Brief introduction to SYZ conjecture

Ju Tan

Abstract

This note is dedicated to giving a brief introduction to the famous SYZ conjecture.

1 Basic Concepts

Before introducing the ground-breaking proposal of Strominger, Yau and Zaslow, let's briefly recall some basic concepts. Readers who are familiar with symplectic geometry can feel free to skip this section.

Definition 1.1. *A symplectic manifold is a smooth manifold M, equipped with a nondegenerate closed 2-form ω.*

The existence of such nondegenerate two forms will force the smooth manifold to have even dimensions. Hence, it makes sense to introduce the following definition.

Definition 1.2. *Let (M, ω) be a 2n-dimensional symplectic manifold. A submanifold Y of M is a Lagrangian submanifold if Y has dimension n and $\omega|_Y \equiv 0$.*

One important source of symplectic manifolds and Lagrangians comes from the cotangent bundle.

Definition 1.3. *Given any smooth manifold X. Let's denote the cotangent bundle $M := T^*X$. It has a projection map $\pi : M \to X$. The tautological 1-form α may be defined pointwise by $\alpha_p = (d\pi_p)^*\xi \in T_p^*M$, for $\xi \in T_x, p = (x, \xi)$.*

Example 1.1. *One good exercise is show that the cotangent bundle $M = T^*X$ has a symplectic form, defined by $\omega = -d\alpha$, which makes the zero section of the cotangent bundle a Lagrangian.*

In the setting of symplectic manifold, people would like to consider a special kind of diffeomorphisms.

Definition 1.4. *Let (M_1, ω_1) and (M_2, ω_2) be 2n-dimensional symplectic manifolds, and let $\varphi : M_1 \to M_2$ be a diffeomorphism. Then φ is a symplectomorphism if $\varphi^*\omega_2 = \omega_1$.*

Definition 1.5. *A Kalher manifold is a symplectic manifold (M, ω) equipped with an integrable almost-complex structure J which is compactible with the symplectic form, meaning that the bilinear form $g(v, w) := \omega(v, Jw)$ on the tangent space at each point is symmetric and positive definite.*

Example 1.2. *A very great example of Kahler manifold is $\mathbb{C}^n$.*

Definition 1.6. *Let (X, ω, J) be a smooth compact Kahler manifold. Suppose it has a nowhere vanishing holomorphic top form Ω. A Lagrangian submanifold L is called special if $Im(e^{-i\theta}\Omega)|_L = 0$. for some phase θ.*

An example of a special Lagrangian will be provided in the next section.

Special Lagrangians are area minimizers. In particular, they are calibrated by $Re(e^{-i\theta})$ and calibrated submanifolds are minimizers.

Conjecture 1.1. *Special Lagrangians are (semi)stable objects of the Fukaya categories.*

Remark. In general, the existence of global nowhere vanishing holomorphic volume form is a strong assumption. It's in fact one of the definitions of Calabi-Yau manifold.

Here we also recall some constructions of Calabi-Yau manifolds.

1. A way to get a nowhere-vanishing holomorphic top form is to consider the complement of the anticanonical bundle in a Fano manifold. For example, we can consider $\mathbb{P}^2 \setminus C$, where C is a cubic curve in $\mathbb{P}^2$.

2. Besides, degree $n+1$ hypersurface in $\mathbb{P}^n$ also provides a source of Calabi-Yau manifolds, because by adjunction formula, the canonical bundle is trivial. More generally, one can consider the hypersurface in a Fano variety.

3. Furthermore, if we allow the holonomy forms a subgroup of $SU(n)$, complex torus of complex dimension n are also Calabi-Yau manifolds.

4. If we would like get noncompact Calabi-Yau manifolds, we can consider the total space of the canonical bundle over some toric Fano varieties. For example, the total space of canonical bundle over $\mathbb{P}^2$.

