Harmonic Analysis : Recitations

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1 Instructor Information

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Part I
Fourier Series

1 Fourier Series

Definition 1 (Real Fourier series)**.** Let $f : [-L, L] \in \mathbb{C}$ be a piecewise continuous function.
The series

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

is called the Fourier series of $f(x)$, where

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \,\mathrm{d}x$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(nx) \,\mathrm{d}x$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(nx) \,\mathrm{d}x$$

Theorem 1. *If $f(x)$ is an even function, then the appropriate Fourier series is*

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

If $f(x)$ is an odd function, then the appropriate Fourier series is

If $f(x)$ is odd, its graph always passes through the origin. Therefore, it can be represented by a summation of sine functions, which also pass through the origin, and there is no need for a term, i.e. $\frac{a_0}{2}$, to change its position at the origin.

$$f(x) \approx \sum_{n=1}^{\infty} a_n \sin(nx)$$

Definition 2 (Complex Fourier series)**.** Let $f : [-L, L] \in \mathbb{C}$ be a piecewise continuous function.
The series

$$f(x) \approx \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

is called the complex Fourier series of $f(x)$, where

$$c_n = \frac{1}{2L} \int\limits_{-L}^{L} f(x) e^{-inx} \, \mathrm{d}x$$

Recitation 1 – Exercise 1.
Calculate the real Fourier series of

$$f(x) = 2x - 2\pi$$

Recitation 1 – Solution 1.
As x is an odd function, the real Fourier series of x, in the interval $[-\pi, \pi]$ is

$$x \approx \sum_{n=1}^{\infty} b_n \sin(nx)$$

where

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int\limits_{-\pi}^{\pi} x \sin(nx) \, \mathrm{d}x \\
&= \frac{1}{\pi} \left(x \int \sin(nx) \, \mathrm{d}x - \int 1 \left(\int \sin(nx) \, \mathrm{d}x \right) \mathrm{d}x \right) \Bigg|_{-\pi}^{\pi} \\
&= \frac{1}{\pi} \left(- \frac{x \cos(nx)}{n} \right) \Bigg|_{-\pi}^{\pi} + \frac{1}{\pi} \int\limits_{-\pi}^{\pi} \frac{\cos(nx)}{n} \, \mathrm{d}x \\
&= \frac{1}{\pi} \left(- \frac{\pi \cos(n\pi) + \pi \cos(-n\pi)}{n} \right) + \cancelto{0}{\frac{1}{\pi} \frac{\sin(nx)}{n^2} \Bigg|_{-\pi}^{\pi}} \\
&= - \frac{\cos(n\pi) + \cos(n\pi)}{n} \\
&= -2 \frac{\cos(n\pi)}{n} \\
&= -2 \frac{(-1)^n}{n}
\end{aligned}$$

Therefore,

$$x \approx 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

Therefore,

$$2x - 2\pi \approx \left(4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)\right) - 2\pi$$

2 Bessel's Inequality

Definition 3 (Piecewise continuous functions)**.** $f : \mathbb{R} \to \mathbb{R}$ is said to be piecewise continuous if, for every finite interval $[a, b]$ there is a finite number of discontinuity points, and the one-sided limits at each of these points are also finite.

Definition 4 (Piecewise continuously differentiable functions)**.** $f : \mathbb{R} \to \mathbb{R}$ is said to be piecewise continuously differentiable if it is piecewise continuous, and

$$\lim_{h\to 0^+} \frac{f(x+h) - f(x^+)}{h} < \infty$$

and

$$\lim_{h\to 0^-} \frac{f(x+h) - f(x^-)}{h} < \infty$$

Theorem 2 (Bessel's Inequality)**.** *Let $f(x)$ be a piecewise continuous function defined on $[-L, L]$. Then*

$$\frac{1}{2}{a_0}^2 + \sum_{n=1}^{\infty} {a_n}^2 + {b_n}^2 \leq \frac{1}{L}\int_{-L}^{L} f(x)^2 \, \mathrm{d}x$$

3 Riemann-Lebesgue's Lemma

Theorem 3 (Riemann-Lebesgue's Lemma)**.** *If $f(x)$ is piecewise continuous on $[-L, L]$, then*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$$

4 Dirichlet's Kernel

Definition 5 (Dirichlet kernel)**.**

$$\begin{aligned} D_m(t) &= \frac{1}{2}\sum_{n=-m}^{m} e^{-int} \\ &= \frac{1}{2} + \sum_{n=1}^{m} \cos(nt) \\ &= \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{2\sin\frac{t}{2}} \end{aligned}$$

is called the Dirichlet kernel of order m.

