Numerical Analysis

Aakash Jog

2015-16

Contents

1 Lecturer Information

Naftali Landsberg

Office: Kitot 215
Telephone: +972 3-640-6422
E-mail: naftali.landsberg@gmail.com
Office Hours: Sundays, 12:00

2 Recommended Reading

1. B. P. Lathi, Linear Systems and Signals, Oxford University Press (2nd Edition), 2005
2. Di Stefano et al, Feedback and Control Systems (SchaumâĂŹs Outline Series)
3. DâĂŹAzzo, J. and C. Houpis, Linear Control System Analysis & Design. 4th ed., McGraw Hill, 1995
4. K. Ogata, Modern Control Engineering, Prentice Hall (5th edition 2005)
5. K. Ogata, Discrete-time control systems, Prentice Hall (2nd Edition 1995)

3 Classification of Systems

1. Linear and Non-linear
2. Causal and Non-causal
3. Time invariant and Time variant

Definition 1. A system is said to be linear if it satisfies the following criteria.

1. Superposition
   If $u_1 \to y_1$ and $u_2 \to y_2$, then $(u_1 + u_2) \to (y_1 + y_3)$.
2. Homogenity
   If $u \to y$, then $\alpha u \to \alpha y$, where α is a constant.

Theorem 1. *Every linear system can be described by an ODE of the type*

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y = b_m u^{(m)} + \cdots + b_1 u^{(1)} + b_0 u$$

where $m < n$.

4 Time-domain Analysis of Linear Time-invariant Systems

Definition 2 (Step function)**.**

$$\delta_{-1}(t) = \begin{cases} 0 & ; \quad t < 0 \\ 1 & ; \quad t > 0 \end{cases}$$

Definition 3 (Delta function)**.**

$$\delta(t) = \begin{cases} 0 & ; \quad t \neq 0 \\ \to \infty & ; \quad t = 0 \end{cases}$$

Definition 4 (Ramp function)**.**

$$\delta_{-2}(t) = \begin{cases} 0 & ; \quad t < 0 \\ t & ; \quad t \geq 0 \end{cases}$$

Theorem 2.

$$f(t)\delta(t) = f(0)\delta(t)$$

Theorem 3.

$$\int_{0^-}^{t} f(\tau)\delta(\tau) = \int_{0^-}^{t} f(0)\delta(\tau)\,\mathrm{d}\tau$$
$$= f(0) \quad , \quad t > 0$$

Theorem 4.

$$\int_{0^-}^{t} f(\tau)\delta(t-\tau)\,\mathrm{d}\tau = \int_{0^-}^{t} f(t)\delta(t-\tau)\,\mathrm{d}\tau$$
$$= f(t)\int_{0^-}^{t} \delta(t-\tau)\,\mathrm{d}\tau$$
$$= f(t)$$

Theorem 5.

$$\int_{0^-}^{t} f(\tau)\delta(t-\tau)\,\mathrm{d}\tau = f(t)$$
$$= f(t) * \delta(t)$$

Exercise 1.
Find the solution for

$$y^{(2)} + 5y^{(1)} + 6y = u(t)$$
$$y(0^-) = 1$$
$$y'(0^-) = 2$$
$$u(t) = \delta_{-1}(t)$$

Solution 1.

$$y^{(2)} + 5y^{(1)} + 6y = u(t)$$

Therefore, the corresponding homogeneous ODE is

$$y^{(2)} + 5y^{(1)} + 6y = 0$$

Therefore, the corresponding characteristic equation is

$$\lambda^2 + 5\lambda + 6 = 0$$

Therefore,

$$\lambda_1 = -2$$
$$\lambda_2 = -3$$

Therefore, the ZIR solution is

$$\begin{aligned} y_{\text{ZIR}}(t) &= Ae^{\lambda_1 t} + Be^{\lambda_2 t} \\ &= Ae^{-2t} + Be^{-3t} \end{aligned}$$

