Moduli Space of Framed Torsion-free Sheaves over ADE Surfaces

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1 Introduction

This note serves as a final project for MA 822, which is intended to give a brief introduction to ADE-type singularities. The main goal is to understand the ADHM construction, given by Nakajima [Nak99][Nak07], of moduli space of framed torsion-free sheaves over the ALE spaces. In order to understand this fascinating object, we first have a glimpse of the minimal resolution of ADE-type singularities, which admits a hyper-kalher metric that is ALE (asymptotically locally Euclidean) [Kro89]. From algebraic geometric perspective, the minimal resolution can also be constructed via blowing up. The incident graph of the exceptional divisors is the corresponding Dynkin diagram. This phenomenon can be understood via the (derived) Mckay correspondence [KV98], which provides important examples of noncommutative crepant resolutions. Using the technique developed by Cho, Hong and Lau [CHL15], the authors also realize such constructions as mirror symmetry phenomenons [HLT24].

2 Finite Subgroups of $\mathbf{SL}(2,\mathbb{C})$

The story starts by considering the action of a finite subgroup Γ on $\mathbb{C}^2$.

Given the standard hermitian inner product on $\mathbb{C}^2$ defined by $< \mathbf{z}, \mathbf{w} >:= \mathbf{z}\cdot\bar{\mathbf{w}}$ and a finite subgroup $\Gamma \subset SL(2,\mathbb{C})$. Notice that Γ acts naturally on $\mathbb{C}^2$. Averaging the inner product by the group Γ, we arrive at a hermitian inner product which is invariant with respect to Γ. This shows that Γ is conjugate to a finite subgroup of the special unitary group $SU(2)$. Hence, the classification of finite subgroups of $SL(2,\mathbb{C})$ is equivalent to the classification of finite subgroups of $SU(2)$.

The idea to classify the finite subgroups of $SU(2)$ is to consider the double cover

$$\pi : SU(2) \twoheadrightarrow SO(3)$$

Thus any finite subgroup G of $SU(2)$ defines a finite subgroup $\bar{G}$ of rotations of $\mathbb{R}^3$. Conversely, every $\bar{G} \subset SO(3)$ can be lifted to a finite subgroup of $SU(2)$ such that the kernel is of order ≤ 2. From this and the classical classification of finite subgroups of $SO(3)$ as symmetry groups of regular polyhedra, we obtain the following.

Proposition 2.1. *Any finite subgroup of $SU(2)$ is one of the following groups:*

1. *The cyclic group $\mathbb{Z}/n\mathbb{Z}$ for $n > 1$.*
2. *The binary dihedral group $\mathbb{B}D_{2n}$ for $n > 1$, the preimage of the dihedral group D_{2n} under π.*
3. *The binary tetrahedral group $\mathbb{BT}$, the preimage of the tetrahedral group $\mathbb{T}$ under π.*
4. *The binary octahedral group $\mathbb{BO}$, the preimage of the octahedral group $\mathbb{O}$ under π.*

5. *The binary dodecahedral group $\mathbb{BD}$, the preimage of the dodecahedral group $\mathbb{D}$ under π.*

To be more precise, here we choose a basis of $\mathbb{C}^2$ and write down the generators of the action explicitly. Let $\epsilon_n := e^{2\pi i/n}$.

1. Γ is a cyclic group of order n. A generator is given by the matrix

$$g_1 = \begin{bmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{-1} \end{bmatrix}$$

2. Γ is a binary dihedral group of order $4n$. Its generators are given by the matrices

$$g_1 = \begin{bmatrix} \epsilon_{2n} & 0 \\ 0 & \epsilon_{2n}^{-1} \end{bmatrix}, g_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

3. Γ is a binary tetrahedral group of order 24. Its generators are given by the matrices

$$g_1 = \begin{bmatrix} \epsilon_4 & 0 \\ 0 & \epsilon_4^{-1} \end{bmatrix}, g_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, g_3 = \frac{1}{1-i} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

4. Γ is a binary octahedral group of order 48. Its generators are

$$g_1 = \begin{bmatrix} \epsilon_8 & 0 \\ 0 & \epsilon_8^{-1} \end{bmatrix}, g_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, g_3 = \frac{1}{1-i} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

5. Γ is a binary icosahedra group of order 120. Its generators are

$$g_1 = \begin{bmatrix} \epsilon_{10} & 0 \\ 0 & \epsilon_{10}^{-1} \end{bmatrix}, g_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, g_3 = \frac{1}{\sqrt{5}} \begin{bmatrix} \epsilon_5 - \epsilon_5^4 & \epsilon_5^2 - \epsilon_5^3 \\ \epsilon_5^2 - \epsilon_5^3 & -\epsilon_5 + \epsilon_5^4 \end{bmatrix}$$

The McKay correspondence, named after John McKay, states that there is a one-to-one correspondence between the McKay graphs of the finite subgroups of $SL(2, \mathbb{C})$ and the extended Dynkin diagrams, which appear in the ADE classification of the simple Lie algebras. Here we recall the definition of McKay graphs.

Definition 2.1. *Let Γ be a finite subgroup and ρ_0 be its linear representation. The McKay graph of the pair (Γ, ρ_0) is defined to be a graph, where the vertices correspond to irreducible representations ρ_i of Γ. A vertex ρ_i is connected to the vertex ρ_j by an edge pointing to ρ_j if ρ_j is a direct summand of $\rho_0 \otimes \rho_i$. Then the weight m_{ij} of the arrow is the number of times this constituent appears in $\rho_0 \otimes \rho_i$.*

The McKay correspondence classifies the possible groups Γ via their McKay graphs. More precisely, we have the following.

Theorem 2.1 (J. McKay)**.** *Let Γ be a nontrivial finite subgroup of $SU(2)$ and ρ_0 be its natural 2-dimensional representation defined by the inclusion. Then, the McKay graph of (Γ, ρ_0) is an affine ADE type Dynkin diagram.*

Here we provide an explicit calculation of the cyclic group.

Example 2.1. *Let $G = C_n =< g_0 >$ be a cyclic group of order n. Since C_n is an abelian group, every linear representation $\rho : C_n \to GL(V)$ decomposes into the direct sum of 1-dimensional representations*

$$V = \sum_{0}^{n-1} V_k,$$

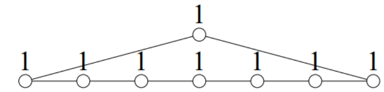
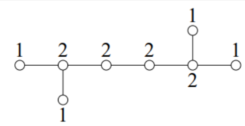
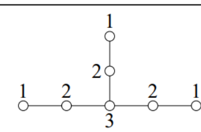
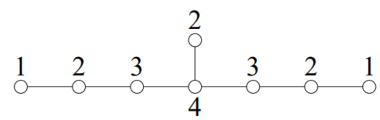
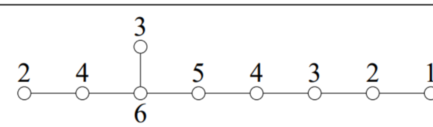
Finite subgroup of $SU(2)$		Affine simply laced Dynkin diagram	
$\mathbb{Z}/n\mathbb{Z}$	$\langle x \mid x^n = 1 \rangle$	$\widetilde{A}_{n-1}$	
$\mathbb{B}D_{2n}$	$\langle x, y, z \mid x^2 = y^2 = y^n = xyz \rangle$	$\widetilde{D}_{n-2}$	
$\mathbb{BT}$	$\langle x, y, z \mid x^2 = y^3 = z^3 = xyz \rangle$	$\widetilde{E}_6$	
$\mathbb{BO}$	$\langle x, y, z \mid x^2 = y^3 = z^4 = xyz \rangle$	$\widetilde{E}_7$	
$\mathbb{BD}$	$\langle x, y, z \mid x^2 = y^3 = z^5 = xyz \rangle$	$\widetilde{E}_8$	

Figure 1: The McKay Correspondence

where $V_k := \{v \in V : \rho_0(g_0)(v) = e^{2\pi i k/n} v\}$*. So* C_n *has* n *irreducible representations.*

If we consider $\rho_0 : C_n \to SU(2)$ *given by the matrix*

$$\begin{bmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{-1} \end{bmatrix},$$

we find that $\rho_0 = \rho_1 \oplus \rho_{-1}$*. Thus* $\rho_0 \otimes \rho_k = \rho_{k-1} \oplus \rho_{k+1}$*. Hence, the Mckay graph* (C_n, ρ_0) *is the Dynkin diagram of affine* $\tilde{A}_{n-1}$*.*

3 Construction of ALE spaces

3.1 Quiver Varieties

In this subsection, we fix notaions for quiver varieties. Let (I, E) be a finite graph of an affine type, where I is the set of vertices and E the set of edges. Let **A** be the adjacency matrix of the graph. Then **C=2I-A** is a (symmetric) Cartan matrix of an affine type.

