APPLICATIONS OF MATLAB IN ENGINEERING

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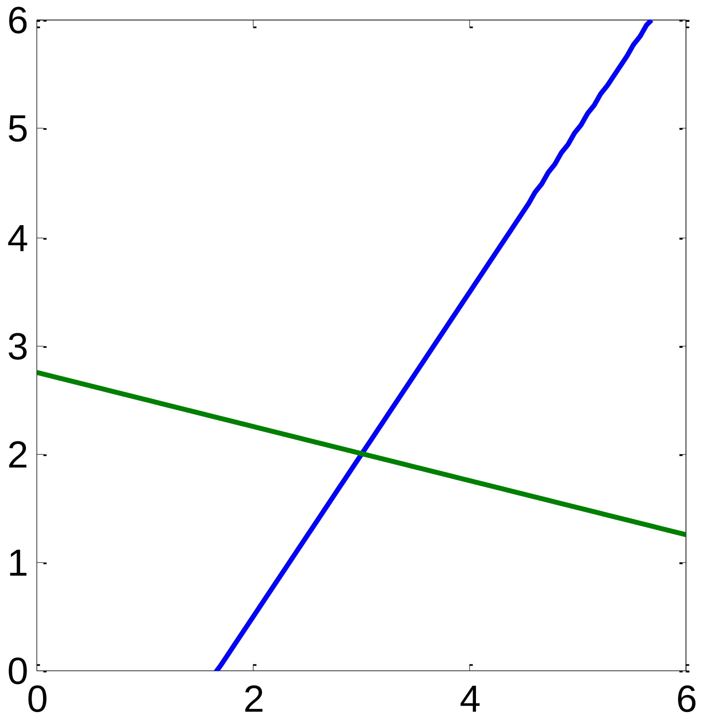
Today:

- Linear equation
- Linear system

Linear Equation

- Suppose you are given linear equations:

$$\begin{cases} 3x - 2y = 5 \\ x + 4y = 11 \end{cases}$$

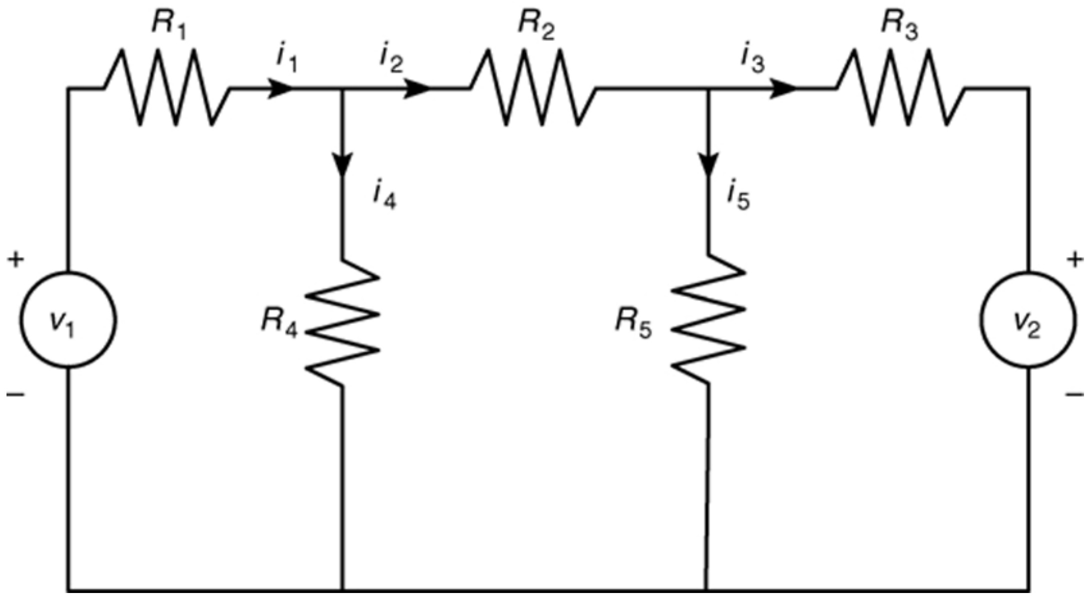


- Matrix notation:

$$\underbrace{\begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 5 \\ 11 \end{bmatrix}}_{\boldsymbol{b}}$$

Why Matrix Form?

