Mirror Construction for Nakajima Quiver Varieties

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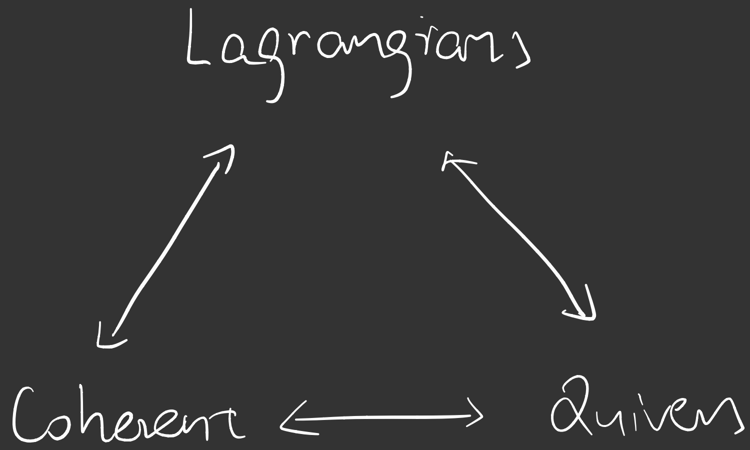
Recent Advances in Mirror Symmetry and Degenerations

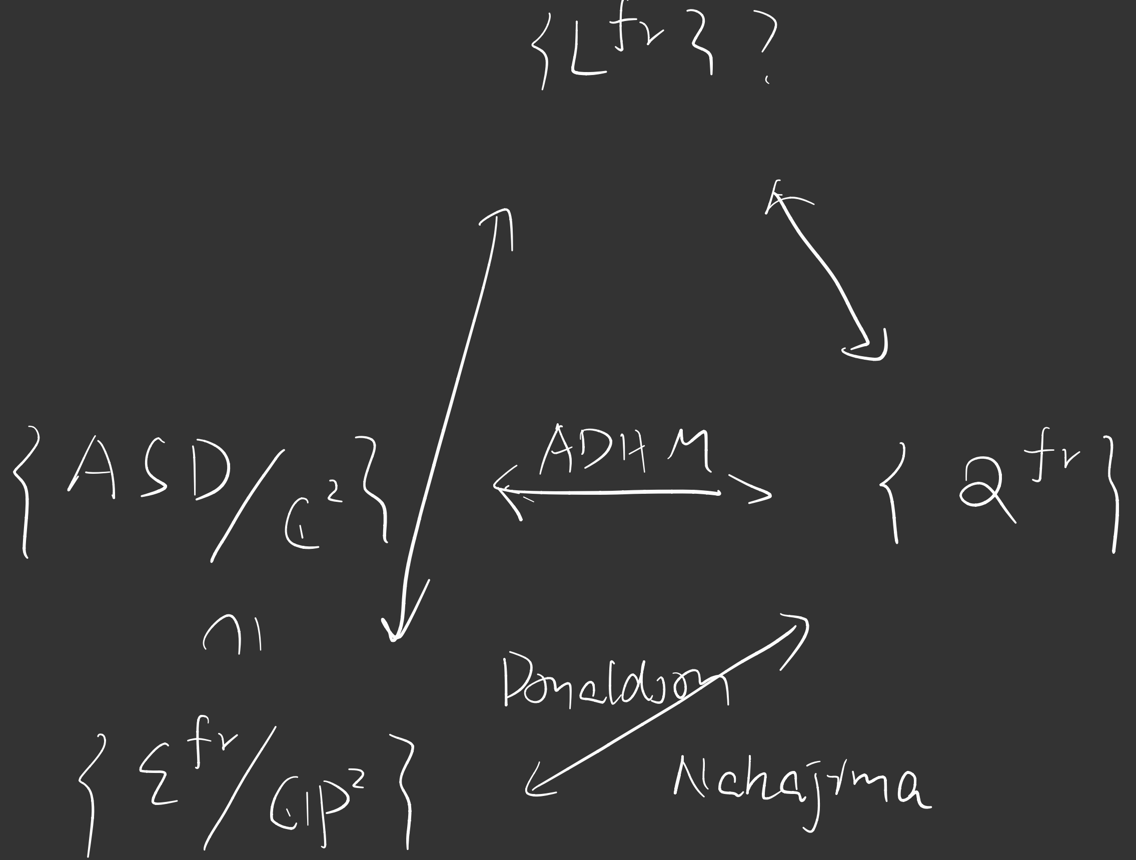
Nov. 18 - 22, IMSA

Focus on



base on ADHM construction & Mirror Symmetry

§ Motivation



Q: L^{fr} ? Dimension vector ? Stability ? Metric ...