Example 3.1. *Let* (I, E) *be the graph of affine* A_1*. Then the cartan matrix is*

$$\boldsymbol{C} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

where the elements 2 on the diagonal imply the graph has no self-loops.

Let H be the set of pairs consisting of an edge together with its orientation. For $h \in H$, we denote $s(h)$ (resp. $t(h)$) the source (resp. target) vertex of h. For $h \in H$, we denote $\bar{h}$ the same edge as h with the reverse orientation. An orientation Ω of the graph is a subset $\Omega \subset H$ such that $\bar{\Omega} \cup \Omega = H, \bar{\Omega} \cap \Omega = \varnothing$. The orientation defines a function $\epsilon : H \to \{\pm 1\}$ given by $\epsilon(h) = 1$ if $h \in \Omega$ and $= -1$ if $h \in \bar{\Omega}$.

Let $V = \oplus_{i \in I} V_i$ be an I-graded vector space. We define its dimension vector by $dimV := (dimV_i)_{i \in I} \in \mathbb{Z}^I_{\geq 0}$. If V^1, V^2 are I-graded vector spaces, we introduce vector spaces

$$L(V^1, V^2) := \oplus_{i \in I} Hom(V^1_i, V^2_i)$$

which is the space of linear maps among the vector spaces over the same vertices. Note that this is not the morphism of representations of a quiver. We also introduce

$$E(V^1, V^2) := \oplus_{h \in H} Hom(V^1_{s(h)}, V^2_{t(h)}).$$

Example 3.2. *Let (I, E) be the graph of affine A_1. If we take an I-graded vector space V, then $E(V, V)$ is the representation space of the doubling of affine A_1. In general, this is also true for a fixed graph and an I-graded vector space.*

For $B = (B_h) \in E(V^1, V^2), C = (C_h) \in E(V^2, V^3)$, we define a multiplication of B and C by

$$CB = (\sum_{t(h)=i} C_h B_{\bar{h}})_i \in L(V^1, V^3).$$

Multiplications ba, Ba of $a \in L(V^1, V^2), b \in L(V^2, V^3), B \in E(V^2, V^3)$ are defined in similar manner. If $a \in L(V^1, V^1)$, its trace $tr(a)$ is understood as $\sum_k tr(a_k)$.

Let V, W be I-graded vector spaces. We define

$$M(V, W) := E(V, V) \oplus L(V, W) \oplus \oplus L(W, V).$$

Remark. $M(V, W)$ can be understood as the representation space of the double quiver with framing at each vertex. Doubling of a quiver plays an important role since it's isomorphic to the cotangent space of the original quiver representation.

The elements in $M(V, W)$ will be denoted by B, a, b respectively. This space has a holomorphic symplectic form given by

$$\omega((B, a, b), (B', a', b')) := tr(\epsilon BB) + tr(ab' - a'b),$$

where ϵB is an element of $E(V, V)$ defined by $(\epsilon B)_h = \epsilon(h) B_h$.

Let $G := G_V$ be the Lie group $\prod_i GL(V_i)$. It acts on $M(V, W)$ via

$$g \cdot (B, a, b) \mapsto (gBg^{-1}, ag^{-1}, gb).$$

Hence it preserves the symplectic form. The moment map is given by

$$\mu(B, a, b) = \epsilon BB + ab \in L(V, V),$$

where the dual of the Lie algebra of G is identified with $L(V, V)$ via the trace.

Let $\zeta_{\mathbb{C}} = (\zeta_{\mathbb{C},i}) \in \mathbb{C}^I$. We define a corresponding element in the center of $LieG$ by $\oplus_i \zeta_{\mathbb{C},i} id_{V_i}$, where we delete the summand corresponding to i if $V_i = 0$. Let $\mu^{-1}(\zeta_{\mathbb{C}})$ be an affine algebraic variety (not necessarily irreducible) defined as the zero set of $\mu - \zeta_{\mathbb{C}}$. The group G acts on $\mu^{-1}(\zeta_{\mathbb{C}})$.

We now define the stability conditions.

For $\zeta_{\mathbb{R}} = (\zeta_{\mathbb{R},i})_{i \in I} \in R^I$, let $\zeta_{\mathbb{R}} \cdot dimV := \sum_i \zeta_{\mathbb{R},i} dimV_i$.

Definition 3.1. *A point $(B, a, b) \in M$ is $\zeta_{\mathbb{R}}$-semistable if the following two conditions are satisfied:*

1. *If an I-graded subspace S of V is contained in $Ker\, b$ and B-invariant, then $\zeta_{\mathbb{R}} \cdot dimS \leq 0$.*

2. *If an I-graded subspace T of V contains in $Im\, a$ and B-invariant, then $\zeta_{\mathbb{R}} \cdot dimT \leq \zeta_{\mathbb{R}} \cdot dimV$.*

We say (B, a, b) is $\zeta_{\mathbb{R}}$-stable if the strict inequalities hold in $1, 2$ unless $S = 0$, $T = V$ respectively.

Remark. Note that (B, a, b) is $\zeta_{\mathbb{R}}$-(semi)stable if and only if (B^*, a^*, b^*) is $-\zeta_{\mathbb{R}}$-(semi)stable.

When defining the resolution of ADE singularities, we also need the stability condition for $B \in E(V, V)$, i.e. when $W = 0$.

Definition 3.2. *Suppose that $\zeta_{\mathbb{R}} \cdot dimV = 0$. A point $B \in E(V, V)$ is $\zeta_{\mathbb{R}}$-semistable if for any I-graded subspace S of V that is $B-$invariant, then $\zeta_{\mathbb{R}} \cdot dimS \leq 0$.*

A point B is $\zeta_{\mathbb{R}}$-stable if the strict equality holds unless $S = 0$ or $S = V$.

This definition coincides with the above for $\zeta_{\mathbb{R}}$-semistable, but not for $\zeta_{\mathbb{R}}$-stable.

Now we can define the quiver varieties.

Let $H^s_{\zeta_{\mathbb{R}},\zeta_{\mathbb{C}}}$ (resp. $H^{ss}_{\zeta_{\mathbb{R}},\zeta_{\mathbb{C}}}$ be the set of $\zeta_{\mathbb{R}}$-stable (resp. $\zeta_{\mathbb{R}}$-semistable) points in $\mu^{-1}(\zeta_{\mathbb{C}})$.