5. Local mirror symmetry. Let $f = f(z_1, \cdots, z_{n-1}) \in \mathbb{C}[z_1^{\pm 1}, \cdots, z_{n-1}^{\pm 1}]$ be a Laurent polynomial in $n-1$ variables. Then the hypersurface

   $$X := \{(x, y, z_1, \cdots, z_{n-1}) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} \mid xy = f(z_1, \cdots, z_{n-1})\}$$

   is a noncompact Calabi-Yau variety, since it admits the following holomorphic volume form: $\Omega := \mathrm{Res}[\frac{dx \wedge dy \wedge dlog\, z_1 \wedge \cdots \wedge dlog\, z_{n-1}}{xy - f}]$.

6. Sometimes the Calabi-Yau varieties come from the crepant resolution. Suppose X is normal, then we know X is regular in codimension 1. Hence, the singular locus of X is codimension ≥ 2 in X.

   Let $i : X_{ns} \to X$ be the inclusion of the nonsingular part of X. Then the canonical bundle over the nonsingular part $\omega_{X_{ns}}$ is defined, and we can put $\omega_X := i_* \omega_{X_{ns}}$, where i_* denotes the push-forward of sheaves. Notice that ω_X is not necessarily a line bundle. We say X is Gorenstein if ω_X is a line bundle. As the adjuntion formular still holds even when the divisor is singular, any hypersurface in a non-singular variety is Gorenstein. An important family of examples is the ADE surfaces.

Theorem 1.1 (Lefschetz hyperplane theorem)**.** *Let X be a smooth complex projective variety of dimension n, and let D be an effective ample divisor on X. Then the restriction map*

$$r_i : H^i(X, \mathbb{Z}) \to H^i(D, \mathbb{Z})$$

is an isomorphism for $i \leq n-2$ and injective for $i < n-1$.

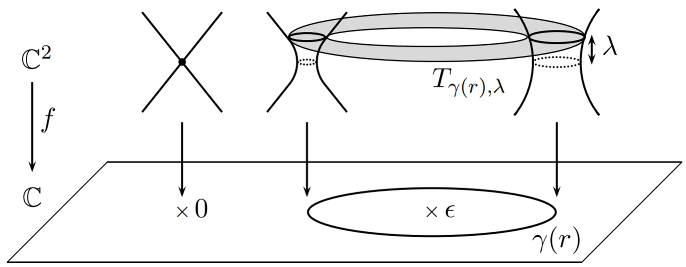
Corollary 1.1. *The Calabi-Yau hypersurface in $\mathbb{P}^n$ has vanishing H^1 for $n \geq 3$.*

Definition 1.7. *Given a fibration $\pi : M \to B$ with fiber a symplectic manifold F with symplectic form σ. $\pi : M \to B$ is called a symplectic fibration if the structure group is contained in the group of symplectomorphisms $Symp(F, \sigma)$.*

2 Important Example

This section highly relies on [Aur07]. Let's consider an open subset $X := \{(x, y) \in \mathbb{C}^2 | xy \neq \epsilon\}$ for some $\epsilon \neq 0$ with a projection map $f : X \to \mathbb{C} \setminus \{\epsilon\}$ defined by $f(x, y) = xy$. The fiber of f over any nonzero complex number is a smooth conic. To visualize the fiber, we can make use of the polar coordinate of x in the fiber, namely $x := r_1 e^{\theta_1}$. Then for any nonzero complex number $a := re^{\theta} \in \mathbb{C} \setminus \{\epsilon\}$, $y = \frac{r_1}{r} e^{\theta - \theta_1}$. If we fix θ_1, we first get a branch of a hyperbola. Taking θ_1 from 0 to 2π, we attain the smooth conic. Topologically, it looks like a cylinder. In fact, this also reveals the structure of S^1-orbit that we will mention later.

Similarly, we know the fiber over 0 is the union of two complex planes (the x and y planes). Topologically, it looks like two cones intersecting at the cone points. With these preparations, we achieve the following nice picture.