Substituting the initial conditions,

$$\begin{aligned} A + B &= 1 \\ -2A - 3B &= 2 \end{aligned}$$

Therefore, the matrix form of the system of equations is

$$\begin{pmatrix} 1 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Therefore, solving,

$$A = 5$$
$$B = -4$$

Therefore,

$$y_{\text{ZIR}}(t) = 5e^{-2t} - 4e_{-3t}$$

The ZSR solution is,

$$y_{\text{ZSR}}(t) = \alpha e^{-2t} + \beta e^{-3t} + y_p$$

As $u(t) = \delta_{-1}(t)$,

$$y = c$$

Therefore, substituting into the ODE, considering zero initial conditions, for $t > 0$,

$$6c = 1$$

Therefore,

$$y_{\text{ZSR}}(t) = \left(\alpha e^{-2t} + \beta e^{-3t} + \frac{1}{6} \right) \delta_{-1}(t)$$

As this is a ZSR case, the solution is zero for $t < 0$. Hence, $\delta_{-1}(t)$ can be written on the right side. This is not necessarily true for the ZIR case.
Therefore,

$$y_{\text{ZSR}}(0) = \frac{1}{6} + \alpha + \beta$$

Also, as the ZSR solution is zero at zero,

$$0 = \frac{1}{6} + \alpha + \beta$$

Differentiating $y_{\text{ZSR}}(t)$,

$$y'_{\text{ZSR}}(0) = (-2\alpha - 3\beta)\delta_{-1}(t) + \left(\frac{1}{6} + \alpha e^{-2t} + \beta e^{-3t}\right)\delta(t)$$

As $f(t)\delta(t) = f(0)\delta(t)$,

$$\begin{aligned} y'_{\text{ZSR}}(0) &= (-2\alpha - 3\beta)\delta_{-1}(t) + \left(\frac{1}{6} + \alpha + \beta\right)\delta(t) \\ &= (-2\alpha - 3\beta)\delta_{-1}(t) \end{aligned}$$

Therefore, solving,

$$\alpha = -\frac{1}{2}$$
$$\beta = \frac{1}{3}$$

Therefore,

$$\begin{aligned} y_{\text{total}}(t) &= y_{\text{ZSR}} + y_{\text{ZIR}} \\ &= \left(\frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t}\right)\delta_{-1}(t) + 5e^{-2t} - 4e^{-3t} \\ &= \frac{1}{6} + \frac{9}{2}e^{-2t} - \frac{11}{3}e^{-3t} \quad , \quad t > 0 \end{aligned}$$

The same solution can be found by solving for $u(t) = \delta(t)$ and then convolving the solution thus found, and the actual input $u(t) = \delta_{-1}(t)$.
Therefore, the new ODE is

$$\begin{aligned} y^{(2)} + 5y^{(1)} + 6y &= \delta(t) \\ y(0^-) &= 0 \\ y-(0^-) &= 0 \end{aligned}$$

The impulse response y_δ can be calculated by finding the response for the step function $y_{\delta_{-1}\text{ZSR}}(t)$, and then differentiating it.

$$\begin{aligned}
y_{\delta_{-1}\text{ZSR}}(t) &= \left(\frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t}\right)\delta_{-1}(t) \\
\therefore y_{\delta\text{ZSR}}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\, y_{\delta_{-1}\text{ZSR}}(t) \\
&= \left(e^{-2t} - e^{-3t}\right)\delta_{-1}(t) + \left(\frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t}\right)\delta(t)
\end{aligned}$$

Else, the impulse response y_δ can be calculated by integrating the ODE around zero and finding the new initial conditions for $t = 0^+$.
Therefore,

$$y^{(2)} + 5y^{(1)} + 6y = \delta(t)$$

$$\therefore \int\limits_{0^-}^{0^+} y''\,\mathrm{d}t + \int\limits_{0^-}^{0^+} 5y'\,\mathrm{d}t + \int\limits_{0^-}^{0^+} 6y\,\mathrm{d}t = 1$$

