Brief Introduction to the McKay Correspondence

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1 Introduction

This note aims at giving a brief survey to McKay Correspondence, with an emphasis on the Γ-Hilbert schemes, where Γ is a finite group.

2 Finite Subgroups of SL$(2, \mathbb{C})$

The story starts by considering the classification or the representation of a finite subgroup $\Gamma \subset SL(2, \mathbb{C})$. Given the standard Hermitian metric on $\mathbb{C}^2$. Γ acts naturally on $\mathbb{C}^2$. Averaging the inner product by the group Γ, we arrive at a hermitian inner product which is invariant with respect to Γ. This shows that Γ is conjugate to a finite subgroup of the special unitary group $SU(2)$. Hence, the classification of finite subgroups of $SL(2, \mathbb{C})$ is equivalent to the classification of finite subgroups of $SU(2)$.

The idea to classify the finite subgroups of $SU(2)$ is to consider the double cover

$$\pi : SU(2) \twoheadrightarrow SO(3).$$

This double cover is defined via the multiplication structure in the quaternion. Thus any finite subgroup Γ of $SU(2)$ defines a finite subgroup $\bar{G}$ of rotations of $\mathbb{R}^3$. Conversely, every $\bar{\Gamma} \subset SO(3)$ can be lifted to a finite subgroup of $SU(2)$ such that the kernel is of order 2. From this and the classical classification of finite subgroups of $SO(3)$ as symmetry groups of regular polyhedra, we obtain the following.

Proposition 2.1. *Any finite subgroup of $SU(2)$ is one of the following groups:*

1. *The cyclic group $\mathbb{Z}/n\mathbb{Z}$ for $n > 1$.*
2. *The binary dihedral group $\mathbb{B}D_{2n}$ for $n > 1$, the preimage of the dihedral group D_{2n} under π.*
3. *The binary tetrahedral group $\mathbb{BT}$, the preimage of the tetrahedral group $\mathbb{T}$ under π.*
4. *The binary octahedral group $\mathbb{BO}$, the preimage of the octahedral group $\mathbb{O}$ under π.*
5. *The binary dodecahedral group $\mathbb{BD}$, the preimage of the dodecahedral group $\mathbb{D}$ under π.*

To be more precise, here we choose a basis of $\mathbb{C}^2$ and write down the generators of the action explicitly. Let $\epsilon_n := e^{2\pi i/n}$.

1. Γ is a cyclic group of order n. A generator is given by the matrix

$$g_1 = \begin{bmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{-1} \end{bmatrix}.$$

2. Γ is a binary dihedral group of order $4n$. Its generators are given by the matrices

$$g_1 = \begin{bmatrix} \epsilon_{2n} & 0 \\ 0 & \epsilon_{2n}^{-1} \end{bmatrix}, g_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

3. Γ is a binary tetrahedral group of order 24. Its generators are given by the matrices

$$g_1 = \begin{bmatrix} \epsilon_4 & 0 \\ 0 & \epsilon_4^{-1} \end{bmatrix}, g_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, g_3 = \frac{1}{1-i} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}.$$

4. Γ is a binary octahedral group of order 48. Its generators are

$$g_1 = \begin{bmatrix} \epsilon_8 & 0 \\ 0 & \epsilon_8^{-1} \end{bmatrix}, g_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, g_3 = \frac{1}{1-i} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}.$$

5. Γ is a binary icosahedra group of order 120. Its generators are

$$g_1 = \begin{bmatrix} \epsilon_{10} & 0 \\ 0 & \epsilon_{10}^{-1} \end{bmatrix}, g_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, g_3 = \frac{1}{\sqrt{5}} \begin{bmatrix} \epsilon_5 - \epsilon_5^4 & \epsilon_5^2 - \epsilon_5^3 \\ \epsilon_5^2 - \epsilon_5^3 & -\epsilon_5 + \epsilon_5^4 \end{bmatrix}.$$

The McKay correspondence, named after John McKay, states that there is a one-to-one correspondence between the finite subgroups of $SL(2, \mathbb{C})$ and the extended Dynkin diagrams, which appear in the ADE classification of the simple Lie algebras. This is done by the McKay graphs. Here we recall the construction.