We say two $\zeta_{\mathbb{R}}$-semistable points $(B, a, b), (B', a', b')$ are S-equivalent when the closures of G_V-orbits intersect in $H^{ss}_{\zeta_{\mathbb{R}},\zeta_{\mathbb{C}}}$. We denote the pair $(\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$ by ζ for brevity. We define

$$\mathcal{M}_\zeta := \mathcal{M}_\zeta(V, W) := H^{ss}_{\zeta_{\mathbb{R}},\zeta_{\mathbb{C}}}/\sim$$

$$\mathcal{M}^{reg}_\zeta := \mathcal{M}^{reg}_\zeta(V, W) := H^{s}_{\zeta_{\mathbb{R}},\zeta_{\mathbb{C}}}/\sim$$

where $\sim$ denotes the $S-$equivalence relation. These can be defined as quotients in the geometric invariant theory.

One may curious about how the space changes according to the parameter ζ. In order to answer this question, we need to introduce the chamber structure. Since the chamber structure comes from the wall defined by the root system, let's briefly recall some related notions to gain a better understanding.

Definition 3.3. *A root system Φ of a set of vectors in $\mathbb{R}^n$ such that*

1. *Φ spans $\mathbb{R}^n$ and $0 \notin \Phi$.*

2. *If $\alpha \in \Phi$ and $\lambda\alpha \in \Phi$, then $\lambda = \pm 1$.*

3. *Φ is closed under reflection through the hyperplane normal to α.*

4. *If $\alpha, \beta \in \Phi$, then $\langle \alpha, \beta \rangle := \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$.*

Definition 3.4. *A set of positive roots of a root system Φ is a set $\Phi^+ \subset \Phi$ such that*

1. *For every $\alpha \in \Phi$, exactly one of α and $-\alpha$ is in Φ^+.*

2. *If $\alpha, \beta \in \Phi^+$, and if $\alpha + \beta$ is a root, then $\beta \in \Phi^+$.*

Definition 3.5. *$\alpha \in \Phi^+$ is a simple root if it's not the sum of other two positive roots.*

Given a root system. We can build a Dynkin diagram that remembers the simple roots' information. We first take the set of simple roots and draw one node for each simple root. For every pair of simple roots α and β, we draw a number of lines between the vertices equal to $< \alpha, \beta >< \beta, \alpha >$, which is guaranteed to be an integer. If one simple root is longer than another, we draw an arrow pointing to the shorter one. If two simple roots have the same length, we omit the arrow.

Fix a dimension vector $\mathbf{v}$. Let

$$R_+ := \{\theta = (\theta_i) \in \mathbb{Z}^I_{\geq 0} | \theta^t \mathbf{C} \theta \leq 2\} \setminus \{0\}$$

$$R_+(\mathbf{v}) := \{\theta = (\theta_i) \in R_+ | \theta_i \leq dim_{\mathbb{C}} V_i \, for \, all \, i\}$$

$$D_\theta := \{x = (x_i) \in R^I | x \cdot \theta = 0\} \; for \; \theta \in R_+$$

When the graph is of affine type, R_+ is the set of positive roots of the corresponding Dynkin diagram, and D_θ is the wall defined by the root θ. If the parameter ζ is generic i.e. not on the wall defined by the roots, we have the following proposition.

Proposition 3.1. *[Nak94] Suppose*

$$\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) \in (\mathbb{R} \oplus \mathbb{C})^I \setminus \bigcup_{\theta \in R_+(v)} (\mathbb{R} \oplus \mathbb{C}) \otimes D_\theta.$$

Then every semistable point is stable, so that the regular locus $\mathcal{M}^{reg}_\zeta$ coincides with $\mathcal{M}_\zeta$.

Fix a complex parameter $\zeta_{\mathbb{C}}$ and move a real parameter $\zeta_{\mathbb{R}}$. A connected component of

$$R^I \setminus \bigcup_{\theta \in R_+(\mathbf{v}), \zeta_{\mathbb{C}} \cdot \theta = 0} D_\theta$$

is called a *chamber*.

Moreover, we have the following relation for the real parameter in the same chamber.

Lemma 3.1. *Take two real parameters $\zeta_{\mathbb{R}}$ and $\zeta'_{\mathbb{R}}$ so that both $(\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$ and $(\zeta'_{\mathbb{R}}, \zeta_{\mathbb{C}})$ are generic. If $\zeta_{\mathbb{R}}$ and $\zeta'_{\mathbb{R}}$ are in the same chamber, the corresponding stability conditions are equivalent.*

3.2 Resolution of Kleinian Singularities

3.2.1 Algebraic Geometry Approach

In this subsection, we will consider the minimal resolution of $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup in $SU(2)$. The quotient space has a double singularity at the origins. This feature is essential for a lot of reasons. Before constructing the resolution, let's define the multiplicity of a singular point.

Definition 3.6. *Let X be a hypersurface in a neighborhood of x, i.e. X is given by a single equation $f = 0$ in an affine space Z. Then the multiplicity of X at x, denoted by $\mu(X, x)$, is defined to be the integer n such that $f \in m^n \setminus m^{n+1}$, where m is the maximal ideal of $\mathscr{O}_{Z,x}$.*

Let Γ be a finite subgroup of $SU(2)$. It acts on $\mathbb{C}^2$ naturally. It's interesting to consider the quotient space $\mathbb{C}^2/\Gamma$. Based on McKay correspondence, see theorem **??**, we have the following explicit classifications of $\mathbb{C}^2/\Gamma$.

Theorem 3.2. *Let Γ be a finite subgroup of $SU(2)$ and $\mathbb{C}^2 := Spec\, \mathbb{C}[x, y]$. Then the quotient space $\mathbb{C}^2/\Gamma$ has the following forms:*

1. *A_n case: $\mathbb{C}^2/\Gamma \cong Spec\, \mathbb{C}[X, Y, Z]/(XY - Z^n)$.*
2. *D_n case: $\mathbb{C}^2/\Gamma \cong Spec\, \mathbb{C}[X, Y, Z]/(X^2 + ZY^2 + Z^{n-1})$, $n \geq 4$.*

3. E_6 *case:* $\mathbb{C}^2/\Gamma \cong Spec\,\mathbb{C}[X,Y,Z]/(X^2+Y^3+Z^4)$.

4. E_7 *case:* $\mathbb{C}^2/\Gamma \cong Spec\,\mathbb{C}[X,Y,Z]/(X^2+Y^3+YZ^3)$.

5. E_8 *case:* $\mathbb{C}^2/\Gamma \cong Spec\,\mathbb{C}[X,Y,Z]/(X^2+Y^3+Z^5)$.

Proof. For simplicity, we will only prove the quotient space of a cyclic subgroup. The proof of other cases can be found in [**?**].

Notice that the generator g_1 of $\Gamma := \mathbb{Z}/n\mathbb{Z}$ acts on (x,y) via $g_1\cdot(x,y) = (\epsilon_n x, \epsilon_n^{-1}y)$. Hence, $X := x^n, Y := y^n, Z := xy$ are invariant under the group action.

On the other hand, suppose $f := x^a y^b$ is a monomial invariant under the group action. Then $g_1 \cdot f = \epsilon_n^{a-b}x^ay^b = x^ay^b$. Hence, $a-b \equiv 0 \ (mod\, n)$. Thus f is a multiple of X,Y,Z. Therefore, we know $\mathbb{C}[x,y]^\Gamma \cong \mathbb{C}[X,Y,Z]/(XY-Z^n)$. In particular, $\mathbb{C}^2/\Gamma \cong Spec\,\mathbb{C}[X,Y,Z]/(XY-Z^n)$.

The other cases are similar to the A type, but the generators are more complicated to find. □

Remark. By definition of the multiplicity, we know $\mathbb{C}^2/\Gamma$ **has a double singularity at the origin.** The resolution of these kinds of singularity is crepant.

Lemma 3.3. *Let $P \in X \subset \mathbb{A}^3$ be an isolated double point, and let $\sigma : X_1 \to X$ be the blowup of X at the isolated double point P. Then $\sigma^*K_X \cong K_{X_1}$.*

Proof. This formally comes from the adjunction formula and the double point.