- An electrical network:



- $V_1 = R_1 i_1 + R_4 i_4$
- $R_4 i_4 = R_2 i_2 + R_5 i_5$
- $R_5 i_5 = R_3 i_3 + V_2$
- $i_1 = i_2 + i_4$
- $i_2 = i_3 + i_5$

- Given the voltages V_1 and V_2 and the resistances $R_1 \dots R_5$
- Solve the currents $i_1 \dots i_5$

Formulation for the Electrical Network

$$\underbrace{\begin{bmatrix} R_1 & 0 & 0 & R_4 & 0 \\ 0 & R_2 & 0 & -R_4 & R_5 \\ 0 & 0 & -R_3 & 0 & R_5 \\ 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} V_1 \\ 0 \\ V_2 \\ 0 \\ 0 \end{bmatrix}}_{\boldsymbol{b}}$$

- Usually when solving linear equations:
  1. $\boldsymbol{A}$ and $\boldsymbol{b}$ are know
  2. $\boldsymbol{x}$ is unknown

Solving Linear Equations

1. Successive elimination (through factorization)
2. Cramer’s method

Gaussian Elimination

- Suppose given:

$$\begin{cases} x + 2y + z = 2 \\ 2x + 6y + z = 7 \\ x + y + 4z = 3 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 6 & 1 & 7 \\ 1 & 1 & 4 & 3 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ & 2 & -1 & 3 \\ & -1 & 3 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ & 2 & -1 & 3 \\ & & 5/2 & 5/2 \end{array}\right]$$

Gaussian Elimination – `rref()`

$$\begin{cases} x + 2y + z = 2 \\ 2x + 6y + z = 7 \\ x + y + 4z = 3 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 6 & 1 & 7 \\ 1 & 1 & 4 & 3 \end{array}\right]$$

```
A = [1 2 1;2 6 1;1 1 4];
b = [2; 7; 3];
R = rref([A b])
```

LU Factorization

- Suppose we want to solve: $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A} \in \Re^{m \times m}$
- Decompose $\boldsymbol{A}$ into 2 triangular matrices: $\boldsymbol{A} = \boldsymbol{L}^{-1}\boldsymbol{U}$
- The problem become: $\boldsymbol{Ax} = \boldsymbol{b} \quad \Rightarrow \quad \boldsymbol{L}^{-1}\underbrace{\boldsymbol{Ux}}_{\boldsymbol{y}} = \boldsymbol{b}$
- Strategies:
    1. Solve $\boldsymbol{L}^{-1}\boldsymbol{y} = \boldsymbol{b}$ to obtain $\boldsymbol{y}$
    2. Then solve $\boldsymbol{Ux} = \boldsymbol{y}$

Lower and Upper Triangular Matrices

- Lower triangular matrix $\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ \vdots & \ddots & 0 \\ \vdots & \cdots & 1 \end{bmatrix} \in \Re^{m \times m}$
- Upper triangular matrix $\boldsymbol{U} = \begin{bmatrix} \vdots & \cdots & \vdots \\ 0 & \ddots & \vdots \\ 0 & 0 & \vdots \end{bmatrix} \in \Re^{m \times m}$

How to Obtain $\boldsymbol{L}$ and $\boldsymbol{U}$?

- The matrices $\boldsymbol{L}$ and $\boldsymbol{U}$ are obtained by using a serious of left-multiplication, i.e.,

$$\underbrace{\boldsymbol{L}_m \ldots \boldsymbol{L}_2 \boldsymbol{L}_1}_{\boldsymbol{L}} \boldsymbol{A} = \boldsymbol{U}$$

LU Factorization Example

- For example: $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}$

$$\boldsymbol{L}_1\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\boldsymbol{L}_2(\boldsymbol{L}_1\boldsymbol{A}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} = \boldsymbol{U}$$

LU Factorization – `lu()`

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}$$

```
A = [1 1 1;2 3 5;4 6 8];
[L, U, P] = lu(A);
```

- Solving: $\begin{cases} \boldsymbol{L}^{-1}\boldsymbol{y} = \boldsymbol{b} \\ \boldsymbol{U}\boldsymbol{x} = \boldsymbol{y} \end{cases}$

```
inv(L)
U
```

