W$, then*

$$(T_1 + T_2)^* = T_1^* + T_2^*$$

Theorem 23. If $T : V \rightarrow W$, $\alpha \in \mathbb{F}$, then

$$(\alpha T)^* = \bar{\alpha} T^*$$

Theorem 24.

$$(T^*)^*$$

Theorem 25. If $T : V \rightarrow W$, $S : W \rightarrow U$, then

$$(S \circ T)^* = T^* \circ S^*$$

13 Special Linear Operators

Definition 53. Let $T : V \rightarrow V$ be a linear operator, and let $T^* : V \rightarrow V$ is the adjoint operator.

T is said to be

1. normal if $T^* \circ T = T \circ T^*$
2. self-adjoint if $T^* = T$ (If $\mathbb{F} = \mathbb{R}$, T is called symmetric.)
3. unitary if $T^* = T^{-1}$ (If $\mathbb{F} = \mathbb{R}$, T is called symmetric.)

Remark 8. The same terminology is used for square matrices.

Remark 9. If B is orthonormal basis of V , $A = [T]_B$, then A is the normal, self-adjoint or unitary according to T .

Theorem 26. Let $v \in V$. T is normal if and only if

$$\|T(v)\| = \|T^*(v)\|$$

Corollary 26.1. Let $T : V \rightarrow V$ be normal, let λ be its eigenvalue, and let v be an eigenvector of T corresponding to λ . Then $\bar{\lambda}$ is an eigenvalue of T^* , and v is an eigenvector of T^* corresponding to $\bar{\lambda}$.

Theorem 27. If T is normal, λ_1, λ_2 are its eigenvalues, v_1, v_2 are eigenvectors corresponding to λ_1, λ_2 respectively. If $\lambda_1 \neq \lambda_2$, then $v_1 \perp v_2$.

Theorem 28. Let T be a self-adjoint operator. Then any eigenvalue λ of T is real.

Theorem 29. Let $T : V \rightarrow V$ be a unitary operator. Then

1. T preserves inner products.
2. T preserves norms.
3. T preserves distances.
4. T preserves angles (in real case).