We abuse the notation, let $\sigma : B \to \mathbb{A}^3$ be the blowup of $\mathbb{A}^3$ at P. Then X_1 is the strict transform of X under this blowup. Since P has codimension 3 in $\mathbb{A}^3$, we know

$$K_B = \sigma^*K_{\mathbb{A}^3} + 2E,$$

where E is the exceptional divisor.

Note that X has a double singularity at P. Thus as a divisor

$$\sigma^*(X) = X_1 + 2E.$$

By adjunction formula, we know

$$K_{X_1} = (K_B + X_1)|_{X_1} = (\sigma^*K_{\mathbb{A}^3} + 2E + \sigma^*(X) - 2E)|_{X_1} = (\sigma^*((K_{\mathbb{A}^3} + X)|_X)) = \sigma^*K_X.$$

□

Therefore, the resolutions of $\mathbb{C}^2/\Gamma$ are crepant resolutions.

Based on this important feature, we know the exceptional fibers are all $(-2)-$curve.

Corollary 3.1. *Let $\sigma : X_1 \to X$ be the blowup of X at the isolated double point P. Then the exceptional fiber E has self-intersection number equals to -2.*

Proof. This also follows from the adjunction formula.

By adjunction formula, we get $\omega_E \cong \omega_{X_1} \otimes \mathscr{O}_{X_1}(E) \otimes \mathscr{O}_E$. Taking the degree on both sides implies

$$2g - 2 = E.(E + K_{X_1}) = E.(E + \sigma^*K_X).$$

Since E has genus 0, and $E.\sigma^*K_X = 0$, we attain $E.E = -2$. □

With this preparation, we can start to consider the resolution of $\mathbb{C}^2/\Gamma$. The idea is simple. We will do successive blowing up at the singular points. And then we will show the first smooth variety we get is the minimal resolution of $\mathbb{C}^2/\Gamma$.

Theorem 3.4. *The quotient space $\mathbb{C}^2/\Gamma$ can be resolved by successively blowing up the singular points.*

In this note, we won't try to prove this theorem, but some convincing examples will be computed concretely.

Example 3.3 (A_1 case)**.** *The first example we consider is $\mathbb{C}^2//(\mathbb{Z}/2\mathbb{Z})$. By theorem 3.2, we know $X := \mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z}) \cong Spec\,\mathbb{C}[x_1, x_2, x_3]/(x_1x_2 - x_3^2)$.*

First, we consider blowup of $\mathbb{A}^3$ at the origin, denoted by $Y := Bl_O(\mathbb{A}^3)$. We consider the blowup as the closure of the graph of φ, where $\varphi : \mathbb{A}^3 \setminus \{0\} \to \mathbb{P}^2$ via $\varphi(x_1, x_2, x_3) = [x_1, x_2, x_3]$. In other words, φ takes a point to the line containing the point and the origin. Therefore, it's not hard to see that

$$Y := \overline{graph\,\varphi} = \{((x_1, x_2, x_3) \times [y_1, y_2, y_3]) \in \mathbb{A}^3 \times \mathbb{P}^2 | x_iy_j = x_jy_i, \forall i, j\}.$$

Since X_1 is the strict transform of X,

$$X_1 \cong \overline{\varphi^{-1}(X \setminus 0)} = \{((x_1, x_2, x_3) \times [y_1, y_2, y_3]) \in \mathbb{A}^3 \times \mathbb{P}^2 | x_iy_j = x_jy_i, x_1x_2 = x_3^2, y_1y_2 = y_3^2, \forall i, j\}.$$

By looking at the Jacobian matrix of X_1, we know X_1 is regular. Denote σ be the blowup map. Then the exceptional curve $E := \sigma^{-1}(0) \subset X_1$ is a degree 2 curve in $\mathbb{P}^2$ with self intersection number -2. In fact, X_1 is isomorphic to $K_{\mathbb{P}^1}$, the total space of canonical bundle over $\mathbb{P}^1$.

Example 3.4 (A_3 case)**.** *Resolving $X := \mathbb{C}^2//(\mathbb{Z}/4\mathbb{Z}) \cong Spec\,\mathbb{C}[x_1, x_2, x_3]/(x_1^2 + x_2^2 - x_3^4)$ is more intereting. This example can represent the standard procedure for resolving the Klein singularities. In this case, we need to blowup twice. The second blowup will be computed locally. To be more rigorous, we need to show that blowup gives a birational morphism, which allows us to perform blowup locally and then glue along with other charts. But that would go beyond the scope of this project.*

Similar to the above example, we first consider the blowup of $\mathbb{A}^3$ at the origin.

$$Y := \overline{graph\,\varphi} = \{((x_1, x_2, x_3) \times [y_1, y_2, y_3]) \in \mathbb{A}^3 \times \mathbb{P}^2 | x_iy_j = x_jy_i, \forall i, j\}.$$

Y can be covered by three affine charts. More precisely, let $Z_i := D(y_i)$ be the open subscheme of Y with y_i doesn't equal zero. Then $Z_1 := \{((x_1, x_2, x_3) \times [1, y_2, y_3]) \in \mathbb{A}^3 \times \mathbb{P}^2 | x_2 = x_1y_2, x_3 = x_1y_3\} \cong Spec\,\mathbb{C}[x_1, y_2, y_3] \cong \mathbb{A}^3$. Similarly, we find that $Z_i \cong \mathbb{A}^3$ for $i = 2, 3$.

We try to analyze X_1 using these local charts. Notice that $X_1 \cap Z_1 \cong \overline{\varphi^{-1}(X \setminus 0) \cap Z_1} = \{((x_1, x_2, x_3) \times [1, y_2, y_3]) \in \mathbb{A}^3 \times \mathbb{P}^2 | x_2 = x_1y_2, x_3 = x_1y_3, x_1^2 + x_2^2 - x_3^4 = 0\} \cong \{((x_1, x_2, x_3) \times [1, y_2, y_3]) \in \mathbb{A}^3 \times \mathbb{P}^2 | x_2 = x_1y_2, x_3 = x_1y_3, x_1^2 + x_1^2y_2^2 - x_1^4y_3^4 = 0\} \cong Spec\,\mathbb{C}[x_1, y_2, y_3]/(1 + y_2^2 - x_1^2y_3^4)$.

Similarly, we can compute

$$X_1 \cap Z_2 \cong Spec\,\mathbb{C}[y_1, x_2, y_3]/(y_1^2 + 1 - x_2^2y_3^4).$$

The most interesting part is $X_1 \cap Z_3 \cong \{((x_1, x_2, x_3) \times [y_1, y_2, 1]) \in \mathbb{A}^3 \times \mathbb{P}^2 | x_1 = x_3y_1, x_2 = x_3y_2, x_1^2 + x_2^2 - x_3^4 = 0\} \cong \{((x_1, x_2, x_3) \times [y_1, y_2, 1]) \in \mathbb{A}^3 \times \mathbb{P}^2 | x_1 = x_3y_1, x_2 = x_3y_2, x_3^2y_1^2 + x_3^2y_2^2 - x_3^4 = x_3^2(y_1^2 + y_2^2 - x_3^2) = 0\} \cong Spec\,\mathbb{C}[x_3, y_1, y_2]/(y_1^2 + y_2^2 - x_3^2)$.