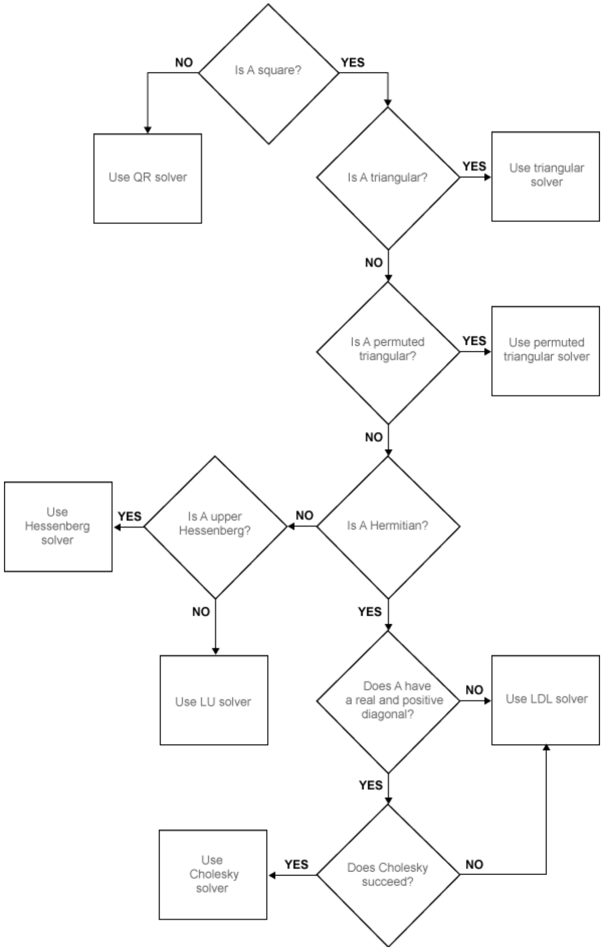
$$\begin{bmatrix} 1 & 0 & 0 \\ -.25 & 1 & 0 \\ -.5 & 0 & 1 \end{bmatrix} \boldsymbol{y} = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 4 & 6 & 8 \\ 0 & -.5 & -1 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \boldsymbol{y}$$

Matrix Left Division: \ or `mldivide()`

- Solving systems of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ using factorization methods:

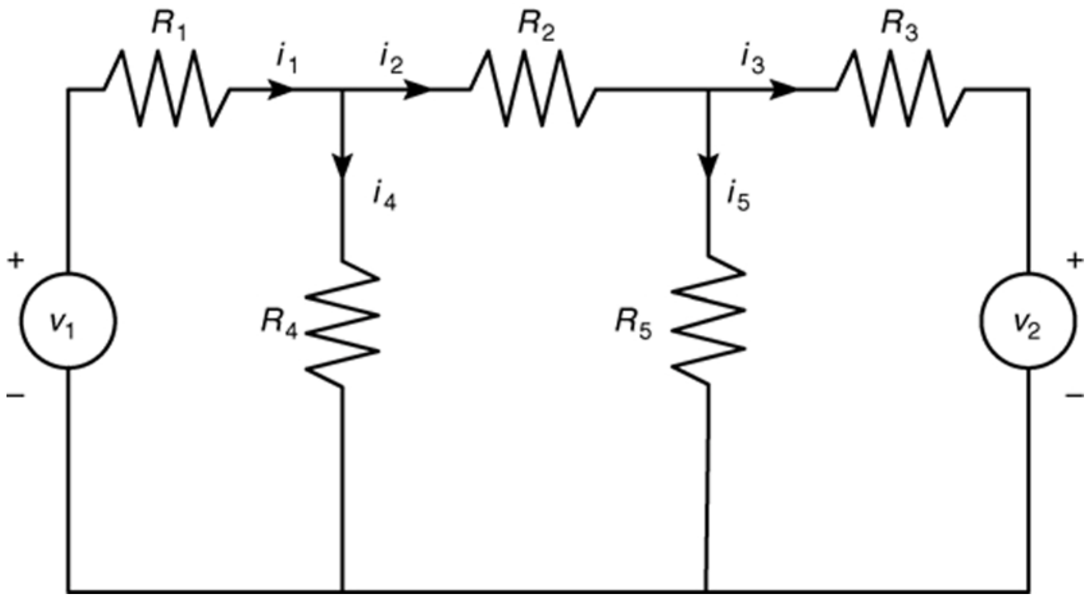
$$\begin{cases} x + 2y + z = 2 \\ 2x + 6y + z = 7 \\ x + y + 4z = 3 \end{cases}$$

```
A = [1 2 1;2 6 1;1 1 4];
b = [2; 7; 3];
x = A\b
```



http://www.mathworks.com/help/matlab/ref/mldivide.html?searchHighlight=mldivide

Exercise



- $V_1 = R_1 i_1 + R_4 i_4$
- $R_4 i_4 = R_2 i_2 + R_5 i_5$
- $R_5 i_5 = R_3 i_3 + V_2$
- $i_1 = i_2 + i_4$
- $i_2 = i_3 + i_5$