Theorem 4 (Second representation of Dirichlet's kernel)**.** *Let $m \in \mathbb{N}$. Then, for $t \neq 2\pi k$, where $k \in \mathbb{Z}$,*

$$\begin{aligned} D_m(t) &= \frac{1}{2} + \cos(t) + \cos(2t) + \cdots + \cos(mt) \\ &= \frac{\sin\left(\left(m+\frac{1}{2}\right)t\right)}{2\sin\left(\frac{1}{2}t\right)} \end{aligned}$$

Theorem 5. *Let*

$$S_m(f,x) = \frac{1}{2}a_0 + \sum_{n=1}^{m} a_n\cos(nx) + b_n\sin(nx)$$

Then,

$$S_m(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\left(\frac{1}{2}\sum_{n=1}^{m}\cos(nt)\right)\mathrm{d}t$$

Theorem 6 (Dirichlet Theorem)**.** *Let $f : [-\pi,\pi] \to \mathbb{R}$ be a piecewise continuously differentiable function.*
Then, $\forall x \in (-\pi,\pi)$,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx) = \frac{f(x^-)+f(x^+)}{2}$$

and for $x = \pi$ or $x = -\pi$,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx) = \frac{f(\pi^-)+f(-\pi^+)}{2}$$

Recitation 2 – Exercise 1.
The Fourier series of x^2 of given to be

$$x^2 \approx \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

Calculate

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Recitation 2 – Solution 1.
As x^2 is continuous, with a continuous derivative, Dirichlet Theorem is applicable.
Therefore, let

$$x = \pi$$

Therefore, by Dirichlet Theorem,

$$\begin{aligned}
\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) &= \frac{\left(\pi^-\right)^2 + \left((-\pi)^+\right)^2}{2} \\
\therefore \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n &= \pi^2 \\
\therefore \frac{\pi^2}{4} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} &= \pi^2 \\
\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{1}{4}\left(\pi^2 - \frac{\pi^2}{3}\right) \\
&= \frac{\pi^2}{6}
\end{aligned}$$

Recitation 2 – Exercise 2.
The Fourier series of

$$f(x) = \begin{cases} x & ; \quad 0 \leq x \leq \pi \\ 0 & ; \quad -\pi \leq x \leq 0 \end{cases}$$

is given to be

$$f(x) \approx \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}}{n} \sin(nx) - \frac{2}{\pi(2n-1)^2} \cos\left((2n-1)x\right) \right)$$

Let this Fourier series be denoted by $S(x)$.
Calculate

1. $1+\frac{1}{3^2}+\frac{1}{5^2}+\dots$

2. $S\left(\frac{\pi}{2}\right)$

Recitation 2 – Solution 2.

1.

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\dots=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

Therefore, for $x=0$, by Dirichlet Theorem,

$$\begin{aligned}\frac{\pi}{4}+\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n}\sin(0)-\frac{2}{\pi(2n-1)^2}\cos(0)\right)&=\frac{f(0^-)+f(0^+)}{2}\\ \therefore\frac{\pi}{4}-\sum_{n=1}^{\infty}\left(\frac{2}{\pi(2n-1)^2}\right)&=0\\ \therefore\frac{\pi}{4}-\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{(2n-1)^2}\right)&=0\\ \therefore\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}&=\frac{\pi^2}{8}\end{aligned}$$

2. By Dirichlet Theorem,

$$\begin{aligned}S\left(\frac{\pi}{2}\right)&=\frac{f\left(\frac{\pi}{2}^-\right)+f\left(\frac{\pi}{2}^+\right)}{2}\\ &=\frac{\pi}{2}\end{aligned}$$

Theorem 7. *If f is a piecewise continuous and periodic function with period of 2π, then*

$$\begin{aligned}S_m(x)&=\frac{a_0}{2}+\sum_{n=1}^{m}\left(a_n\cos(nx)+b_n\sin(nx)\right)\\ &=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)D_m(t)\,\mathrm{d}t\end{aligned}$$

Recitation 3 – Exercise 1.
Calculate the limit

$$L = \lim_{n\to\infty} \int\limits_{-n}^{n} \sin\left(\frac{2n+1}{2}t\right) \frac{\cos^2\left(\frac{\pi}{4}+t\right)+\pi^2}{\sin\left(\frac{t}{2}\right)}\,\mathrm{d}t$$

Recitation 3 – Solution 1.