Definition 2.1. *Let Γ be a finite subgroup and ρ be its linear representation. The McKay graph of the pair (Γ, ρ) is defined to be a graph, where the vertices correspond to irreducible representations ρ_i of Γ. A vertex ρ_i is connected to the vertex ρ_j by an edge pointing to ρ_j if ρ_j is a direct summand of $\rho \otimes \rho_i$. Then the weight m_{ij} of the arrow is the number of times this constituent appears in $\rho \otimes \rho_i$.*

The classical McKay correspondence classifies the possible groups Γ via their McKay graphs. More precisely, we have the following.

Theorem 2.1 (J. McKay)**.** *Let Γ be a nontrivial finite subgroup of $SU(2)$ and ρ be its natural 2-dimensional representation defined by the inclusion. Then, the McKay graph of (Γ, ρ_0) is an affine ADE Dynkin diagram.*

Here we provide an explicit calculation of the cyclic group.

Example 2.1. *Let $G = C_n =< g_0 >$ be a cyclic group of order n. Since C_n is an abelian group, every linear representation $\rho : C_n \to GL(V)$ decomposes into the direct sum of 1-dimensional representations*

$$V = \sum_{0}^{n-1} V_k,$$

where $V_k := \{v \in V : \rho_0(g_0)(v) = e^{2\pi i k/n} v\}$. So C_n has n irreducible representations.

If we consider $\rho_0 : C_n \to SU(2)$ given by the matrix

$$\begin{bmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{-1} \end{bmatrix},$$

we find that $\rho_0 = \rho_1 \oplus \rho_{-1}$. Thus $\rho_0 \otimes \rho_k = \rho_{k-1} \oplus \rho_{k+1}$. Hence, the Mckay graph (C_n, ρ_0) is the Dynkin diagram of affine $\tilde{A}_{n-1}$.

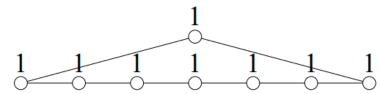
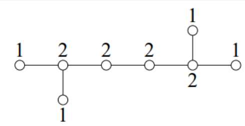
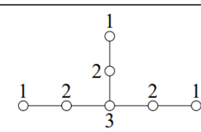
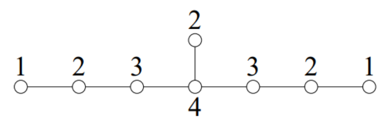
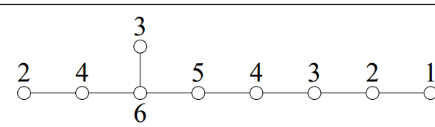
Finite subgroup of $SU(2)$		Affine simply laced Dynkin diagram	
$\mathbb{Z}/n\mathbb{Z}$	$\langle x \mid x^n = 1 \rangle$	$\widetilde{A}_{n-1}$	
$\mathbb{B}D_{2n}$	$\langle x, y, z \mid x^2 = y^2 = y^n = xyz \rangle$	$\widetilde{D}_{n-2}$	
$\mathbb{BT}$	$\langle x, y, z \mid x^2 = y^3 = z^3 = xyz \rangle$	$\widetilde{E}_6$	
$\mathbb{BO}$	$\langle x, y, z \mid x^2 = y^3 = z^4 = xyz \rangle$	$\widetilde{E}_7$	
$\mathbb{BD}$	$\langle x, y, z \mid x^2 = y^3 = z^5 = xyz \rangle$	$\widetilde{E}_8$	

Figure 1: The McKay Correspondence

3 Geometric McKay correspondence

The slogan of the geometric McKay is that the representation theory of the finite subgroup Γ is 'equivalent' to the geometry or topology of the minimal/crepant resolution of $\mathbb{C}^2/\Gamma$, [Rei97, Rei02].

3.1 The main statements

Let's start by classifying the quotient space. Let Γ be a finite subgroup as before. It acts on $\mathbb{C}^2$ naturally. It's interesting to consider the quotient space $\mathbb{C}^2/\Gamma$. Based on McKay correspondence, see Theorem 2, we have the following explicit classifications of $\mathbb{C}^2/\Gamma$.