· Prop (Hu-Lau-T.) (Sketch)

The Maurer-Cartan def spaces of $(L^{fr}, \bar{E})$

为 a Nakajima quiver var (. . .)

§ Coherent sheaves $\longleftrightarrow$ Quiver

· Def: A framed torsion-free sheaf of rk r over $\mathbb{CP}^2$ is a pair

$(\mathcal{E}^{fr}, \Phi)$ s.t. ① $\mathcal{E}^{fr}$: torsion-free

② $\Phi : \mathcal{E}^{fr}|_{l_\infty} \xrightarrow{\simeq} \mathcal{O}_{l_\infty}^{\oplus r}$

$l_\infty \subseteq \mathbb{CP}^2$

$\overset{②}{\Longrightarrow}$ $c_1(\mathcal{E}^{fr}) = 0$.

§ Coh $\longleftrightarrow$ Quiver

Recall Nakajima's ADHM construction for $\mathrm{rk}\, \mathcal{E}^{fr} = 1$, $c_2(\mathcal{E}^{fr}) = n$.

$\mathcal{E}^{fr}|_{\mathbb{C}^2}$ = torsion-free of rk 1 / $\mathbb{C}^2$ $\quad \mathbb{C}^2 = \mathrm{Spec}\, \mathbb{C}[z_1, z_2]$.

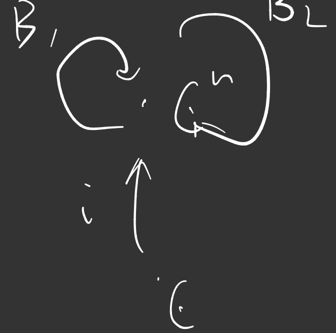
$\mathbb{C}^2 = \mathbb{CP}^2 \setminus \ell_\infty$

= ideal sheaf of n pts I

(1) $z_i \curvearrowright \mathbb{C}[z_1,z_2]/I \cong \mathbb{C}^n \circlearrowleft B_i$

(2) $\mathbb{C} \hookrightarrow \mathbb{C}[z_1,z_2] \twoheadrightarrow \mathbb{C}[z_1,z_2]/I \cong \mathbb{C}^n$

$\overset{1:1}{\longleftrightarrow}$



(1) $[B_1, B_2] = 0$

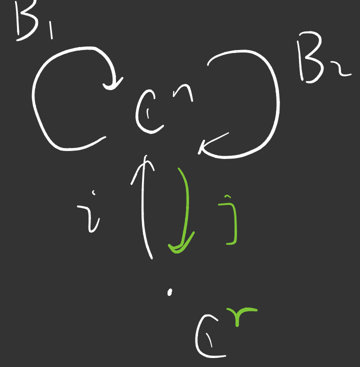
(2) (Stability) There's no $V \subsetneq \mathbb{C}^n$ that's (B_1, B_2)-inv & contains $\mathrm{Im}\, i$

$/GL_n(\mathbb{C})$

- **Thm** (Nakajima)

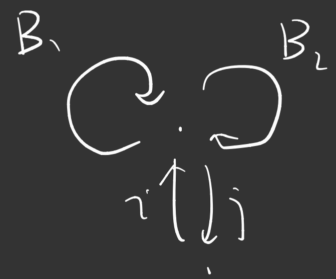
There's a 1-1 correspondence between

① $\left\{ (\mathcal{E}^{fr}, \Phi) \;\middle|\; \begin{array}{l} \Phi : \mathcal{E}^{fr}|_{\ell_\infty} \xrightarrow{\simeq} \mathcal{O}_{\ell_\infty}^{\oplus r} \\ c_2(\mathcal{E}^{fr}) = n \end{array} \right\} \Big/ \sim$

② $\{ (B_1, B_2, i, j) \in \mathrm{Rep}\Big($ 

$\Big) \;|\;$ (1) $[B_1, B_2] + ij = 0$

(2) (stability) There's no proper subspace of $\mathbb{C}^n$ that's (B_1, B_2)-inv and contains $\mathrm{Im}\, i$ $\} \big/ GL_n(\mathbb{C})$

$\{(\Sigma^{fr}, \Phi)\}/\sim \quad \overset{1:1}{\longleftrightarrow} \quad$ monads $\quad \overset{1:1}{\longleftrightarrow} \quad \{$  $\mid \cdots \}/\sim$

(Three terms complexes)

- Rmk:

(1) (Kronheimer, Kronheimer – Nakajima, Nakajima)

Similar results also hold for ADE surfaces or ALE spaces,

$\Gamma \underset{\text{finite}}{\subseteq} SL_2(\mathbb{C})$, $\Gamma \curvearrowright \mathbb{C}^2$ $\quad \mathbb{C}^2/\Gamma \longleftarrow \widetilde{\mathbb{C}^2/\Gamma}$

(2) It can also be generalized to noncommutative case.