$$\begin{aligned}
L &= \lim_{n\to\infty} \int\limits_{-n}^{n} \sin\left(\frac{2n+1}{2}t\right) \frac{\cos^2\left(\frac{\pi}{4}+t\right)+\pi^2}{\sin\left(\frac{t}{2}\right)}\,\mathrm{d}t \\
&= 2\lim_{n\to\infty} \int\limits_{-\pi}^{\pi} \left(\cos^2\left(\frac{\pi}{4}+t\right)+\pi^2\right) \frac{\sin\left(n+\frac{1}{2}t\right)}{2\sin\left(\frac{t}{2}\right)}\,\mathrm{d}t \\
&= 2\lim_{n\to\infty} \int\limits_{-\pi}^{\pi} \left(\cos^2\left(\frac{\pi}{4}+t\right)+\pi^2\right) D_n(t)\,\mathrm{d}t
\end{aligned}$$

Let

$$f(x) = \cos^2 x + \pi^2$$

Let S_n be the partial sum of the Fourier series.
Therefore,

$$S_n = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} D_n(t)\left(\cos^2(x+t)+\pi^2\right)\mathrm{d}t$$

Therefore,

$$\begin{aligned}
L &= 2\pi \lim_{n\to\infty} \frac{1}{\pi} \int\limits_{-\pi}^{\pi} \left(\cos^2\left(\frac{\pi}{4}+t\right)+\pi^2\right) D_n(t)\,\mathrm{d}t \\
&= 2\pi \lim_{n\to\infty} S_n\left(\frac{\pi}{4}\right) \\
&= 2\pi \frac{f\left(\frac{\pi}{4}^+\right)+f\left(\frac{\pi}{4}^-\right)}{2} \\
&= 2\pi f\left(\frac{\pi}{4}\right) \\
&= 2\pi\left(\cos^2\left(\frac{\pi}{4}\right)+\pi^2\right) \\
&= \pi + 2\pi^3
\end{aligned}$$

5 Fourier Series in a General Interval

Definition 6. Let f be a piecewise continuous function defined on $[a, b]$. The Fourier series over $[a, b]$ is defined as

$$\begin{aligned} f(x) &\approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n x}{b-a}\right) + b_n \sin\left(\frac{2\pi n x}{b-a}\right) \right) \\ &\approx \sum_{-\infty}^{\infty} c_n e^{\frac{2\pi i n x}{b-a}} \end{aligned}$$

where

$$\begin{aligned} a_0 &= \frac{1}{b-a} \int_a^b f(x) \, \mathrm{d}x \\ a_n &= \frac{2}{b-a} \int_a^b f(x) \cos \frac{2\pi n x}{b-a} \, \mathrm{d}x \\ b_n &= \frac{2}{b-a} \int_a^b f(x) \sin \frac{2\pi n x}{b-a} \, \mathrm{d}x \\ c_n &= \frac{1}{b-a} \int_a^b f(x) e^{\frac{2\pi i n x}{b-a}} \, \mathrm{d}x \end{aligned}$$

Recitation 3 – Exercise 2.
Develop the Fourier series for $\operatorname{sign}(x)$ over $[0, \pi]$.

Recitation 3 – Solution 2.

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} \operatorname{sign}(x) \, \mathrm{d}x \\ &= 2 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} \operatorname{sign}(x) \cos\left(\frac{2\pi n x}{\pi}\right) \mathrm{d}x \\ &= \frac{2}{\pi} \int_0^{\pi} \cos(2nx) \, \mathrm{d}x \\ &= 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{\pi} \int\limits_0^{\pi} \operatorname{sign}(x) \sin\left(\frac{2\pi n x}{\pi}\right) \mathrm{d}x \\ &= \frac{2}{\pi} \int\limits_0^{\pi} \sin(2nx) \, \mathrm{d}x \\ &= 0 \end{aligned}$$

Therefore, over $[0, \pi]$,

$$\begin{aligned} \operatorname{sign}(x) &= \frac{2}{2} + \sum_{n=1}^{\infty} 0 \\ &= 1 \end{aligned}$$

Theorem 8. *Let f be continuous in $[-\pi, \pi]$, with piecewise continuous derivative, and $f(-\pi) = f(\pi)$. Then, the Fourier series converges uniformly on $[-\pi, \pi]$.*

Theorem 9 (Percival Equality)**.** *Let f be a piecewise continuous function in $[-\pi, \pi]$. Then,*

$$\begin{aligned} \frac{1}{\pi} \int\limits_{-\pi}^{\pi} |f(x)|^2 \, \mathrm{d}x &= \frac{{a_0}^2}{2} + \sum_{n=1}^{\infty} \left({a_n}^2 + {b_n}^2\right) \\ &= 2 \sum_{n=-\infty}^{\infty} |c_n|^2 \end{aligned}$$

Recitation 4 – Exercise 1.