Theorem 3.1. *Let Γ be a finite subgroup of $SU(2)$ and $\mathbb{C}^2 := Spec\,\mathbb{C}[x, y]$. Then the quotient space $\mathbb{C}^2/\Gamma$ has the following forms:*

1. *A_n case: $\mathbb{C}^2/\Gamma \cong Spec\,\mathbb{C}[X, Y, Z]/(XY - Z^n)$.*
2. *D_n case: $\mathbb{C}^2/\Gamma \cong Spec\,\mathbb{C}[X, Y, Z]/(X^2 + ZY^2 + Z^{n-1})$, $n \geq 4$.*
3. *E_6 case: $\mathbb{C}^2/\Gamma \cong Spec\,\mathbb{C}[X, Y, Z]/(X^2 + Y^3 + Z^4)$.*
4. *E_7 case: $\mathbb{C}^2/\Gamma \cong Spec\,\mathbb{C}[X, Y, Z]/(X^2 + Y^3 + YZ^3)$.*
5. *E_8 case: $\mathbb{C}^2/\Gamma \cong Spec\,\mathbb{C}[X, Y, Z]/(X^2 + Y^3 + Z^5)$.*

In particular, these spaces only have the singularity at the orgin.

Proof. For simplicity, we will only prove the quotient space of a cyclic subgroup.

Notice that the generator g_1 of $\Gamma := \mathbb{Z}/n\mathbb{Z}$ acts on (x, y) via $g_1 \cdot (x, y) = (\epsilon_n x, \epsilon_n^{-1} y)$. Hence, $X := x^n, Y := y^n, Z := xy$ are invariant under the group action.

On the other hand, suppose $f := x^a y^b$ is a monomial invariant under the group action. Then $g_1 \cdot f = \epsilon_n^{a-b} x^a y^b = x^a y^b$. Hence, $a - b \equiv 0 \ (mod\, n)$. Thus f is a multiple of X, Y, Z. Therefore, we know $\mathbb{C}[x, y]^\Gamma \cong \mathbb{C}[X, Y, Z]/(XY - Z^n)$. In particular, $\mathbb{C}^2/\Gamma \cong Spec\, \mathbb{C}[X, Y, Z]/(XY - Z^n)$.

The other cases are similar to the A type, but the generators are more difficult to find. □

Remark. The orgins are sometimes called the Kleinnian singularity, Du Val singularity or simple singularity.

The geometric McKay correspondence examines how the topology of the minimal resolution of $X := \mathbb{C}^2/\Gamma$ reflects the representation theory of Γ. A general theory of the algebraic surface tells us that the minimal resolution of X exists and is unique. However, minimal resolutions do not necessarily exist in higher dimensions (even in dimension 3). Instead, we seek a crepant resolution. For $X := \mathbb{C}^2/\Gamma$, the crepant resolution is also minimal.

Definition 3.1. *A resolution of scheme $f : \tilde{X} \to X$ is called crepant if $f^* K_X = K_{\tilde{X}}$, where $K_{\tilde{X}}$ is the canonical bundle over $\tilde{X}$.*

Since $X := \mathbb{C}^2/\Gamma$ is a hypersurface in $\mathbb{C}^3$ and the adjunction formula holds even for singular divisor, we know that the canonical line bundle K_X exists over X and X has Gorenstein singularity at the origin. Furthermore, by adjunction formula, the canonical bundle K_X is trivial, i.e. $K_X \cong \mathscr{O}_X$, since there are no nontrivial line bundles over $\mathbb{C}^3$.

In the following, we want to compute two explicit examples, which give a sense of how to resolve the singularity. In fact, $\mathbb{C}^2/\Gamma$ can be resolved by successively blowing up the singular points. These examples are insightful and shed light on the Geometric McKay correspondence. Readers who are not interested in this can skip to Theorem 3.2.

Let's start by recalling the definition of the incidence graph (dual graph).