By computing the Jacobian matrix, we know the first two charts are smooth, but $X_1 \cap Z_3$ still has a singularity at the origin. Furthermore, we claim that $\sigma_1^{-1}(0)$ ***consists of two intersecting exceptional curves!*** *This is because $\sigma_1^{-1}(0) \cap Z_3 \cong Spec\,\mathbb{C}[y_1, y_2]/(y_1^2 + y_2^2)$ contains two irreducible components $V_1 := V(y_1 + iy_2)$ and $V_2 := V(y_1 - iy_2)$. Notice that $\sigma_1^{-1}(0) \cap Z_1 = Spec\,\mathbb{C}[y_3]$. The transition map of $\mathbb{P}^2$ tells us that this curve is glued with V_1 to get an exceptional curve $E_1 \cong \mathbb{P}^1$. Similarly for V_2, so we have two exceptional curves $E_1 \cup E_2$.*

To resolve the remaining singular point, we do the blowing up again in the chart $X_1 \cap Z_3$. This is the same as A_1 case. Hence, we get one more exceptional curve E_3.

In the end, we get three exceptional divisors, which are E_3 and the strict transformation of E_1 and E_2, denoted by $\tilde{E_1}, \tilde{E_2}$ respectively. Since E_1 and E_2 intersect at the blowup point, $\tilde{E_1}, \tilde{E_2}$ no longer intersect. But they all intersect E_3. Therefore, if we draw a node of each irreducible component of the exceptional divisors and draw an edge if they intersect, we will attain the A_3 type Dynkin diagram.

We can resolve other Klein singularities using similar techniques. Each time, we only blow up the isolated double point. **Based on previous discussions, the subsequent blowups are crepant resolutions and the exceptional fiber only contains (-2)-curves.** In fact, the first nonsingular complex surface we get is the minimal resolution of the singularities. Furthermore, they are all Calabi-Yau variety! Let's first show they are minimal resolutions using $Castelnuovo's$ theorem and then prove the minimal resolutions are all Calabi-Yau variety.

Theorem 3.5 (Castelnuovo)**.** *If Y is a curve on a nonsingular projective surface X, with $Y \cong \mathbb{P}^1$ and $Y^2 = -1$, then there exists a morphism $f : X \to X_0$ to a nonsingular projective surface X_0, and a point $P \in X_0$, such that X is isomorphic via f to the blowup of X_0 with center P, and Y is the exceptional curve.*

Corollary 3.2. *A nonsingular projective surface X is a minimal surface if and only if it doesn't have the exceptional curve of the first type, i.e. (-1)-curve.*

Proof. If X doesn't have any (-1) curves but not minimal, then there exists a nonsingular projective surface Y such that $f : X \to Y$ is the blowup of Y with center at some point P. However, since Y is nonsingular, $f^{-1}(P)$ is a (-1) curve. Contradiction.

On the other hand, if X is a minimal surface, then by $Castelnuovo's$ theorem, it cannot have (-1) curve, otherwise we can contract the (-1) curve to obtain another nonsingular projective surface. □

Corollary 3.3. *Let Γ be a finite subgroup of $\mathbb{C}^2$. Then the first nonsingular surface we obtain when resolving $\mathbb{C}^2/\Gamma$ is a minimal surface.*

Proof. By construction, we know the resolution only contains (-2)-curves in its exceptional fiber. Hence it's minimal. □

Another important corollary is that the minimal resolution of $\mathbb{C}^2/\Gamma$ is a Calabi-Yau variety. By this, we mean the canonical bundle of the resolution space is trivial.

Corollary 3.4. *Let $\sigma : \tilde{X} := \widetilde{\mathbb{C}^2/\Gamma} \to X := \mathbb{C}^2/\Gamma$ be the minimal resolution. Then the canonical bundle $\omega_{\tilde{X}}$ is trivial.*

Proof. According to the construction, we know X is attained by successive blowing up at the double singularities. Hence, $\sigma^*\omega_X \cong \omega_{\tilde{X}}$.

But X is an affine variety. Any line bundles over X are trivial. Thus $\omega_{\tilde{X}}$ is also trivial. □

Remark. Later, using the hyperkahler construction, the minimal resolution is automatically Calabi-Yau, since the hyperkahler manifolds are all Calabi-Yau.

The behavior of the exceptional fiber in the A_3 case is not a coincidence. In fact, we have the following surprising fact:

Theorem 3.6. *Let Γ be a finite subgroup in $SU(2)$. Let $\pi : \widetilde{\mathbb{C}/\Gamma} \to \mathbb{C}/\Gamma$ be the minimal resolution. Then the exceptional fiber of $\pi^{-1}(0)$ is a tree of (-2)-curves, whose incidence graph is the Dynkin diagram of Γ.*

This theorem tells us the behavior of the exceptional divisors. They form a cycle of $\mathbb{P}^1$, which has self-intersection number equal to -2 and each curve only intersects the adjacent curve.

3.2.2 Toric Geometry Approach

Notice that only A_n type singularities are toric varieties. The corresponding cone of A_n singularity is generated by the arrows $(0,1)$ and $(n+1,1)$ in $\mathbb{R}^2$.

This toric variety $X := \mathbb{C}/(\mathbb{Z}/(n+1)\mathbb{Z})$ has double singularity at the toric origin. Resolution corresponds to inserting the arrows $(i,1)$ for $i = 1, 2, \cdots, n$ in the cone picture. By the $Orbit - Cone$ correspondence, we know these n arrows correspond to n toric divisors, which are exactly the exceptional curves $\mathbb{P}^1$.

Using the toric geometry technique, we can verify that each exceptional curve has self-intersection number equals to -2. And they have intersection number 1 for the adjacent divisors.

3.2.3 Differential Geometry Approach

Another approach to get the minimal resolution of $\mathbb{C}^2/\Gamma$ is introduced by Kronheimer [Kro89] using quiver representation and Hyperkahler quotient. In this section, we briefly summarize the construction so are some main results.

We first take and fix an affine Dynkin graph. Let I be the set of vertexes. Let $0 \in I$ be the vertex corresponding to the simple root, which is the negative of the highest weight root of the corresponding simple Lie algebra. Let δ be the vector in the kernel of the affine Cartan matrix whose $0-$component is equal to 1. Let G_δ be the complex Lie group using the symbol above.

Example 3.5. *If we consider A type Dynkyn diagram,* $\delta = (1, 1, \cdots, 1)$. $G_\delta \cong \mathbb{C}^* \times \cdots \times \mathbb{C}^*$.

Choose the hyperkahler parameter $\zeta^0 = (\zeta^0_{\mathbb{R}}, \zeta^0_{\mathbb{C}}) \in \mathbb{R}^3 \otimes Z$, where $Z \subset \mathbb{R}[I]$ is the level 0 hyperplane $\{x \in \mathbb{R}[I] | x \cdot \delta = 0\}$. We further assume that ζ^0 is generic, i.e., it's not contained in any $\mathbb{R}^3 \otimes D_\theta$ where D_θ is a real root hyperplane. (ζ^0 is not on the wall).

As before, we consider the representation of the double quiver with framing at each vertex. But this time, we take zero-dimensional representation at the framing vertexes. Namely, we consider $M(\delta, 0)$.

Since $\mathbb{C}^*$ still acts trivially on $M(\delta, 0)$, we define

$$X_{\zeta^0} = \{\xi \in M(\delta, 0) | \mu(\xi) = -\zeta^0_{\mathbb{C}}\} //_{(-\zeta^0_{\mathbb{R}})} G_\delta \tag{1}$$

where $'//'_{(-\zeta^0_{\mathbb{R}})}$ means the GIT quotient with respect to the parameter $(-\zeta^0_{\mathbb{R}})$. As we assume ζ^0 is generic, by lemma 3.1, we have

$$X_{\zeta^0} = H^s_{(-\zeta^0_{\mathbb{R}}, -\zeta^0_{\mathbb{C}})} / (G_\delta / \mathbb{C}^*),$$

where $H^s_{(-\zeta^0_{\mathbb{R}}, -\zeta^0_{\mathbb{C}})}$ is the set of $(-\zeta_{\mathbb{R}})-$stable points in $\mu_{\mathbb{C}}^{-1}(-\zeta^0_{\mathbb{C}})$ as before. Kronheimer showed the following result of X_{ζ^0}.