Let

$$\begin{aligned}
y'' &= \frac{1}{a}\delta(t) \\
\therefore y' &= \frac{1}{a}\delta_{-1}(t) \\
\therefore y' &= \frac{1}{a}\delta_{-2}(t)
\end{aligned}$$

Therefore, substituting,

$$y'(0^+) - y'(0^-) + 5\left(y(0^+) - y(0^-)\right) + 6\left(\int y\Big|_{0^+} - \int y\Big|_{0^-}\right) = 1$$

Substituting the initial conditions,

$$y'(0^+) = 1$$

Similarly for $t > 0$.

5 Impulse Response

5.1 Finding the General Solution for an ODE

$$\begin{aligned}\sum_{k=0}^{n} a_k y^{(k)}(t) &= \sum_{k=0}^{m} b_k u^{(k)}(t) \\ y(0^-) &= 0 \\ y^{(1)}(0^-) &= 0 \\ &\vdots \\ y^{(n-1)}(0^-) &= 0\end{aligned}$$

where

$$u(t) = f(t)\delta_{-1}(t)$$

1. Solve

$$\begin{aligned}\sum_{k=0}^{n} a_k y^{(k)}(t) &= u(t) \\ y(0^-) &= 0 \\ y^{(1)}(0^-) &= 0 \\ &\vdots \\ y^{(n-1)}(0^-) &= 0\end{aligned}$$

   where

$$u(t) = f(t)\delta_{-1}(t)$$

   (a) Solve

$$\begin{aligned}\sum_{k=0}^{n} a_k y^{(k)}(t) &= \delta(t) \\ y(0^-) &= 0 \\ y^{(1)}(0^-) &= 0 \\ &\vdots \\ y^{(n-1)}(0^-) &= 0\end{aligned}$$

   where

$$u(t) = f(t)\delta_{-1}(t)$$

i. Solve

$$\begin{aligned}\sum_{k=0}^{n} a_k y^{(k)}(t) &= 0\\ y(0^-) &= 0\\ y^{(1)}(0^-) &= 0\\ &\vdots\\ y^{(n-2)}(0^-) &= 0\\ y^{(n-1)}(0^-) &= \frac{1}{a_n}\end{aligned}$$

ii. Let this solution be y_δ

2. Let this solution be $y(t)$.
Therefore,

$$y_f(t) = \int_0^t f(\tau) y_\delta(t-\tau)\,\mathrm{d}\tau$$

3. The solution for

$$\begin{aligned}\sum_{k=0}^{n} a_k y^{(k)}(t) &= \sum_{k=0}^{m} b_k u^{(k)}\\ y(0^-) &= 0\\ y^{(1)}(0^-) &= 0\\ &\vdots\\ y^{(n-1)}(0^-) &= 0\end{aligned}$$

where

$$u(t) = f(t)\delta_{-1}(t)$$

is

$$y_{\mathrm{ZSR}}(t) = \sum_{k=0}^{n} b_k {y_f}^{(k)}(t)$$

Definition 5 (Convolution)**.**

$$h(t) * f(t) = \int_{-\infty}^{\infty} h(\tau) f(t-\tau)\,\mathrm{d}\tau$$
$$= \int_{-\infty}^{\infty} h(t-\tau) f(\tau)\,\mathrm{d}\tau$$

Theorem 6. *If*

$$h(t<0) = 0$$
$$f(t<0) = 0$$

then

$$h(t) * f(t) = \int_{0}^{t} h(\tau) f(t-\tau)\,\mathrm{d}\tau$$
$$= \int_{0}^{t} h(t-\tau) f(\tau)\,\mathrm{d}\tau$$

Exercise 2.
Let

$$h(t) = \left(Ae^{-\alpha t} + Be^{-\beta t}\right)\delta_{-1}(t)$$
$$f(t) = \sin(at)\delta_{-1}(t)$$

Find $h(t) * f(t)$.