Definition 3.2. *Let E be a subvariety which is a tree of $\mathbb{P}^1$. The incidence graph of E is constructed by assigning each irreducible component of E a vertex, assigning an edge between two vertices if the corresponding irreducible components intersect.*

Example 3.1 (A_1 case)**.** *The first example we consider is $\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})$. By theorem 3.1, we know $X := \mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z}) \cong Spec\, \mathbb{C}[x_1, x_2, x_3]/(x_1 x_2 - x_3^2)$.*

First, we consider blowup of $\mathbb{A}^3$ at the origin, denoted by $Y := Bl_O(\mathbb{A}^3)$. We consider the blowup as the closure of the graph of φ, where $\varphi : \mathbb{A}^3 \setminus \{0\} \to \mathbb{P}^2$ via $\varphi(x_1, x_2, x_3) = [x_1, x_2, x_3]$. In other words, φ takes a point to the line containing the point and the origin. Therefore, it's not hard to see that

$$Y := \overline{graph\, \varphi} = \{((x_1, x_2, x_3) \times [y_1, y_2, y_3]) \in \mathbb{A}^3 \times \mathbb{P}^2 \mid x_i y_j = x_j y_i, \forall i, j\}.$$

The resolution X_1 is the strict transform of X. In other words,

$$X_1 \cong \overline{\varphi^{-1}(X \setminus 0)} = \{((x_1, x_2, x_3) \times [y_1, y_2, y_3]) \in \mathbb{A}^3 \times \mathbb{P}^2 \mid x_i y_j = x_j y_i, x_1 x_2 = x_3^2, y_1 y_2 = y_3^2, \forall i, j\}.$$

By looking at the Jacobian matrix of X_1, we know X_1 is regular. Denote π be the blowup map. Then the exceptional curve $E := \pi^{-1}(0) \subset X_1$ is a degree 2 curve in $\mathbb{P}^2$ with self intersection number -2. More precisely, X_1 is isomorphic to $K_{\mathbb{P}^1}$, the total space of canonical bundle over $\mathbb{P}^1$.

In particular, ***the incidence graph of E is the A_1 Dynkin diagram.***

Example 3.2 (A_3 case)**.** *Resolving* $X := \mathbb{C}^2/(\mathbb{Z}/4\mathbb{Z}) \cong Spec\,\mathbb{C}[x_1, x_2, x_3]/(x_1^2 + x_2^2 - x_3^4)$ *is more intereting. This example can represent the standard procedure for resolving the Klein singularities. In this case, we need to blowup twice. The second blowup will be computed locally.*

Similar to the above example, we first consider the blowup of $\mathbb{A}^3$ *at the origin.*

$$Y := \overline{graph\,\varphi} = \{((x_1, x_2, x_3) \times [y_1, y_2, y_3]) \in \mathbb{A}^3 \times \mathbb{P}^2 \mid x_i y_j = x_j y_i, \forall i, j\}.$$

Y *can be covered by three affine charts. More precisely, let* $Z_i := D(y_i)$ *be the open subscheme of* Y *with* y_i *doesn't equal zero. Then* $Z_1 := \{((x_1, x_2, x_3) \times [1, y_2, y_3]) \in \mathbb{A}^3 \times \mathbb{P}^2 | x_2 = x_1 y_2, x_3 = x_1 y_3\} \cong Spec\,\mathbb{C}[x_1, y_2, y_3] \cong \mathbb{A}^3$*. Similarly, we find that* $Z_i \cong \mathbb{A}^3$ *for* $i = 2, 3$*.*

We try to analyze X_1 *using these local charts. Notice that* $X_1 \cap Z_1 \cong \overline{\varphi^{-1}(X \setminus 0) \cap Z_1} = \{((x_1, x_2, x_3) \times [1, y_2, y_3]) \in \mathbb{A}^3 \times \mathbb{P}^2 \mid x_2 = x_1 y_2, x_3 = x_1 y_3, x_1^2 + x_2^2 - x_3^4 = 0\} \cong \{((x_1, x_2, x_3) \times [1, y_2, y_3]) \in \mathbb{A}^3 \times \mathbb{P}^2 \mid x_2 = x_1 y_2, x_3 = x_1 y_3, x_1^2 + x_1^2 y_2^2 - x_1^4 y_3^4 = 0\} \cong Spec\,\mathbb{C}[x_1, y_2, y_3]/(1 + y_2^2 - x_1^2 y_3^4)$*.*