- Write a function to solve $i_1 \ldots i_5$ for given V_1, V_2, and $R_1 \ldots R_5$

Matrix Decomposition Functions

`qr`	Orthogonal-triangular decomposition
`ldl`	Block LDL' factorization for Hermitian indefinite matrices
`ilu`	Sparse incomplete LU factorization
`lu`	LU matrix factorization
`chol`	Cholesky factorization
`gsvd`	Generalized singular value decomposition
`svd`	Singular value decomposition

Cramer's (Inverse) Method

- Given the problem:

$$\underbrace{\begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 5 \\ 11 \end{bmatrix}}_{\boldsymbol{b}}$$

- Suppose there exists the $\boldsymbol{A}^{-1} \in \Re^{m \times m}$ such that

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$$

- The variable $\boldsymbol{x}$ is: $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$

Inverse Matrix

- For a matrix $\boldsymbol{A}$, the inverse is defined as:

$$\boldsymbol{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(\boldsymbol{A})} adj(\boldsymbol{A}) = \frac{1}{\det(\boldsymbol{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $\det(\boldsymbol{A})$ is the determinant:

$$\det(\boldsymbol{A}) = |ad - bc|$$

- Properties: $\boldsymbol{A} = (\boldsymbol{A}^{-1})^{-1}$, $(k\boldsymbol{A})^{-1} = k^{-1}\boldsymbol{A}^{-1}$

Solving Equations Using Cramer's Method

- Given equation:

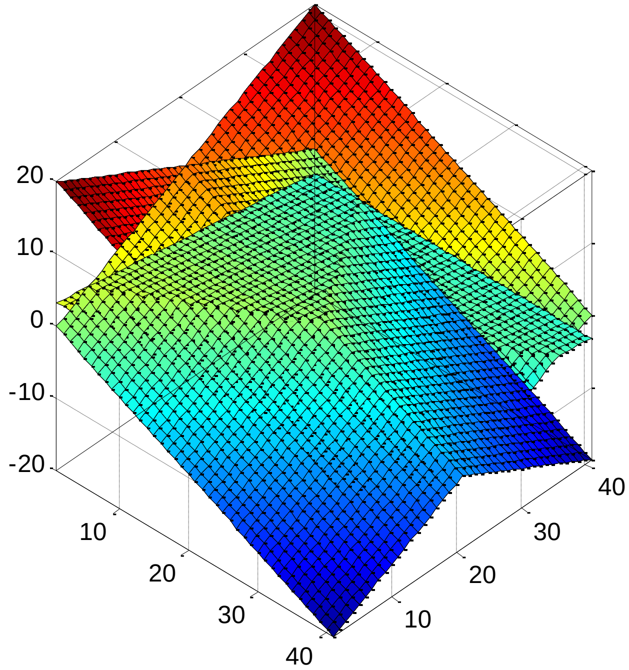
$$\begin{cases} x + 2y + \ \ z = 2 \\ 2x + 6y + \ \ z = 7 \\ x + \ \ y + 4z = 3 \end{cases} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}$$