(Kapustin – Kuznetsov – Orlov, Baranovsky - Ginzburg - Kuznetsov)

§ Lagrangian $\longleftrightarrow$ Quiver

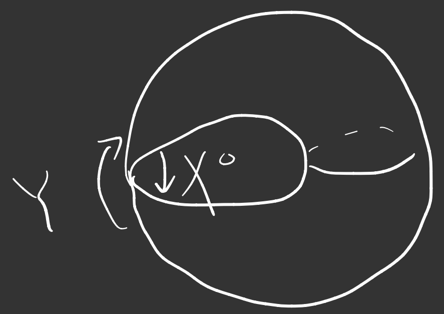
Cho – Hong – Lau

Lag immersion $L \subseteq (M, \omega)$ $\rightsquigarrow$ Quiver Q

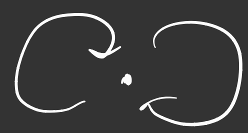
irred comp in $\tilde{L}$ $\rightsquigarrow$ vertex

deg 1 intersection $\rightsquigarrow$ arrow

· Eg:

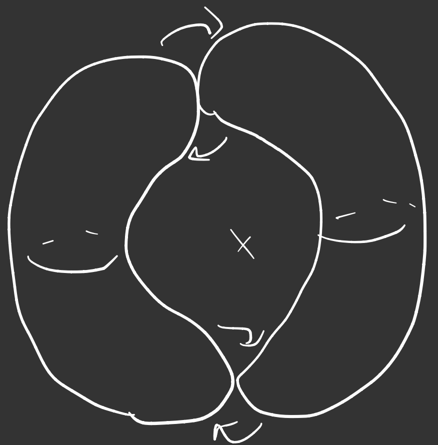
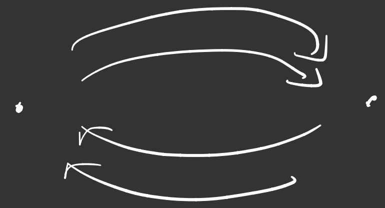


$\rightsquigarrow$



$X = (N, S)$ $\quad$ $Y = (S, N)$

· Eg:

 ⇝ 

$(CF^{\bullet}(L), \{m_k^b\})$

obstruction ⇝ Relations

Formal Def of L ⇝ Path alg A_L of Q

· Thm (Cho-Hong-Lau):

There exists an A_∞-functor $Fuk(M) \longrightarrow dg\ A_L\text{-mod}$

$\check{M}$ = moduli of Lags ∂p $\longleftrightarrow$ $L \subseteq M$

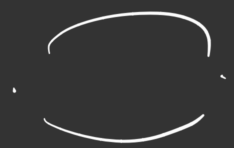
Plumbing Construction

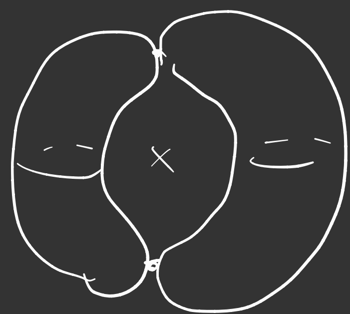
Graph D $\rightsquigarrow$ Symplectic mfd (M, ω)

vertex v $\rightsquigarrow$ Riemannian mfd S^2_v

Edge $\rightsquigarrow$ Gluing of $D^*S^2_v\big|_{U_1 \ni p_1}$ & $D^*S^2_w\big|_{U_2 \ni p_2}$ in a nbd of base pts

$\Phi : (x, y) \longmapsto (-y, x)$



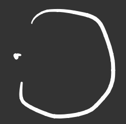
$\tilde{A}_1$ $\rightsquigarrow$ $T^*\Big($  $\Big)$

The Fukaya category of plumbing has been studied by Abouzaid, Abouzaid & Smith, Etgü & Lekili, Karabas & Lee ...

- Thm (Hu - Lau - T.): Let D be a diagram, M be the Liouville mfd obtained by the plumbing of S^2. Let $\mathbb{L}$ be the zero section. Then the associated quiver Q is the double of D, and the formal def space $A_{\mathbb{L}}$ is a preprojective algebra i.e.

$$\widehat{kQ} \Big/ \left\langle \sum_{t(a)=v} x_{\bar{a}} x_a \right\rangle_v$$

up to a change of coordinate & sign.

In particular, when $D =$ ,

$L =$ 

$\mathrm{Def}(L) = \mathbb{C}[x,y]$ if we restrict to the full subcategory that has no convergent issues.