Similarly, we can compute

$$X_1 \cap Z_2 \cong Spec\,\mathbb{C}[y_1, x_2, y_3]/(y_1^2 + 1 - x_2^2 y_3^4).$$

The most interesting part is $X_1 \cap Z_3 \cong \{((x_1, x_2, x_3) \times [y_1, y_2, 1]) \in \mathbb{A}^3 \times \mathbb{P}^2 \mid x_1 = x_3 y_1, x_2 = x_3 y_2, x_1^2 + x_2^2 - x_3^4 = 0\} \cong \{((x_1, x_2, x_3) \times [y_1, y_2, 1]) \in \mathbb{A}^3 \times \mathbb{P}^2 \mid x_1 = x_3 y_1, x_2 = x_3 y_2, x_3^2 y_1^2 + x_3^2 y_2^2 - x_3^4 = x_3^2(y_1^2 + y_2^2 - x_3^2) = 0\} \cong Spec\,\mathbb{C}[x_3, y_1, y_2]/(y_1^2 + y_2^2 - x_3^2)$*.*

By computing the Jacobian matrix, we know the first two charts are smooth, but $X_1 \cap Z_3$ *still has a singularity at the origin. Furthermore, we claim that* $\sigma_1^{-1}(0)$ ***consists of two intersecting exceptional curves!*** *This is because* $\sigma_1^{-1}(0) \cap Z_3 \cong Spec\,\mathbb{C}[y_1, y_2]/(y_1^2 + y_2^2)$ *contains two irreducible components* $V_1 := V(y_1 + iy_2)$ *and* $V_2 := V(y_1 - iy_2)$*. Notice that* $\sigma_1^{-1}(0) \cap Z_1 = Spec\,\mathbb{C}[y_3]$*. The transition map of* $\mathbb{P}^2$ *tells us that this curve is glued with* V_1 *to get an exceptional curve* $E_1 \cong \mathbb{P}^1$*. Similarly for* V_2*, so we have two exceptional curves* $E_1 \cup E_2$*.*

To resolve the remaining singular point, we do the blowing up again in the chart $X_1 \cap Z_3$*. This is the same as* A_1 *case. Hence, we get one more exceptional curve* E_3*.*

In the end, we get three exceptional divisors, which are E_3 *and the strict transformation of* E_1 *and* E_2*, denoted by* $\tilde{E_1}, \tilde{E_2}$ *respectively. Since* E_1 *and* E_2 *intersect at the blowup point,* $\tilde{E_1}, \tilde{E_2}$ *no longer intersect. But they all intersect* E_3*. Therefore,* ***the incidence graph of the exceptional divisor*** E ***is the*** A_3 ***Dynkin diagram.***

The corresponding intersection matrix $(E_i \cdot E_j)_{ij} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$*, which is the negative of the Cartan matrix of* A_3 *Dynkin diagram.*

The phenomenon that the incidence graph of the exceptional divisor corresponds to the ADE type Dynkin diagram holds in general. We have the following theorem:

Theorem 3.2 (Geometric McKay Correspondence)**.** *Let* Γ *be a finite subgroup in* $SL_2(\mathbb{C})$*. The quotient space* $\mathbb{C}^2/\Gamma$ *admits a crepant resolution* $\widetilde{\mathbb{C}^2/\Gamma}$*, which is also minimal.*

Besides, the exceptional divisor $\pi^{-1}(0)$ *is a tree of* $\mathbb{P}^1$*, whose incidence graph is the Dynkin diagram of* Γ*. Here* $\pi : \widetilde{\mathbb{C}^2/\Gamma} \to \mathbb{C}^2/\Gamma$ *is the crepant resolution. In addition, the intersection matrix of the exceptional divisor is the negative of the corresponding Cartan matrix.*

Furthermore, the representation ring $R(\Gamma)$ *of* Γ *is isomorphic to the* K *ring* $K(\widetilde{\mathbb{C}^2/\Gamma})$*.*

Proof. The proof can be found in for example [Nak99], [IN96]. □

Proposition 3.1. *The rank of $H^2(\widetilde{\mathbb{C}^2/\Gamma})$ equals the number of irreducible components in the exceptional divisors, which also equals the number of nontrivial irreducible representations of Γ.*

Proof. In fact, $\widetilde{\mathbb{C}^2/\Gamma}$ is homotopic to the exceptional divisors by a general result of Nakajima quiver variety [Nak94]. Hence, $H^2(\widetilde{\mathbb{C}^2/\Gamma})$ is generated by the fundamental class of each irreducible component, which is $\mathbb{CP}^1$.