Use the Fourier series

$$x^2 \approx \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos(nx)$$

to calculate $\sum\limits_{n=1}^{\infty} \frac{1}{n^4}$.

Recitation 4 – Solution 1.
As x^2 is continuous, by Percival Equality,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left|x^2\right|^2 \mathrm{d}x = \frac{{a_0}^2}{2} + \sum_{n=1}^{\infty} {a_n}^2 + {b_n}^2$$
$$= \frac{2\pi^4}{9} + \sum_{n=1}^{\infty}\frac{16}{n^2}$$

Therefore,

$$16\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{\pi}\int_{\pi}^{\pi} x^4 \, \mathrm{d}x - \frac{2\pi^4}{9}$$
$$= \frac{1}{\pi}\left.\frac{x^5}{5}\right|_{-\pi}^{\pi} - \frac{2\pi^4}{9}$$
$$= \frac{2\pi^4}{5} - \frac{2\pi^4}{9}$$
$$= \frac{8}{45}\pi^4$$
$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

Recitation 4 – Exercise 2.
Use the Fourier series

$$e^x \approx \sum_{n=\infty}^{\infty}(-1)^n\frac{e^{\pi}-e^{-\pi}}{2\pi(1-in)}e^{inx}$$

to calculate $\sum\limits_{n-\infty}^{\infty}\frac{1}{n^2+1}$.

Recitation 4 – Solution 2.
As x^2 is continuous, by Percival Equality,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|e^x|^2 \,\mathrm{d}x = 2\sum_{n=-\infty}^{\infty}|c_n|^2$$
$$= 2\sum_{n=-\infty}^{\infty}\frac{\left(e^{\pi}-e^{-\pi}\right)^2}{4\pi^2|1-in|^2}$$
$$= \frac{2\left(e^{\pi}-e^{-\pi}\right)^2}{4\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{n^2+1}$$

Therefore,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1} &= \frac{4\pi}{2\left(e^{\pi}-e^{-\pi}\right)^2} \int_{-\pi}^{\pi} |e^x|^2 \, \mathrm{d}x \\
&= \frac{4\pi}{2\left(e^{\pi}-e^{-\pi}\right)^2} \left.\frac{e^{2x}}{2}\right|_{-\pi}^{\pi} |e^x|^2 \, \mathrm{d}x \\
&= \frac{4\pi}{2\left(e^{\pi}-e^{-\pi}\right)^2} \frac{e^{2\pi}-e^{-2\pi}}{2} \\
&= \frac{e^{2\pi}-e^{-2\pi}}{\left(e^{\pi}-e^{-\pi}\right)^2} \\
&= \frac{\left(e^{\pi}+e^{-\pi}\right)\left(e^{\pi}-e^{-\pi}\right)}{\left(e^{\pi}-e^{-\pi}\right)^2} \\
&= \frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}
\end{aligned}$$

Recitation 5 – Exercise 1.
Use

$$|x| \approx \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \cos\left((2n-1)x\right)$$

to compute the Fourier series of

$$\operatorname{sign}(x) = \begin{cases} 1 & ; \quad x \geq 0 \\ -1 & ; \quad x < 0 \end{cases}$$

Recitation 5 – Solution 1.
As $|x|$ is continuous, piecewise differentiable, and $|-\pi| = |\pi|$, its Fourier series converges uniformly. Hence, the Fourier series can be differentiated term by term.
Therefore,

$$|x|' = \operatorname{sign}(x)$$

Therefore,

$$\begin{aligned}
\operatorname{sign}(x) &\approx \left(\frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \cos\left((2n-1)x\right)\right)' \\
&\approx \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \sin\left((2n-1)x\right)
\end{aligned}$$

Recitation 5 – Exercise 2.
f is given to be continuous, piecewise differentiable, and periodic with period 2π. Also

$$f(x) \approx \frac{a_0}{2} \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Determine whether the following are true or false.

$$\lim_{n\to\infty} na_n = \lim_{n\to\infty} nb_n = 0$$

True or false.

Recitation 5 – Solution 2.
na_n and nb_n are the Fourier coefficients for $f'(x)$. Therefore, as f is piecewise differentiable, and by Riemann-Lebesgue's Lemma,

$$\lim_{n\to\infty} na_n = \lim_{n\to\infty} nb_n = 0$$

Hence, the statement is true.