- $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$

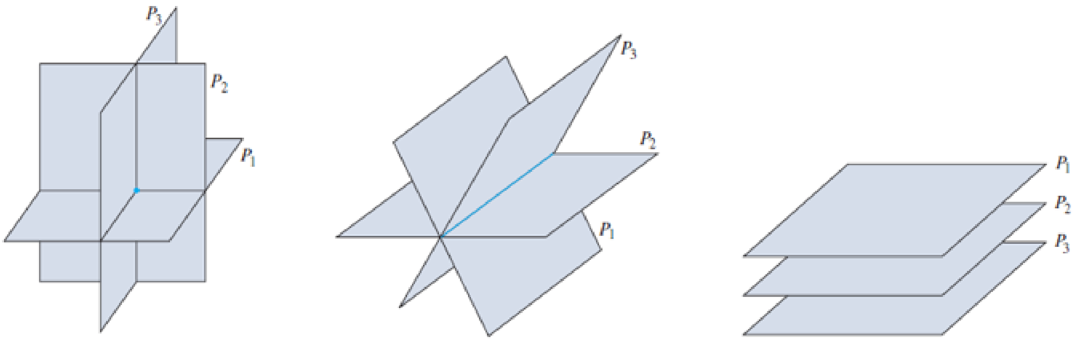
```
A = [1 2 1;2 6 1;1 1 4];
b = [2; 7; 3];
x = inv(A)*b
```

Exercise

- Plot the planes in 3D: $$\begin{cases} x + y + z = 0 \\ x - y + z = 0 \\ x + 3z = 0 \end{cases}$$



Singular



- The inverse matrix does not exist

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 9 & 8 & 7 & 6 \\ 1 & 3 & 2 & 8 \end{bmatrix}$$

```
A = [ 1 2 3 4; 2 4 6 8; ...
      9 8 7 6; 1 3 2 8];
inv(A)
det(A)
```

Problem with Cramer's Method

- The determinant is zero if the equations are singular, i.e., `det(A)=0`
- The accuracy is low when the determinant is very close to zero, i.e., `det(A)~0`
- Recall that

$$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})} adj(\boldsymbol{A})$$

Functions to Check Matrix Condition

cond	Matrix condition number
rank	Matrix rank

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4.0001 & 6 \\ 9 & 8 & 7 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 9 & 8 & 7 \end{bmatrix}$$

- $\boldsymbol{Ax} = \boldsymbol{b}$
- Check the change in $\boldsymbol{x}$ if $\boldsymbol{A}$ changes by a "small" amount $\delta\boldsymbol{A}$:

$$\frac{||\delta\boldsymbol{x}||}{||\boldsymbol{x}||} < \kappa(\boldsymbol{A})\frac{||\delta\boldsymbol{A}||}{||\boldsymbol{A}||},$$

  where $\kappa(\boldsymbol{A})$ is the condition number of $\boldsymbol{A}$
- A smaller $\kappa(\boldsymbol{A})$ indicate a well-conditioned matrix

```
A = [ 1 2 3; 2 4.0001 6; 9 8 7];  cond(A)
B = [ 1 2 3; 2 5 6; 9 8 7];  cond(B)
```

Linear System

- Suppose you are given linear equations:

$$\begin{cases} 2 \cdot 2 - 12 \cdot 4 = x \\ 1 \cdot 2 - \ \ 5 \cdot 4 = y \end{cases}$$

$$\begin{cases} 3x - 2y = 5 \\ x + 4y = 11 \end{cases}$$

<Linear equation>

- Matrix notation:

$$\underbrace{\begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} 2 \\ 4 \end{bmatrix}}_{\boldsymbol{b}} = \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\boldsymbol{y}}$$

$$\underbrace{\begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 5 \\ 11 \end{bmatrix}}_{\boldsymbol{b}}$$

- Note the difference between the two formulation

Eigenvalues and Eigenvectors

- For a system $\boldsymbol{A} \in \Re^{m\times m}$, matrix multiplication $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{b}$ is complicated
- Want to find vector(s) $\boldsymbol{v}_i \in \Re^m$ such that

$$\boldsymbol{A}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i, \qquad \text{where } \lambda_i \in \Re$$

- Then we decompose $\boldsymbol{b} = \sum \alpha_i \boldsymbol{v}_i\,,\ \ \alpha_i \in \Re$
- The multiplication becomes:

$$\boldsymbol{A}\boldsymbol{b} = \sum \alpha_i \boldsymbol{A}\boldsymbol{v}_i = \sum \alpha_i \lambda_i \boldsymbol{v}_i$$