Remark. In fact, X is the complement of a canonical divisor in $\mathbb{CP}^2$. More precisely, let $D := \{(x : y : z) \in \mathbb{CP}^2 | (xy - \epsilon z^2)z = 0\}$ (the union of a conic and a line), for the same ϵ as before. We know $X = \mathbb{CP}^2 \setminus D$. The Fubini-study metric on $\mathbb{CP}^2$ naturally induces a symplectic structure ω on X. Then the projection map we consider is the rational map $f : \mathbb{CP}^2 \to \mathbb{CP}^1$ defined by $[x, y, z] \mapsto [xy, z^2]$. Besides, X can be equipped with a nowhere vanishing holomorphic volume form, which is given by

$$\Omega = \frac{dx \wedge dy}{xy - \epsilon}.$$

This pair $X = \mathbb{CP}^2 \setminus D$ is also called a log Calabi Yau surface.

Furthermore, this fibration is equipped with the Hamiltonian S^1-action on (X, ω) given by

$$S^1 \times \mathbb{C}^2 \to \mathbb{C}^2, (\theta, (x, y)) \mapsto (e^{i\theta} x, e^{-i\theta} y),$$

whose moment map for this S^1-action is $\mu(x, y) = \frac{1}{2}(|x|^2 - |y|^2)$.

We will consider Lagrangian tori which are contained in $f^{-1}(\gamma)$ for some simple closed curve $\gamma \subset \mathbb{C}$, and consist of a single S^1-orbit inside each fiber. Recall that $f : X \to \mathbb{C} \setminus \{1\}$ carries a natural horizontal distribution, given at every point by the symplectic orthogonal to the fiber. More precisely, we first have the vertical direction or the fiber direction given by $Kerdf$. Using the symplectic form mentioned in the remark, we can consider its symplectic orthogonal $(\mathrm{Ker}\, df)^{\Omega} := \{v \in T_p X | \omega(v, w) = 0, \forall w \in \mathrm{Ker}\, df\}$. Parallel transport with this horizontal distribution yields symplectomorphisms between the smooth fibers. Notice that by definition, the tangent vector of the flow line lies in the symplectic orthogonal. Hence, it perserves the moment map level. This suggests the following definition.

Definition 2.1. *Given a simple closed curve $\gamma \subset \mathbb{C}$ and a real number λ, we define*

$$T_{\gamma,\lambda} = \{(x, y) \in f^{-1}(\gamma), \mu(x, y) = \lambda\}.$$

By construction $T_{\gamma,\lambda}$ is an embedded torus in X, except when $0 \in \gamma$ and $\lambda = 0$. When $0 \in \gamma$ and $\lambda = 0$, $T_{\gamma,0}$ is an immersed torus with nodal singularity at the origin.

One can check that $T_{\gamma,\lambda}$ is a Lagrangian in X. (Hint: the tangent space is generated by two elements. One is tangent to the S^1-orbit, the other lies in the horizontal distribution.)

Proposition 2.1 (Proposition 5.2 in [Aur07])**.** *The tori $T_{\gamma(r),\lambda} = \{(x, y), |xy - \epsilon| = r, \mu(x, y) = \lambda\}$ are special Lagrangian with respect to $\Omega = \frac{dx\wedge dy}{xy-\epsilon}$.*

Corollary 2.1. *The map $\pi : X \to \mathbb{R}^2$ via $\pi(x, y) = (log|xy - \epsilon|, \frac{1}{2}(|x|^2 - |y|^2))$ defines a special Lagrangian torus fibration.*

Remark. This example is important for many reasons.

1. Special Lagrangian torus fibration in general is hard to find. This example provides a non-toric case.

2. From this example, it's clear that even for a smooth surface. The SYZ fibration may have singular fibers.

There are some other possible ways to construct special Lagrangians.

1. Explicit metric. Let $X = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}$ be a complex tori. $\Omega = dz$. $\omega = \frac{i}{2} dz \wedge d\bar{z}$. In this example, all the lines with rational slopes are special Lagrangians. In this case, the slope is the same as the phase, while the volume is the length of the line. For irrational slope, it covers the torus but it's not regarded as a submanifold of X.