Theorem 10. *If f is piecewise continuous with Fourier series*

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

the, for all $x \in [-\pi, \pi]$,

$$\int_0^x f(t)\,\mathrm{d}t = \frac{a_0}{2}x + \sum_{n=1}^{\infty} \frac{a_n}{n} \sin(nx) - \frac{b_n}{n}\left(\cos(nx) - 1\right)$$

This is not a Fourier series due to the x in $\frac{a_0}{2}x$.
Therefore, substituting the Fourier series of x,

$$\int_0^x f(t)\,\mathrm{d}t = \sum_{n=1}^{\infty} \frac{b_n}{n} + \sum_{n=1}^{\infty} \left(\frac{a_n + (-1)^n a_0}{n} \sin(nx) - \frac{b_n}{n} \cos(nx) \right)$$

Recitation 6 – Exercise 1.
Let f be piecewise continuous with Fourier series

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Prove

$$\sum_{n=1}^{\infty} \frac{b_n}{n} \leq \infty$$

Recitation 6 – Solution 1.
Let

$$F(x) = \int_0^x f(t)\,\mathrm{d}t$$

Therefore, as $f(x)$ is piecewise continuous, $F(x)$ is also piecewise continuous. Therefore,

$$F(x) = \frac{a_0}{2}x + \sum_{n=1}^{\infty} \frac{a_n}{n}\sin(nx) - \frac{b_n}{n}(\cos(nx) - 1)$$

Therefore, the Fourier series of $F(x) - \frac{a_0}{2}x$ is

$$\begin{aligned} F(x) - \frac{a_0}{2}x &\approx \sum_{n=1}^{\infty} \frac{a_n}{n}\sin(nx) - \frac{b_n}{n}(\cos(nx) - 1) \\ &\approx \sum_{n=1}^{\infty} \frac{b_n}{n} + \sum_{n=1}^{\infty} \frac{a_n}{n}\sin(nx) - \frac{b_n}{n}\cos(nx) \end{aligned}$$

Therefore, as $F(x) - \frac{a_0}{2}x$ is piecewise continuous and finite, $\sum\limits_{n=1}^{\infty} \frac{b_n}{n}$ is also finite.

Theorem 11. *If f is 2π periodic and k times differentiable, such that the k derivatives are continuous and $f^{(k+1)}(x)$ is piecewise continuous, then,*

$$\lim_{n\to\infty} \left|n^{k+1}a_n\right| = \lim_{n\to\infty} \left|n^{k+1}b_n\right| = \lim_{n\to\infty} \left|n^{k+1}c_n\right| = 0$$

Theorem 12. *If the Fourier coefficients of a 2π periodic function satisfy*

$$|c_n| \leq \frac{c}{n^{k+1+\varepsilon}}$$

where $\varepsilon > 0$, and c is constant, then f is k times differentiable.

Recitation 7 – Exercise 1.
The Fourier series of f is

$$f(x) = \sum_{n=1}^{\infty} \frac{n^2+1}{n^4} e^{inx}$$

If f differentiable four times?

Recitation 7 – Solution 1.
As $f(x)$ equals it Fourier series, $f(x)$ must also be periodic with period 2π. If possible, let f be differentiable 4 times, with all derivatives being continuous, and the let the fifth derivative be piecewise continuous.
Therefore,

$$\begin{aligned}\lim_{n\to\infty} \left|n^5 c_n\right| &= \lim_{n\to\infty} \left|n^5 \frac{n^2+1}{n^4}\right| \\ &= \infty\end{aligned}$$

Therefore, as the limit is not zero, f is not differentiable 4 times.

Recitation 7 – Exercise 2.
Let

$$f(x) = \sum_{n\neq 0} \frac{1}{n^{2.01}} e^{inx}$$

Give an upper bound for the number of times it is differentiable.

Recitation 7 – Solution 2.
$f(x)$ is 2π periodic.

$$\lim_{n\to\infty} \left|n^3 \frac{1}{n^{2.01}}\right| = \infty$$

Therefore, f is differentiable at most twice. Also,

$$\begin{aligned}|c_n| &= \frac{1}{n^{2.01}} \\ &= \frac{1}{n^{1+1+0.01}}\end{aligned}$$

Therefore, $f(x)$ is differentiable once.

6 Inner Product Spaces

Definition 7 (Norm)**.** Let V be a vector space. A function $\|\cdot\| : V \to \mathbb{R}^+$, such that

1. $\forall v \in V$
$$\|v\| \geq 0$$
and $\|v\| = 0$ if an only if $v = \overrightarrow{0}$.
2. $\forall v \in V$ and $\alpha \in \mathbb{F}$,
$$\|\alpha v\| = |\alpha|\|v\|$$
3. $\forall u, v \in V$,
$$\|u + v\| \leq \|u\| + \|v\|$$

is called a norm.
It is usually defined as

$$\|v\| = \sqrt{\langle v, v\rangle}$$

Theorem 13 (Pythagoras Theorem)**.** *If $u, v \in V$ are orthogonal vectors in an inner product space, then*