Solution 2.

$$h(t) * f(t) = \int_{0}^{t} h(t-\tau) f(\tau)\,\mathrm{d}\tau$$
$$= \int_{0}^{t} \left(Ae^{-\alpha(t-\tau)} + Be^{-\beta(t-\tau)}\right)\sin(\alpha t)\,\mathrm{d}\tau$$

6 Laplace Transform

Definition 6 (One-sided Laplace transform)**.**

$$\mathcal{L}\left\{y(t)\right\} = \int\limits_{0^-}^{\infty} y(t)e^{-st}\,\mathrm{d}t$$
$$= Y(s)$$

Theorem 7. *If* $f(0^-) = 0$*,*

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s)$$

If $f(0^-) \neq 0$*,*

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0)$$

Theorem 8. *If*

$$F(s) = \mathcal{L}\left\{f(t)\right\}$$

$f(t)$	$F(s)$
$\delta(t)$	1
$\delta(t - t_0)$	e^{-st_0}
$\delta_{-1}(t)$	$\frac{1}{s}$
$e^{-at}\delta_{-1}(t)$	$\frac{1}{s+a}$

Exercise 3.
Solve

$$y'' + 5y' + 6y = u(t) + 2u'(t)$$
$$y(0^-) = 1$$
$$y'(0^-) = 2$$

where

$$u(t) = \delta_{-1}(t)$$

Solution 3.

$$
\begin{aligned}
y'' + 5y' + 6y &= u(t) + 2u'(t) \\
\therefore \mathcal{L}\left\{y'' + 5y' + 6y\right\} &= \mathcal{L}\left\{u(t) + 2u'(t)\right\}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
2\left(sU(s) - u(0)\right) &= s^2 Y(s) - sy(0^-) - y'(0^-) + 5\left(sY(s) - y(0^-)\right) + 6Y(s) \\
\therefore Y(s)\left(s^2 + 5s + 6\right) &= U(s)\left(2s + 1\right) + sy(0^-) + 5y(0^-) - 2u(0^-)
\end{aligned}
$$

Therefore,

As $u(t) = \delta_{-1}(t)$, $U(s) = \frac{1}{s}$

$$
\begin{aligned}
Y(s) &= \frac{2s+1}{(s+2)(s+3)} U(s) + \frac{s+7}{(s+2)(s+3)} \\
&= \frac{2s+1}{s(s+2)(s+3)} + \frac{s+7}{(s+2)(s+3)}
\end{aligned}
$$

Let

Therefore, solving,

$$
\begin{aligned}
A &= \frac{1}{6} \\
B &= \frac{3}{2} \\
C &= -\frac{5}{3}
\end{aligned}
$$

Therefore,

Therefore,

Theorem 9. *Let*

$$
F(s) = \frac{A_0}{(s-\lambda)^q} + \frac{A_1}{(s-\lambda)^{q-1}} + \cdots + \frac{A_{q-1}}{s-\lambda}
$$

Then

$$A_k = \frac{1}{k!} \frac{\mathrm{d}^k}{\mathrm{d}s^k} \left(Q(s)\right)\Bigg|_{s=\lambda}$$

where

$$\begin{aligned} k &\in \{0, \dots, q-1\} \\ Q(s) &= (s-\lambda)^q F(s) \end{aligned}$$

Theorem 10. *The solution of*

$$\sum_{k=0}^{n} a_k y^{(k)} = \sum_{k=0}^{m} b_k u^{(k)}$$

where

$$u(t) = f(t)\delta_{-1}(t)$$

is

$$Y(s) = \underbrace{\underbrace{\frac{B(s)}{A(s)}}_{\textit{transfer function}} U(s)}_{\textit{contribution of the input}} + \underbrace{\frac{Q(s)}{A(s)}}_{\textit{contribution of the initial conditions}}$$