2. One can also construct a special Lagrangian using some real structure. In other words, let i be an involution on the Calabi-Yau manifold (X, ω, Ω), i.e. $i^2 = id$. If $i^*\omega = -\omega, i^*\Omega = \bar{\Omega}$, then $Fix(i)$ is a special Lagrangian. This is because if $L = Fix(i)$ has half dimension, we know $i^*(\omega|L) = -\omega|_L = \omega|_L$. Thus $\omega|_L = 0$. Similarly, $i^*\Omega|_L = \Omega|_L = \bar{\Omega}|_L$. Hence, $Im(\Omega)|_L = 0$.

3. Lagrangian mean curvature flow: Given a Lagrangian L in a Calabi-Yau manifold (X, ω, Ω). Let $\theta : L \to S^1$ be a map such that $\Omega|_L = e^{i\theta} vol_L$. Suppose L has Maslov zero, θ admits a lifting to $\mathbb{R}$. The mean curvature is defined to be $\vec{H} := \nabla\theta$. Solving the equation $\frac{dL_t}{dt} = \vec{H}$, which preserves Maslov zero and being a Lagrangian, provides a Lagrangian mean curvature flow. It's conjectured that the limit of Lagrangian mean curvature flow is a special Lagrangian.

4. HyperKahler Rotation of a Calabi-Yau 2-fold: given a HyperKahler 4-fold. There exist three almost complex structures I, J, K satisfying some compatible conditions. Consider the linear combination $aI + bJ + cK$. If the linear combination is also an almost complex structure, we have $a^2 + b^2 + c^2 = 1$, which can be identified as $\mathbb{C}P^1$.

3 SYZ Conjecture

With the above example in mind, we can now introduce the SYZ conjecture, which revealed the intimate relation between a pair of mirror Calabi-Yau manifolds in a geometric manner:

Conjecture 3.1 ([SYZ96])**.** *Suppose that X and $\check{X}$ are Calabi-Yau manifolds mirror to each other. Then*

1. *both X and $\check{X}$ admit special Lagrangian torus fibrations with sections $\mu : X \to B$ and $\check{\mu} : \check{X} \to B$ over the same base. There exists a dense open subset $B_{reg} \subset B$ such that the fibers $\mu^{-1}(b)$ and $\check{\mu}^{-1}(b)$ for every $b \in B_{reg}$ are nonsingular tori, and they are dual tori;*

2. *for each $b \in B \setminus B_{reg}$, the fibers $\mu^{-1}(b)$ and $\check{\mu}^{-1}(b)$ should be singular special Lagrangian submanifolds of X and $\check{X}$ respectively.*

This conjecture provides guidance to find the mirror manifolds, namely, a mirror of any given Calabi-Yau manifold X is given by fiberwise dualizing a special Lagrangian torus fibration on X. However, in general, there are a lot of difficulties to proceed with this construction.

Definition 3.1. *The situation in which $B_{reg} = B$ so that there is no singular locus is called the semi-flat limit of the SYZ conjecture.*

The important geometric feature of a pair of Lagrangian torus fibrations $\mu : X \to B$ and $\check{\mu} : \check{X} \to B$ which encodes mirror symmetry is the dual torus fibers of the fibration. Recall that given a Lagrangian torus $T \subset X$. Let assume it to be an abelian variety, the dual torus is defined as the Jacobian variety of T. This is again a torus of the same dimension. Furthermore, the Jacobian variety has the interpretation as the moduli space of the line bundles with vanishing first Chern class on T.

This duality and the interpretation of the dual torus as a moduli space of sheaves on the original torus is what allows one to interchange the data of submanifolds and subsheaves. There are two nice examples of this phenomenon:

1. If $p \in X$ is a point which lies inside some fiber T of the special Lagrangian torus fibration with even dimension, then since $T = Jac(\check{T})$, this point corresponds to a line bundle supported on $\check{T}$. Furthermore, if there exists a Lagrangian section of this fibration, i.e. $s : B \to X$ such that the image of s is a Lagrangian in X and $\mu \circ \pi = id$, then this section s should associate a line bundle over each dual torus. Consequently, a line bundle on the total space of the mirror $\check{X}$. In this way, we attain an approach to relate the symplectic geometric object to the complex algebraic geometry.