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

Proof. As u and v are orthogonal,

$$\langle u, v\rangle = 0$$

Therefore,

$$\begin{aligned}
\|u + v\|^2 &= \langle u + v, u + v\rangle \\
&= \langle u + v, u\rangle + \langle u + v, v\rangle \\
&= \langle u, u\rangle + \langle v, u\rangle + \langle u, v\rangle + \langle v, v\rangle \\
&= \langle u, u\rangle + \langle v, v\rangle \\
&= \|u\|^2 + \|v\|^2
\end{aligned}$$

□

Recitation 8 – Exercise 1.
Let the inner product of two functions over the vector space $C^0[-\pi, \pi]$, i.e. the set of all continuous functions over $[-\pi, \pi]$ be defined as

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) \, \mathrm{d}x$$

Find the norm of f in this space.
Show

$$\|\sin(nx)\| = 1$$
$$\|\cos(nx)\| = 1$$

Recitation 8 – Solution 1.

$$\begin{aligned} \|f\| &= \sqrt{\langle f, f \rangle} \\ &= \sqrt{\frac{1}{\pi} \int_{-\pi}^{\pi} (f(x))^2 \, \mathrm{d}x} \end{aligned}$$

Therefore,

$$\begin{aligned} \|\cos(nx)\| &= \sqrt{\frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(nx) \, \mathrm{d}x} \\ &= \sqrt{\frac{1}{\pi} \pi} \\ &= 1 \end{aligned}$$

Therefore,

$$\begin{aligned} \|\sin(nx)\| &= \sqrt{\frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(nx) \, \mathrm{d}x} \\ &= \sqrt{\frac{1}{\pi} \pi} \\ &= 1 \end{aligned}$$

Definition 8. A set is said to be orthonormal if $\forall u, v \in V$,

$$\langle u, v \rangle = 0$$

and

$$\|v\| = 1$$

Theorem 14. *In the space $C^0[-\pi, \pi]$ with*

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) \, \mathrm{d}x$$

the set $\left\{ \frac{1}{\sqrt{2}} \right\} \cup \{\cos(nx)\}_{n=1}^{\infty} \cup \{\sin(nx)\}_{n=1}^{\infty}$ *is orthonormal.*

Definition 9 (Orthonormal set)**.** A set $\{c_1, \ldots, c_n\}$ is said to be orthonormal if

$$\langle c_i, c_j \rangle = \delta_{ij}$$

Definition 10 (Projection)**.** Let W be a subspace of a vector space V, such that

$$W = \operatorname{span}\{c_1, \ldots, c_n\}$$

The projection of a vector v with respect to W is defined as

$$\begin{aligned} \operatorname{proj}_W(v) &= v_W \\ &= \sum_{k=1}^{n} \frac{\langle v, c_k \rangle}{\|c_k\|^2} \end{aligned}$$

If W is orthonormal,

$$v_W = \sum_{k=1}^{n} \langle v, c_k \rangle c_k$$

Theorem 15. *Let W be a subspace of a vector space V. Let $v \in V$. Then, v_W is the best approximation for v in W, i.e.*

$$\|v - \operatorname{proj}_W(v)\| = \min_{w \in W} \|v - w\|$$

Theorem 16. *Let W be a subspace of a vector space V. Let $v \in V$. Then, $v - v_W \in W^{\perp}$, and $v_W \in W$.*

Theorem 17. *Let W be a subspace of a vector space V. Let $v \in V$.*

$$\|v\|^2 = \|v_W\|^2 + \|v - v_W\|^2$$

Recitation 9 – Exercise 1.
Let

$$W = \text{span}\, 1, \cos(2x), \sin(x)$$

and

$$f(x) = x$$

with inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,\mathrm{d}x$$

1. Find $g \in W$, which is the best approximation for f.
2. Find α, β, γ, so that the function

   $$F(\alpha, \beta, \gamma) = \Big\| f - (\alpha + \beta\cos(2x) + \gamma\sin(x)) \Big\|$$

   is minimal.