Furthermore, by the Geometric McKay correspondence, we know that there's 1-1 correspondence between the irreducible components of the exceptional divisor and the vertices in the corresponding ADE Dynkin diagram, which also corresponds to the nontrivial irreducible representations of Γ. □

Remark. Nakajima [Nak94] also showed that the exceptional divisors form a holomorphic Lagrangian subvariety

In summary, the Geometric McKay correspondence tells us that the exceptional curves correspond to the irreducible representations of Γ, with intersection data reflecting the structure of the group's McKay graph. This interplay between geometry and representation theory is both elegant and powerful.

3.2 Resolution of Kleinian Singularities: Γ-Hilbert Schemes

The main idea is that one can construct the crepant resolution by keeping track how the Γ-orbits approach the origins, i.e. Γ-Hilbert Schemes. In this subsection, we want to introduce the notions of Γ-Hilbert Schemes of points on $\mathbb{C}^2$ and we will see that the fine moduli spaces are quiver varieties.

Let M be a nonsingular quasiprojective complex variety of dimension n, and Γ be a finite subgroup in the automorphism group of M, with the property that the stabilizer subgroup of any point $x \in M$ acts on T_xM as a subgroup of $SL(T_xM)$. For example, let $M = \mathbb{C}^n$ and Γ be the finite subgroup in $SL_n(\mathbb{C})$.

The Γ-Hilbert scheme $\Gamma-\mathrm{Hilb}(M)$ was introduced by Nakamura [**?**] as a good can didate for a crepant resolution of M/Γ. It parametrises Γ-clusters or 'scheme theoretic Γ-orbits' on M: recall that a cluster $Z \subset M$ is a zero-dimensional subscheme.

Definition 3.3. *A Γ-cluster is a Γ-invariant cluster whose global sections are isomorphic to the regular representation $\mathbb{C}[\Gamma]$ of Γ.*

There is a Hilbert–Chow morphism

$$\pi : \Gamma-\mathrm{Hilb}(M) \to M/\Gamma,$$

which, on closed points, sends a Γ-cluster to the orbit supporting it. Note that π is a projective morphism, is onto and is birational on one component.

From now on, we will focus on $M = \mathbb{C}^2$ and Γ be a finite subgroup of $SL_2(\mathbb{C})$. To give a precise description of Γ-Hilbert Schemes, we have better recall the Hilbert scheme of points over $\mathbb{C}^2$.

Let's recall Nakajima's construction of Hilbert scheme of n points on $\mathbb{C}^2 = \mathrm{Spec}(\mathbb{C}[z_1, z_2])$.

Given an ideal sheaf I of n points on $\mathbb{C}^2$. (We will not distinguish the modules and the associated coherent sheaves, since $\mathbb{C}^2$ is affine.) We have the quotient $\mathbb{C}[z_1, z_2]/I \cong \mathbb{C}^n$. Notice that $\mathbb{C}[z_1, z_2]/I$ is a $\mathbb{C}[z_1, z_2]$-module. Hence, the action of z_i induces endomorphism B_i on $\mathbb{C}^n$. Since z_1 and z_2 commute, we know $[B_1, B_2] = 0$.

Besides, there exists an inclusion of the coefficient ring. More precisely, we have $\mathbb{C} \hookrightarrow \mathbb{C}[z_1, z_2] \twoheadrightarrow \mathbb{C}[z_1, z_2]/I \cong \mathbb{C}^n$. Therefore, we have a morphism $i : \mathbb{C} \to \mathbb{C}^n$. In addition, this implies $z_1^p z_2^q \cdot 1$ forms a basis of $\mathbb{C}^n$ for all $p, q \in \mathbb{Z}$. It corresponds to the fact there's no proper subspace of $\mathbb{C}^n$ that is (B_1, B_2)-invariant and contains $Im(i)$. Furthermore, this construction doesn't depend on the choices of the basis. (B_1, B_2, i) is a quiver representation.