Theorem 3.7 ([Kro89]). *If* ζ^0 *is generic, then*

1. X_{ζ^0} *is a 4-dimensional hyperkahler manifold. In particular, a smooth complex surface.*

2. X_{ζ^0} *is diffeomorphic to the minimal resolution of* $\mathbb{C}^2/\Gamma$*, where* Γ *is the finite subgroup of* $SL_2(\mathbb{C})$ *associated to the affine Dynkin graph.*

3. X_{ζ^0} *has a hyperkahler metric, which is ALE (asymptotically locally Euclidean) of order 4, i.e., there is a compact subset* $K \subset X_{\zeta^0}$ *and a diffeomorphic* $X_{\zeta^0} \setminus K \to (\mathbb{C}^2 \setminus B_r(0))/\Gamma$ *under which the metric is approximated by the standard Euclidean metric on* $\mathbb{C}^2/\Gamma$*; it is written in the Euclidean coordinate* $(x_i)_{i=1}^4$

$$g_{ij} = \delta_{ij} + a_{ij}$$

with $\partial^p a_{ij} = O(r^{-4-p}), p \geq 0$*, where* $r^2 = \sum x_i^2$ *and* ∂ *denotes the differentiation with respect to the coordinates* x_i.

The key point of the first statement is showing that ζ^0 is generic ensuring the group acts freely on the moment map level. Hence, we can apply the hyperkahler reduction.

Here we only give the reason why the resolution is minimal. This is based on the vanishing of the first Chern class. In fact, the first Chern class is zero implying that X_{ζ^0} has no exceptional curve of the first kind.

Lemma 3.8. *Given a rank k locally free sheaf or vector bundle E. Then $c_1(E) = c_1(det\, E)$, where $det\, E$ is the determinant bundle of E.*

Proof. Given an arbitrary vector bundle E of rank k. By splitting principle, we can assume E is the direct sum of k line bundles $L_1 \oplus L_2 \cdots \oplus L_k$. Then $c_1(E) = \sum_i c_1(L_i)$.

On the other hand, $det\, E = \otimes_i L_i$. Hence, $c_1(det\, E) = \sum_i c_1(L_i)$. □

Corollary 3.5. *If a complex surface X has vanishing first Chern class, then it contains no exceptional curves of first kind.*

Proof. If X has zero first Chern class, i.e. $c_1(TX) = 0$, by lemma 3.8, $c_1(\omega_X) = 0$.

Suppose X contains an exceptional curve of the first kind. Then $K_X = f^*K_Y + E$, where $f : X \to Y$ is the monoidal transformation and E is the exceptional curve. But $c_1(\omega_X) = 0$ implies $f^*K_Y = -E$, which is impossible. Thus X doesn't contain the exceptional curve of the first kind. □

4 Moduli Space of Framed Torsion Free Sheaves

In this section, we try to analyze the moduli space of framed torsion free sheaves over $X_{\zeta^0} := \tilde{\mathbb{C}^2/\Gamma}$ following [Nak94]. In fact, we hope to describe the moduli space set theoretically. Though it's possible to give a complex-analytic structure on the framed moduli space, we don't try to do so.

We first introduce the compactification of X_{ζ^0}. When X_{ζ^0} is the minimal resolution of $\mathbb{C}^2/\Gamma$, the compactification $\bar{X_{\zeta^0}}$ is obtained by considering $\mathbb{P}^2/\Gamma$ and resolve the singularity at the origin, but keep the singularity on the line at infinity l_∞ untouched.

For general X_{ζ^0}, we have a coordinate system at the end $X_{\zeta^0} \setminus K \to (\mathbb{C}^2 \setminus B_r(0))/\Gamma$ such that the complex structure is approximated by the standard one on $\mathbb{C}^2/\Gamma$ up to order $O(r^{-4})$. Let $\bar{X_{\zeta^0}} = X_{\zeta^0} \cup l_\infty$, where $l_\infty = \mathbb{P}^1/\Gamma$. We endow a structure of a differential orbifold so that $(X^0_\zeta \setminus K) \cup l_\infty$ is identified with $\mathbb{P}^2 \setminus B_r(0))/\Gamma$ via the coordinate system at the end. Here the 'orbifold' means that we remember the action of Γ on the neighborhood $\tilde{U} = \mathbb{P}^2 \setminus B_r(0)$ of l_∞. According to [Nak94], $\bar{X^0_\zeta}$ can be given a structure of a complex analytic orbifold.

Definition 4.1. *Given a torsion-free sheaf E on $\bar{X^0_\zeta}$. A framing Φ is a fixed isomorphism from $E|_{l_\infty}$ to $(\rho \otimes \mathscr{O}_{\mathbb{P}^1})/\Gamma$, where ρ is a representation of Γ. In other words, framing is a fixed trivialization of E at l_∞.*

Let $\zeta^0_\mathbb{R} \in \mathbb{R}[I]$ be the parameter for the stability condition, in the level 0 hyperplane $\zeta^0_\mathbb{R} \cdot \delta = 0$ as in the previous section. We take a parameter $\zeta_\mathbb{R}$ from the chamber containing $\zeta^0_\mathbb{R}$ in its closure with $\zeta_\mathbb{R} \cdot \delta < 0$. These conditions uniquely determine the chamber containing $\zeta_\mathbb{R}$. As $\zeta_\mathbb{R}$ is not contained in any root hyperplane, the stability and semistability are equivalent for $\zeta_\mathbb{R}$. Then we have the following nice description of the moduli space of framed torsion-free sheaves.

Theorem 4.1. *There is a bijection between $M_{(\zeta_\mathbb{R}, \zeta^0_\mathbb{C})}(V, W)$ and the framed moduli space of torsion-free sheaves (E, Φ) on X_{ζ^0}, where W corresponding to the representation of Γ, and V is given by Chern classes of E by the formula:*

$$c_1(E) = \sum_{i \neq 0} u_i c_1(\mathcal{R}_i), \text{ where } U = W - CV,$$

$$ch_2(E) = \sum_i u_i ch_2(\mathcal{R}_i) + 2V \cdot \delta ch_2(\mathscr{O}(l_\infty)).$$

This bijection is given by a three-term complex that arises from the Beilinson spectral sequence called monad. More details can be found in the Chapter II of [Nak99].

5 Noncommutative Crepant Resolution

5.1 Introduction to noncommutative geometry and NCCRs

In this section, we give a brief introduction to noncommutative crepant resolution (NCCR).

Let's introduce some notations. Let Λ be a ring. We write $\Lambda - Mod$ as the category of Λ-module without any hypothesis of finite generation.

Definition 5.1. *Let Λ be a ring and $M \in \Lambda - Mod$.*

1. *Denote by $add\, M$ the full subcategory of $\Lambda - Mod$ containing all direct summands of finite direct sums of copies of M*
2. *Say M is a generator (for $\Lambda - mod$) if every finitely generated left Λ-module is homomorphic image of a finite direct sum of copies of M. Equivalently, $\Lambda \in add\, M$.*
3. *Say M is a progenerator if M is a finitely generated projective module and M is a generator. Equivalently, $add\, \Lambda = add\, M$.*

Theorem 5.1 (Morita Equivalence)**.** *The following are equivalent for rings Λ and Γ.*

1. *There is an equivalence of abelian categories $\Lambda - mod \cong \Gamma - mod$.*
2. *There exists a progenerator $P \in \Lambda - mod$ such that $\Gamma \cong End_\Lambda(P)$.*
3. *There exists a $(\Lambda - \Gamma)$-bimodule P such that the functor $Hom_\Lambda(P, -) : \Lambda - mod \to \Gamma - mod$ is an equivalence.*

In this case, we say Λ and Γ are Morita equivalence.