Recitation 9 – Solution 1.
1.

$$\text{proj}_W(f) = \frac{\langle f, 1 \rangle}{\|1\|^2} 1 + \frac{\langle f, \cos(2x) \rangle}{\|\cos(2x)\|^2}\cos(2x) + \frac{\langle f, \sin(x) \rangle}{\|\sin(x)\|^2}\sin(x)$$

Comparing to the standard form of the Fourier series,

$$\begin{aligned} a_0 &= \frac{\langle f, 1 \rangle}{\|1\|^2} \\ &= 0 \\ a_2 &= \frac{\langle f, \cos(2x) \rangle}{\|\cos(2x)\|^2} \\ &= 0 \\ b_1 &= \frac{\langle f, \sin(x) \rangle}{\|\sin(x)\|^2} \\ &= 2 \end{aligned}$$

Therefore,

$$g(x) = 2\sin x$$

2. The projection of f onto W is the best approximation of f. Therefore, F is minimal. Hence,

$$\alpha = 0$$
$$\beta = 0$$
$$\gamma = 2$$

Definition 11 (Complete set)**.** An orthonormal set $\{u_k\}_{k=1}^{\infty}$ is said to be complete if the only vector $v \in V$, such that $\forall k$,

$$\langle v, u_k \rangle = 0$$

is the zero vector.

Definition 12 (Hilbert space)**.** A inner product, normed, complete vector space is said to be a Hilbert space.

Theorem 18 (Central Theorem about Complete Orthonormal Systems)**.** *Let V be a Hilbert space. Then, the following conditions are equivalent for an orthonormal set $\{u_k\}_{k=1}^{\infty}$,*

1. *For $v \in V$, if $\forall k$,*

   $$\langle v, u_k \rangle = 0$$

   then

   $$v = \overrightarrow{0}$$

   That is, if any vector is orthogonal to the entire orthonormal set, then the vector must be the zero vector.

2. *For $v \in V$,*

   $$\lim_{n\to\infty} \left\| \sum_{k=1}^{n} \langle v, u_k \rangle u_k - v \right\| = 0$$

3. For $v \in V$,

$$\sum_{k=1}^{\infty} |\langle v, u_k \rangle|^2 = \|v\|^2$$

That is, Perceval's Equality holds.

Recitation 10 – Exercise 1.
Give an example for an infinite orthogonal set in $L^2([-\pi, \pi])$, which is not a complete set, where $L^2([-\pi, \pi])$ is the space of all functions f such that $\int_{-\pi}^{\pi} |f|^2 \, dx$ is finite.

Recitation 10 – Solution 1.
The set $\{\cos(nx)\}_{n=1}^{\infty}$ is an infinite orthogonal set but it is not a complete set as the function

$$f(x) = \sin x$$

satisfies

$$\langle f, \cos(nx) \rangle = 0$$

even though

$$f \neq 0$$

Recitation 10 – Exercise 2.
$\forall n \in \mathbb{N}$, let

$$h_n(t) = e^{int} - e^{i(n+1)t}$$

1. Prove that $\{h_n\}_{n=-\infty}^{\infty}$ satisfies the condition that if $f \in L^2([-\pi, \pi])$ and $\langle f, h_n \rangle = 0$ for all n, then $f = 0$, where $L^2([-\pi, \pi])$ is the space of all functions f such that $\int_{-\pi}^{\pi} |f|^2 \, dx$ is finite.

2. Is $\{h_n\}_{n=-\infty}^{\infty}$ is orthogonal in $L^2([-\pi, \pi])$, where $L^2([-\pi, \pi])$ is the space of all functions f such that $\int_{-\pi}^{\pi} |f|^2 \, dx$ is finite.

Recitation 10 – Solution 2.

1.

$$\begin{aligned}\langle f, h_n\rangle &= \left\langle f, e^{int} - e^{i(n+1)t}\right\rangle \\ &= \left\langle f, e^{int}\left(1 - e^{it}\right)\right\rangle \\ &= \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f e^{-int}\left(1 - e^{-it}\right) \mathrm{d}t \\ \therefore 0 &= \left\langle f\left(1 - e^{-it}\right), e^{int}\right\rangle\end{aligned}$$

Therefore, as the set $\left\{e^{int}\right\}_{n=-\infty}^{\infty}$ is complete, if the dot product of any function with all e^{int} is zero, then the function must be zero. Therefore,

$$f\left(1 - e^{-it}\right) = 0$$

Therefore, $f = 0$.

2.

$$\begin{aligned}\langle h_n, h_{n+1}\rangle &= \left\langle e^{int} - e^{i(n+1)t}, e^{i(n+1)t} - e^{i(n+1)t}\right\rangle \\ &\neq 0\end{aligned}$$

Therefore, it is not orthogonal.

Recitation 11 – Exercise 1.

Let V be a Banach space, i.e. a linear, complete space.

1. Prove that if $\{v_n\} \subset V$, $\{w_n\} \subset W$ converge to the same vector, then

$$\lim_{n\to\infty} \|v_n - w_n\| = 0$$

2. Let $\{v_n\} \subset V$ be a series of vectors, such that

$$\|v_n\| \leq \frac{1}{2^n}$$

Prove that $\sum\limits_{n=1}^{\infty} v_n$ converges.