In summary, given an ideal sheaf I of n points, we obtain an isomorphic class of linear maps $(B_1, B_2, i) \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathbb{C}^2) \oplus \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n)$ satisfying $[B_1, B_2] = 0$ and

(*) there's no proper subspace of $\mathbb{C}^n$ that is (B_1, B_2)-invariant and contains $Im(i)$.

Theorem 3.3 ([Nak94]). *The quiver variety*

$$\mathcal{M}(n,1) := \{(B_1, B_2, i) \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathbb{C}^2) \oplus \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n) \mid [B_1, B_2] = 0, \text{satisfy Condition } (*)\}/GL_n(\mathbb{C})$$

is the Hilbert scheme of n points on $\mathbb{C}^2$.

Remark. Similar constructions can be generalized to $\mathbb{C}^n$, see for example [IN00].

Recall that for Hilbert scheme of points on surfaces, we have the Hilbert-Chow morphisms, which is a resolution:

$$\pi : \mathrm{Hilb}^n(\mathbb{C}^2) \to S^n(\mathbb{C}^2).$$

In the following, we consider the case that $n = |\Gamma|$, the cadinality of Γ. The Γ-action on $\mathbb{C}^2$ naturally induces that on $\mathrm{Hilb}^n(\mathbb{C}^2)$ and on the symmetric product $S^n(\mathbb{C}^2)$. Since the Γ-action on $\mathbb{C}^2 \setminus \{0\}$ is free, the Γ-orbit $\Gamma \cdot p$ of a point $p \in \mathbb{C}^2 \setminus \{0\}$ consists of n distinct points, hence defines a 0-dimensional subscheme $Z \in \mathrm{Hilb}^n(\mathbb{C}^2)$. In addition, the quotient of the corresponding ideal sheaf gives a regular representation of Γ. Conversely, any Γ-fixed point in the open stratum $\pi^{-1}(S^n_{(1,\ldots,1)}(\mathbb{C}^2))$ comes from a Γ-orbit, where $S^n_{(1,\ldots,1)}$ is the open subset that contains n-unordered distinct points.

Let $\tilde{X}$ be the closure of the set of orbits $\Gamma \cdot \mathbb{C}^2 \setminus \{0\}$ and it has dimension 2. Then we have

Theorem 3.4. *[IN96] $\tilde{X}$ is the Γ-Hilbert scheme. Besides, the restriction of the Hilbert-Chow morphism to $\tilde{X}$ is the crepant resolution of $\mathbb{C}^2/\Gamma = (S^n(\mathbb{C}^2))^\Gamma$.*

Let's give an explicit description of $\Gamma - \mathrm{Hilb}(\mathbb{C}^2)$. For our purpose, we denote $\mathbb{C}^n$ (resp. $\mathbb{C}$) by R (resp. W), since $\mathbb{C}^n$ is the regular representation. And we will write $Q := \mathbb{C}^2$, which is the natural representation of Γ. Take the irreducible decomposition of R and W as Γ-module

$$W = W_0 \otimes R_0, \;\; R = \textstyle\bigoplus_k V_k \otimes R_k,$$

where R_k is the irreducible representations of Γ with R_0 be the trivial representation and the dimension of V_k stands for the multiplicities.

Recall that in the construction of the McKay graph, we also consider the decomposition

$$Q \otimes R_i = \otimes m_{ij} R_j.$$

Therefore, we have the following decompositions of the Γ-invariant part $(\mathrm{Hom}(R, R \otimes Q) \oplus \mathrm{Hom}(W, R))^\Gamma = \mathrm{Hom}_\Gamma(R, R \otimes Q) \oplus \mathrm{Hom}_\Gamma(W, R)$:

$$\mathrm{Hom}_\Gamma(R, R \otimes Q) = \bigoplus_{k,l} \mathrm{Hom}_\Gamma(V_l \otimes R_l, V_k \otimes R_k \otimes Q) = \bigoplus_{k,l} m_{kl} \mathrm{Hom}(V_l, V_k).$$

$$\mathrm{Hom}_\Gamma(W, R) = \mathrm{Hom}(W_0, R_0).$$

Notice that this is nothing else but the representations of the double quiver associated to the affine ADE Dynkin diagrams (ignoring the framing $\mathrm{Hom}(W_0, R_0)$).