2. Another example is the Lagrangian torus fibre T itself together with a flat unitary connection ∇ on the trivial complex line bundle. Indeed, the gauge equivalence class of the connection ∇ is determined by its holonomy $hol_\nabla \in Hom(\pi_1(T), U(1)) = Hom(H_1(T), U(1)) = Hom(\mathbb{Z}^{2g}, S^1) \cong \check{T}$! In other words, there's a one-to-one correspondence between the gauge equivalence classes of flat unitary connection and its holonomy. Hence, the pair (T, ∇) corresponds to a point on the dual torus $\check{T}$, which is the skyscraper sheaf at that point. With this perspective, the mirror $\check{X}$ should be identified with the moduli space of a certain pair (L, ∇) on X, where $L \subset X$ is a special Lagrangian submanifold, ∇ is a flat unitary connection on L.

These two examples produce the most extreme kinds of coherent sheaf, locally free sheaves (of rank 1) and torsion sheaves supported on points. By more careful construction one can build up more complicated examples of coherent sheaves. Similarly, one would expect that a Lagrangian multisection (a union of k Lagrangian sections) should be mirror to a rank k vector bundle on the mirror manifold. If the Lagrangian sections with nontrivial intersection points are unobstructed, one would also expect that it's mirror to a nontrivial vector bundle. More details about the semi-flat case SYZ conjecture can be found in [LYZ00] and [Leu01].

Remark. If the mirror torus fibrations are not in the semi-flat limit, then special care must be taken when crossing over singular sets of the base B. Roughly speaking, we will have the following difficulties caused by the singular fiber:

1. We can only take dual for tori. If we only take dual for smooth torus fibers, then we only attain a subset of the mirror.

2. It emanates a wall of Maslov index zero holomorphic discs bounded by torus fibers.

3. If we walk around the wall caused by the singular point, we have nontrivial monodromy.

To get a more complete story of mirror symmetry, we have better introduce Kontsevich's remarkable Homological mirror symmetry conjecture, which was introduced in 1994 earlier than SYZ conjecture.

Conjecture 3.2 (HMS conjecture)**.** *If two Calabi-Yau manifolds X and $\check{X}$ are mirror to each other, then $DFuk(X)$ is equivalent to $D^b(\check{X})$ as a triangulated category.*

If the HMS conjecture is true, the mirror pairs that come from SYZ are expected to satisfy this categorical correspondence. In particular, the triangulated structure should be preserved on the object levels. Therefore, the connected sum of Lagrangians in $DFuk(X)$ should correspond to a certain complex of coherent sheaves over $\check{X}$. For example, the ideal sheaf of a point should correspond to the connected sum of lagrangian section and the lagrangian fiber up to shifting.

Besides, on the structure level, the autoequivalence of $DFuk(X)$ is also expected to correspond to the autoequivalence of $D^b(\check{X})$. Furthermore, we also expect there are some relations on the operators.

Here are some examples of expectations:

1. The Dehn twist, which is a symplectomorphism along a sphere, should correspond to a certain twist functor along a spherical object on the B-side. For more details, please see [ST00].

2. The Lagrangian translation, which is a symplectomorphism realized via a lagrangian section, should correspond to tensoring with a line bundle. Some explanations can be found in [HK21].

References

[Aur07] Denis Auroux. Mirror symmetry and t-duality in the complement of an anticanonical divisor, 2007.

[HK21] Paul Hacking and Ailsa Keating. Symplectomorphisms of mirrors to log calabi-yau surfaces, 2021.

[Leu01] Naichung Conan Leung. Mirror symmetry without corrections, 2001.

[LYZ00] Naichung Conan Leung, Shing-Tung Yau, and Eric Zaslow. From special lagrangian to hermitian-yang-mills via fourier-mukai transform, 2000.

[ST00] Paul Seidel and R. P. Thomas. Braid group actions on derived categories of coherent sheaves, 2000.

[SYZ96] Andrew Strominger, Shing-Tung Yau, and Eric Zaslow. Mirror symmetry is t-duality. *Nuclear Physics B*, 479(1-2):243–259, nov 1996.