Remark. Notice that $\Lambda^{\oplus n}$ is a progenerator. Thus by the theorem, we know Λ is Morita equivalent to $\Gamma = End_\Lambda(\Lambda^{\oplus n}) = M_n(\Lambda)$.

Corollary 5.1. *Let Λ be a ring and M, N two Λ-modules such that $add\, M = add\, N$. Then $End_\Lambda(M)$ and $End_\Lambda(N)$ are Morita equivalent via the functors $Hom_\Lambda(M, N) \otimes_{End_\Lambda(M)} -$ and $Hom_\Lambda(N, M) \otimes_{End_\Lambda(N)} -$.*

The following cousin of Morita equivalence will be essential later.

Proposition 5.1. *Let Λ be a ring and M a finitely generated Λ-module which is a generator. Set $\Gamma = End_\Lambda(M)$. Then the functor*

$$Hom_\Lambda(M, -) : \Lambda - mod \to \Gamma - mod$$

is fully faithful, and restricts to an equivalence

$$Hom_\Lambda(M, -) : add\, M \to add \Gamma.$$

In particular, the indecomposable projective Γ-modules are precisely the modules of the forms $Hom_\Lambda(M, N)$ for N an indecomposable module in $add\, M$.

On the geometric side, it has long been standard operating procedure to study a variety or scheme X by investigating the sheaves on X, particularly those of algebraic origin, the quasicoherent sheaves. Now one may ask what information, if any, is lost in the passage from the geometric object X to the category $coh(X)$ or $Qcoh(X)$. In fact, we have the following:

Theorem 5.2 (Gabriel-Rosenberg Reconstruction)**.** *A scheme X can be reconstructed up to isomorphism from the abelian category $Qcoh(X)$.*

For projective schemes, Serre's fundamental construction describes the quasicoherent sheaves on X in terms of the graded modules over the homogenous coordinate ring. Explicitly, let A be a finitely generated graded algebra over a field, and set $X = Proj\, A$. Let $GrMod$, resp. $grmod\, A$, denote the category of graded, resp. finitely generated graded, A-modules. The graded modules annihilated by $A_{\geq n}$ for $n >> 0$ form a subcategory $Tors\, A$, resp. $tors\, A$, and

$$Tails\, A := GrMod\, A/Tors\, A,\ tails\, A = grmod\, A/tors\, A$$

are defined to be the quotient categories. This means two graded modules M and N are equivalent isomorphic in $Tails\, A$ if and only if $M_{\geq n}$ and $N_{\geq n}$ agree as graded modules for large enough n.

Theorem 5.3. *Let A be a finitely generated graded algebra over $A_0 = k$, a field, and set $X = Proj\, A$. Then the functor $\Gamma_* : coh\, X \to tails\, A$, defined by sending a coherent sheaf $\mathscr{F}$ to the image in $tails\, A$ of $\oplus_{n=-\infty} H^0(X, \mathscr{F}(n))$, defines an equivalence of categories $coh(X) \cong tails\, A$.*

Remark. This is why people define coherent sheaves over a noncommuatative graded algebra as $tails\, A$.

Definition 5.2. *If (R, m) is a local ring and $M \in mod\, R$, we define the depth of M to be*

$$depth_R M := min\{i \geq 0 |\, Ext^i_R(R/m, M) \neq 0\}.$$

For $M \in mod\, R$ it's always true that $depth\, M \leq dim\, M \leq dim\, R \leq dim_{R/m}\, m/m^2 < \infty$. We say M is a (maximal) Cohen-Macaulay (CM) module if $depth\, M = dim\, R$. We say R is a CM ring if $R_R \in CM\, R$, and we say R is Gorenstein if it's CM and furthermore $inj.dim\, R < \infty$.

Definition 5.3. *When R is not necessarily local, we define M to be CM module if all its localizations at the maximal ideal are CM R_m-module.*

Definition 5.4. *Let R be a (equicodimensional normal) Cohen-Macaulay (CM) ring. A noncommutative crepant resolution (NCCR) of R is by definition a ring of the form $End_R(M)$ for some reflexive R-module M such that*

1. *$End_R(M) \in CM\, R$*
2. *$gl.dim$ $End_R(M) = dim\, R$*

In the definition, the first condition turns to correspond to the geometric property of crepancy.

Theorem 5.4. *Suppose $f : Y \to Spec\, R$ is a projective birational map, where Y and R are both normal Gorenstein of dimension d. If Y is derived equivalent to a ring Λ, then the following are equivalent:*

1. *f is crepant*
2. *$\Lambda \in CM\, R$.*

In this case, $\Lambda \cong End_R(M)$ for some reflexive R-module M.

This theorem has the following important corollary.

Corollary 5.2. *Suppose $f : Y \to Spec\, R$ is a projective birational map between d-dimensional Gorenstein normal varieties. If Y is derived equivalent to a ring Λ, then the following are equivalent:*

1. *Λ is a noncommutative crepant resolution.*

2. *$f : Y \to Spec\, R$ is a crepant resolution.*

Remark. As the motivation of the second condition, we recall that R is a regular local commutative ring if and only if $gl.dim\, R < \infty$. For an irreducible variety V, V is nonsingular if and only if $gl.dim\, V < \infty$ if and only if $gl.dim\, V = dim\, O(V)$. Although the definition is made in the CM setting, to get any relationship with the geometry it turns out to be necessary to require that R is Gorenstein. This is because for a CM ring R, there might exist $M \in CM\, R$ with $gl.dim\, End_R(M) < \infty$ but $gl.dim\, End_R(M) \neq dim\, R$. This will not happen for Gorenstein ring.

Definition 5.5. *A canonical module or dualizing module for a Noetherian commutative ring R is a finitely generated module M such that for any maximal ideal m, $Ext^n_R(R/m, M)$ has dimension 1 if $n = height(m)$, otherwise zero.*

In all the geometric situations, we will be interested in (or when R is Gorenstein) a canonical module does exist.

Lemma 5.5. *Suppose that (R, m) is a local CM normal domain of dimension d with a canonical module, and let $M \in ref\, R$.*

1. *(Reflexive equivalence) M induces equivalences of categories*

$$\begin{array}{ccc} add\, M & \longrightarrow & proj\, End_R(M) \\ \downarrow & & \downarrow \\ ref\, R & \longrightarrow & ref_R\, End_R(M) \end{array}$$

where the horiztontal arrows are equivalences induced by $Hom_R(M, -)$.

2. *(The Auslander- Buchsbaum Formula)*

   (a) *If $\Lambda := End_R(M)$ is a NCCR, then for all $X \in mod\Lambda$ we have*

   $$depth_R X + proj.dim_\Lambda X = dim\, R.$$

   In particular, this implies $CM\, R \cong Proj\, \Lambda$.

   (b) *If R is Gorenstein and $\Lambda := End_R(M) \in CM\, R$, then for all $X \in mod\, \Lambda$ with $proj.dim \Lambda M < \infty$ we have*

   $$depth_R X + proj.dim_\Lambda X = dim\, R.$$

The special case $M = R$, namely $\Lambda := End_R(R) \cong R$, gives the classical Auslander-Buchsbaum formula. As a first application, we have:

Lemma 5.6. *Suppose R is a local Gorenstein normal domain, $M \in CM\, R$ with $End_R(M) \in CM\, R$. Then $gl.dim\, End_R(M) < \infty$ if and only if $gl.dim\, End_R(M) = dim\, R$.*

The second application of the Auslander-Buchsbaum formula is the uniqueness theorem of 2-dimensional NCCR.

Theorem 5.7. *Let (R, m) be a local CM normal domain of dimension 2 with a canonical module. If R has a NCCR, then all NCCRs of R are Morita equivalent.*

Remark. All these theorems hold in the non-local setting, provided that we additionally assume that R is equicodimensional. And the theorem gives uniqueness of 2-dimensional NCCRs, but NCCRs do not exist for all local CM normal domain of dimension 2.