Let's recall what is the double quiver and the preprojective algebra.

Definition 3.4. *Given a graph D. The double quiver Q of D is a quiver with the same vertices and with the set of oriented edges $H := (e, o(e))$, where e is an edge of D and $o(e)$ is the orientation of e. Thus, each edge e connecting vertices v_i and v_j gives rises to two oriented arrows $a : v_i \to v_j$ and $\bar{a} : v_j \to v_i$.*

Definition 3.5. *A preprojective algebra of the double quiver Q is the path algebra $\mathbb{C}Q/I$, where I is the two-sided ideal generated by $\sum_{t(a)=v} X_{\bar{a}} X_a$ for all vertices v up to sign. Here X_a is the element in the path algebra $\mathbb{C}Q$ associated with the arrow a.*

Notice that the relations actually happen at every vertex.

Proposition 3.2. *The Γ-Hilbert scheme $\tilde{X}$ is the quiver variety that parametrizes the rank $(V_0, \ldots, V_n)$ representations of Q satisfying the preprojective algebra relations, where Q is the double quiver of the McKay graph associated to Γ.*

Remark.

1. By a standard fact of regular representations, we know $dim_{\mathbb{C}} V_k = dim_{\mathbb{C}} R_k$.
2. Furthermore, the tautological bundles of the quiver variety $\tilde{X}$ form a basis of $K(\tilde{X})$.
3. This is one of the key step in the HyperKahler construction of the ALE spaces [Kro89].

4 Derived McKay Correspondence

Theorem 4.1 ([KV98] Derived $SL_2(\mathbb{C})$ McKay correspondence)**.** *Let Γ be a finite subgroup of $SL_2(\mathbb{C})$ and let $\pi : \tilde{X} := \widetilde{\mathbb{C}^2/\Gamma} \to X := \mathbb{C}^2/\Gamma$ denote the crepant resolution. Then $\mathcal{A}$ is Morita equivalent to $\mathbb{C}[x, y]\#\Gamma$ i.e.*

$$\mathrm{mod} - \mathcal{A} \backsimeq \mathrm{mod} - \mathbb{C}[x, y]\#\Gamma,$$

where $\mathcal{A}$ is the preprojective algebra of the corresponding affine ADE Dynkin diagram. Furthermore,

$$D^b(\tilde{X}) \backsimeq D^b(\mathrm{mod} - \mathcal{A}) \backsimeq D^b(\mathrm{mod} - \mathbb{C}[x, y]\#\Gamma).$$

Remark. In fact, X admits a tilting bundle, which is the direct sum of the tautological bundles.

An important higher-dimensional analog can be found in [BKR01].

Theorem 4.2 ([BKR01])**.** *Let Γ be a finite subgroup of $SL_n(\mathbb{C})$ and $X := \mathbb{C}^n/\Gamma$. Let $\tilde{X}$ be the Γ-Hilbert scheme constructed as before together with the Hilbert-Chow morphism $\pi : \tilde{X} \to X$. Suppose the fiber product*

$$\tilde{X} \times_X \tilde{X} = \{(x_1, x_2) \in \tilde{X} \times \tilde{X} \mid \pi(x_1) = \pi(x_2)\} \subset \tilde{X} \times \tilde{X}$$

has dimension $\leq n + 1$. Then $\tilde{X}$ is a crepant resolution of X and $D^b(\tilde{X}) \backsimeq D^b_\Gamma(X)$.

Since the technical assumption in the previous theorem naturally holds for $n = 2, 3$, we have

Corollary 4.1 (Derived $SL_3(\mathbb{C})$ McKay correspondence)**.** *Let Γ be a finite subgroup of $SL_3(\mathbb{C})$ and $X := \mathbb{C}^3/\Gamma$ and $\tilde{X}$ be the Γ-Hilbert scheme. Then $D^b(\tilde{X}) \backsimeq D^b_\Gamma(X)$.*

Remark. Notice that the crepant resolution need not exist in higher dimensions, such as dimension 4.

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