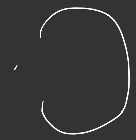
$\Rightarrow F^L : \mathrm{Fuk}^{sub}(M) \longrightarrow dg\ \mathbb{C}[x,y]\text{-mod}.\ (\mathrm{Coh}(\mathbb{C}^2))$

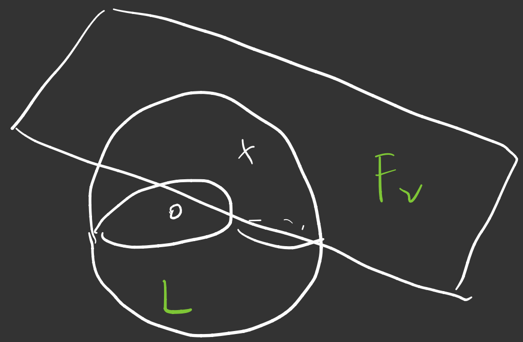
· Rmk: $\mathrm{Def}(L)$ is also computed by Hong-Kim-Lau.

- Q: What's L^{fr}?

- Answer: $L^{fr} = L \cup_v F_v$, where F_v is the cotangent fiber of S^2_v.

- Answer: $L^{fr} = L \underset{v}{\cup} F_v$, where F_v is a cotangent fiber over S^2_v.

- Eg: $D =$ 

$L^{fr} =$ 

$= L \cup F_v$

In good cases, L is a fiber of a Lag torus fibration, and F_v is a section of the Lag torus fibration

Intuition

$$0 \to I \to \mathcal{O}_{\mathbb{C}^2} \to \mathcal{O}_p \to 0$$

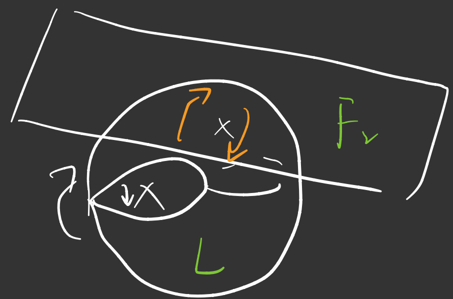
$$I \to \mathcal{O}_{\mathbb{C}^2} \to \mathcal{O}_p \xrightarrow{[1]} I$$

HMS & SYZ

$$\longleftrightarrow$$

$$\text{"}L \cup F_v\text{"} \to F_v \to L \xrightarrow{[1]} \text{"}L \cup F_v\text{"}$$

Aside:



$$E \to L^{F_v}$$

- Thm (Hu - Lau - T.)

(1) $MC\ \mathrm{Def}(L^{fr}, E)$ is

$$\left[\left\{ \left(B_1, B_2 \circlearrowright \mathbb{C}^n,\ i : \mathbb{C}^r \to \mathbb{C}^n,\ j : \mathbb{C}^n \to \mathbb{C}^r \right) \;\middle|\; [B_1, B_2] + ij = 0 \right\} \Big/ GL_n(\mathbb{C}) \right]$$

(2) $b_0 \in MC\ \mathrm{Def}(L^{fr}, E)$ is a stable rep of Q^{fr} iff

$HF^2(L, (L^{fr}, E, b_0)) = 0$

3) $F^L(L^{fr}, E, b_0)$ is monad over $\mathbb{C}^2$, whose cohomology is a (resp. framed) torsion-free sheaf over $\mathbb{C}^2$ (resp. $\mathbb{CP}^2$).

Thm (Hu-Lau-T.)

Let $\mathbb{L}^{fr} = \mathbb{L} \cup_v F_v$, where F_v is one cotangent fiber over S^2_v. Then the associated quiver Q^{fr} is the framed double quiver of D, & the formal def space of $\mathbb{L}^{fr}$ is a framed preprojective algebra, i.e.

$$\widehat{kQ^{fr}} \Big/ \left\langle \sum x_{\bar{a}} x_a + i_v j_v \right\rangle_v$$

up to a change of coordinate.

- Cor: The Maurer-Cartan def space of $(\mathbb{L}^{fr}, E)$ is the Nakajima quiver variety $\mathcal{M}(Q^{fr}, \operatorname{rank} E)$ if we fix a stability condition, where E is the trivial V.B. over the normalization of $\mathbb{L}^{fr}$.

· Rmk: Similar results about framed torsion-free sheaves also hold over ADE surfaces.

· Thm (Lau - T.)

Let E_1, E_2 be flat V.B. over L^{fr} with rank E_1 < rank E_2, then $HF^\circ((L^{fr}, E_1, b_1), (L^{fr}, E_2, b_2))$ is a sheaf over $m(Q^{fr}, \text{rank } E_1) \times m(Q^{fr}, \text{rank } E_2)$, whose support is the Hecke correspondence.

Thank You!