Recitation 11 – Solution 1.

1. Let

$$\lim_{n\to\infty} v_n = u$$
$$\lim_{n\to\infty} w_n = u$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} \|v_n - w_n\| &= \lim_{n\to\infty} \|v_n - u + u - w_n\| \\ &\leq \lim_{n\to\infty} \|v_n - u\| + \lim_{n\to\infty} \|w_n - u\| \\ &\leq 0 + 0 \\ &\leq 0\end{aligned}$$

Also, as the norm must be non-negative,

$$0 \leq \lim_{n\to\infty} \|v_n - w_n\|$$

Therefore,

$$\lim_{n\to\infty} \|v_n - w_n\| = 0$$

2. Let

$$w_m = \sum_{n=1}^{m} v_n$$

Therefore,

$$\begin{aligned}\|w_{m+k} - w_m\| &= \left\| \sum_{n=m+1}^{m+k} v_n \right\| \\ &\leq \sum_{n=m+1}^{m+k} \|v_n\| \\ &\leq \sum_{n=m+1}^{m+k} \frac{1}{2^n} \\ &\leq \frac{1}{2^m} \sum_{n=1}^{k} \frac{1}{2^n} \\ &\leq \frac{1}{2^m}\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \|w_{m+k} - w_m\| = 0$$

Therefore, the series is a Cauchy series.
As V is a Banach space, it is complete. Hence, a Cauchy series converges in it.
Therefore, as the series is Cauchy, and the space is Banach, the series converges.

Recitation 11 – Exercise 2.
Let

$$\lambda_n = \min_{\alpha\in\mathbb{R}} \frac{1}{\pi} \int_{-\pi}^{\pi} \left|\sqrt{\cos x} - \alpha\cos(nx)\right|^2 \mathrm{d}x$$

Calculate $\lim\limits_{n\to\infty} \lambda_n$.

Recitation 11 – Solution 2.

$$\langle f, g\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f\overline{g}\,\mathrm{d}x$$

Therefore,

$$\begin{aligned}\langle f, f\rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} f\overline{f}\,\mathrm{d}x \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} |f|^2\,\mathrm{d}x\end{aligned}$$

Therefore, if

$$f(x) = \sqrt{\cos x}$$

then,

$$\begin{aligned}\lambda_n &= \min_{\alpha\in\mathbb{R}} \left|\sqrt{\cos x} - \alpha\cos(nx)\right|^2 \\ &= \min_{\alpha\in\mathbb{R}} \left|f(x) - \alpha\cos(nx)\right|^2\end{aligned}$$

Therefore, by the best approximation theorem, α will be the coefficient corresponding to the best approximation, i.e.,

$$\alpha \cos(nx) = \text{proj}_W \left(\sqrt{\cos x}\right)$$

Therefore,

$$\alpha = \frac{\left\langle \sqrt{\cos x}, \cos(nx) \right\rangle}{\|\cos(nx)\|^2}$$

Therefore,

$$\begin{aligned}
\lambda_n &= \left\| \sqrt{\cos x} - \alpha \cos(nx) \right\|^2 \\
&= \left\| \sqrt{\cos x} - \frac{\left\langle \sqrt{\cos x}, \cos(nx) \right\rangle}{\|\cos(nx)\|^2} \cos(nx) \right\|^2 \\
&= \left\| \sqrt{\cos x} \right\|^2 - \frac{\left\langle \sqrt{\cos x}, \cos(nx) \right\rangle^2}{\|\cos(nx)\|^2} \|\cos(nx)\|^2 \\
&= \left\| \sqrt{\cos x} \right\|^2 - \left\langle \sqrt{\cos x}, \cos(nx) \right\rangle
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\| \sqrt{\cos x} \right\|^2 &= \frac{1}{\pi} \int\limits_{-\pi}^{\pi} |\cos x| \, \mathrm{d}x \\
&= \frac{2}{\pi} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, \mathrm{d}x \\
&= \frac{2}{\pi} \sin x|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{4}{\pi}
\end{aligned}$$

Therefore,

$$\lambda_n = \frac{4}{\pi} - {a_n}^2$$

Therefore,

$$\begin{aligned}
\lim_{n \to \infty} \lambda_n &= \lim_{n \to \infty} \left(\frac{4}{\pi} - {a_n}^2 \right) \\
&= \frac{4}{\pi} - \lim_{n \to \infty} {a_n}^2
\end{aligned}$$

By Riemann-Lebesgue's Lemma,

$$\lim_{n \to \infty} a_n = 0$$

Therefore,

$$\lim_{n \to \infty} \lambda_n = \frac{4}{\pi}$$