In dimension three, the situation is more complicated, but can still be controlled. In algebraic geometry, when passing from surfaces to 3-folds we often replace the idea of a minimal resolution with a crepant resolution, and these are not unique up to isomorphism. However, by a result of Bridgeland, all crepant resolutions a of a given $Spec\, R$ are unique up to derived equivalence. Using this as motivation, we thus ask whether all NCCRs for a given R are derived equivalent.

Remark. This is the best that we can hope for. In fact, there are examples of $NCCRs$ that are not Morita equivalent.

Theorem 5.8. *Suppose that (R, m) is a normal CM domain with a canonical module, such that $dim\, R = 3$. Then all $NCCRs$ of R are derived equivalent.*

In the commutative world, we have the following theorem:

Theorem 5.9 ([Bri00])**.** *Let X be a complex threefold with terminal singularities. Let $f : Y \to X$ and $f' : Y' \to X$ be crepant resolutions of X. Then $D^b(Coh(Y))$ is equivalent to $D^b(Coh(Y'))$.*

This proved Bondal-Orlov conjecture in dimension 3.

5.2 Tilting

We are interested in possible equivalences between $D^b(coh\, Y)$ and $D^b(mod\, \Lambda)$. To achieve this, we need two other nice subcategories.

Definition 5.6. *We define $Perf(Y) \subset D^b(coh\, Y)$ to be all those complexes that are locally quasi-isomorphic to bounded complexes consisting of vector bundles of finite ranks. We denote $K^b(proj\, \Lambda) \subset D^b(mod\, \Lambda)$ to be all those complexes isomorphic to bounded complexes of finitely generated projective Λ-modules.*

Furthermore, any equivalence between $D^b(coh\, Y)$ and $D^b(mod\, \Lambda)$ must restrict to an equivalence between $Perf(Y)$ and $K^b(proj\, \Lambda)$. Now the point is that $K^b(proj \Lambda)$ has a very special object Λ_Λ, considered as a complex in degree zero. So we need $Perf(Y)$ to contain an object that behaves in the same way as Λ_Λ does. Hence, it's important to analyze the properties of Λ_Λ.

The first property is Hom-vanishing in the derived category. More precisely, $Hom_{D^b(mod \Lambda)}(\Lambda_\Lambda, \Lambda_\Lambda[i]) = 0$ for $i \neq 0$.

Secondly, Λ_Λ 'generates' $K^b(proj\, \Lambda)$.

Definition 5.7. *Let $\mathcal{C}$ be a triangulated category. A full subcategory $\mathcal{D}$ is called a triangulated subcategory if*

1. $0 \in \mathcal{D}$
2. *$\mathcal{D}$ is closed under shifts and finite sums*
3. *If $X \to Y \to Z \to X[1]$ is a triangle in $\mathcal{C}$, then if any two of $\{X, Y, Z\}$ is in $\mathcal{D}$, then so is the third.*

If further $\mathcal{D}$ is closed under direct summands, then we say $\mathcal{D}$ is thick.

Let $\mathcal{C}$ be a triangulated category, $M \in \mathcal{C}$. We denote by $thick(M)$ the smallest full thick triangulated subcategory containing M.

Proposition 5.2. $thick(\Lambda_\Lambda) = K^b(proj\,\Lambda)$.

These properties tell us a necessary condition for $D^b(coh\,Y) \cong D^b(mod\,\Lambda)$ is that there exists a complex $\mathcal{V} \in Perf(Y)$ for which Hom-vanishing and $thick(\mathcal{V}) = Perf(Y)$. Tilting theory tells us that these properties are in fact sufficient.

Definition 5.8. *We say that $\mathcal{V} \in Perf(Y)$ is a tilting complex if $Hom_{D^b(coh\,Y)}(\mathcal{V}, \mathcal{V}[i]) = 0$ for all $i \neq 0$, and further $thick(\mathcal{V}) = Perf(Y)$. If further $\mathcal{V}$ is a vector bundle, then we say that $\mathcal{V}$ is a tilting bundle.*

Theorem 5.10 (Theorem 7.6 in [HdB05])**.** *Let Y be a projective scheme over a commutative noetherian ring of finite type over $\mathbb{C}$. Assume $\mathcal{V}$ is a tilting bundle. Then*

1. *$RHom_Y(\mathcal{V}, -)$ induces an equivalence between $D^b(coh\,Y)$ and $D^b(mod\,End_Y(\mathcal{V}))$.*
2. *Y is smooth if and only if $gl.dim\,End_Y(\mathcal{V}) < \infty$.*

Usually, we use the following trick to verify the second condition of a tilting complex.

Theorem 5.11 (Neeman's Generation Trick)**.** *Say Y has an ample line bundle $\mathscr{L}$. Pick $\mathcal{V} \in Perf(Y)$. If $(\mathscr{L}^{-1})^{\otimes n} \in thick(\mathcal{V})$ for all $n \geq 1$, then $thick(\mathcal{V}) = Perf(Y)$.*

Remark. Theorem 5.10 holds in a more general setting. For example, it's true for Noetherian abelian category, see [KK20]. In particular, it should be true for the quasi-projective scheme Y. The second condition in the tilting object also has to be modified.

Theorem 5.12 (Derived $SL_2(\mathbb{C})$ McKay correspondence)**.** *Let Γ be a finite subgroup of $SL_2(\mathbb{C})$ and let $Y \to \mathbb{C}^2/\Gamma$ denote the minimal resolution. Then*

$$D^b(\mathrm{mod}\mathbb{C}[x,y]\#\Gamma) \backsimeq D^b(Y).$$

An important three-dimensional analog can be found in [BKR01].

5.3 Relative Serre Functors and Calabi-Yau Algebras

Definition 5.9. *Suppose $Z \to Spec\,T$ is a morphism where T is a CM ring with a canonical module C_T. We say that a functor $\mathbb{S} : Perf(Z) \to Perf(Z)$ is a Serre functor relative to C_T if there are functorial isomorphisms*

$$RHom_T(RHom_Z(\mathscr{F}, \mathcal{G}), C_T) \cong RHom_Z(\mathcal{G}, \mathbb{S}(\mathscr{F}))$$

in $D(Mod\,T)$ for all $\mathscr{F} \in Perf(Z), \mathcal{G} \in D^b(coh\,Z)$. If Λ is a module-finite T-algebra, we define a Serre functor $\mathbb{S} : K^b(proj\,\Lambda) \to K^b(proj\,\Lambda)$ relative to C_T in a similar way.

Remark. Recall the classical Serre's duality. Given a locally free sheaf $\mathscr{F}$, we have

$$Ext^i(\mathscr{F}, \mathcal{G}) \cong Ext^i(\mathscr{O}_X, \mathscr{F}^V \otimes \mathcal{G}) \cong Ext^{n-i}(\mathscr{F}^V \otimes \mathcal{G}, \omega_X) \cong Ext^{n-i}(\mathcal{G}, \mathscr{F} \otimes \omega_X)$$

The condition of perfect complex modifies locally free sheaf.

Definition 5.10. *Suppose that $\mathcal{C}$ is a triangulated category in which the Hom spaces are all k-vector spaces. We say that $\mathcal{C}$ is $d-$Calabi Yau if there exists a functorial isomorphism*

$$Hom_{\mathcal{C}}(x, y[d]) \cong Hom_{\mathcal{C}}(y, x)^*$$

for all $x, y \in \mathcal{C}$, where $(-)^*$ denotes the k-dual.

Remark. Given a d-dimensional Calabi-Yau manifold. The Serre duality gives the above duality in the derived category of coherent sheaves.

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