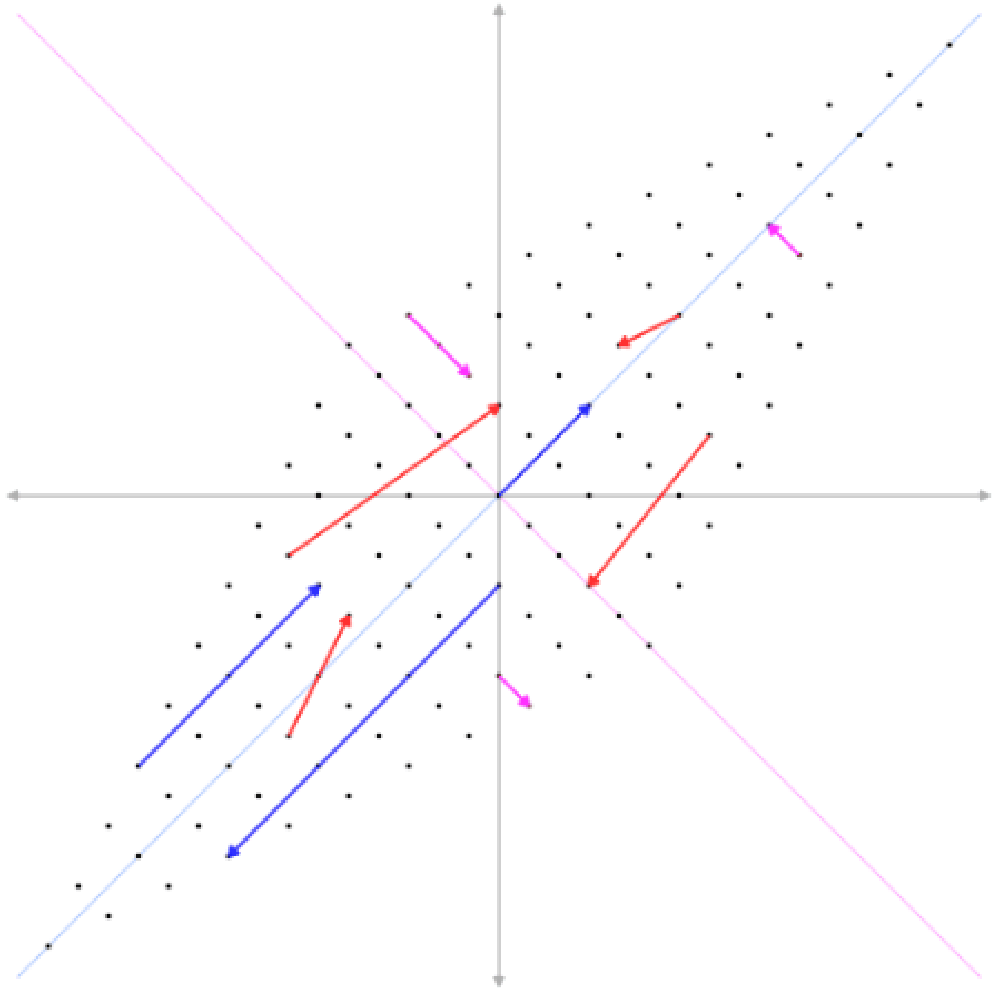
Interpretation of Eigenvalues and Eigenvectors

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{b}$$

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\lambda_1 = 1, \quad \boldsymbol{v}_1 = \begin{bmatrix} -0.71 \\ 0.71 \end{bmatrix}$$

$$\lambda_2 = 3, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0.71 \\ 0.71 \end{bmatrix}$$



Solving Eigenvalues and Eigenvectors

- For given $\boldsymbol{Ab} = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$$\lambda_1 \boldsymbol{v}_1 = -1 \begin{bmatrix} 0.97 \\ 0.24 \end{bmatrix}, \quad \lambda_2 \boldsymbol{v}_2 = -2 \begin{bmatrix} 0.95 \\ 0.32 \end{bmatrix}$$

$$\boldsymbol{b} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 = -41.2\boldsymbol{v}_1 + 44.3\boldsymbol{v}_2$$

$$\boldsymbol{Ab} = \boldsymbol{A}(\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2)$$

$$= \alpha_1 \boldsymbol{A}\boldsymbol{v}_1 + \alpha_2 \boldsymbol{A}\boldsymbol{v}_2 = \alpha_1 \lambda_1 \boldsymbol{v}_1 + \alpha_2 \lambda_2 \boldsymbol{v}_2$$

$$= (-41.2)(-1) \begin{bmatrix} 0.97 \\ 0.24 \end{bmatrix} + (44.3)(-2) \begin{bmatrix} 0.95 \\ 0.32 \end{bmatrix}$$

eig()

- Find the eigenvalues and eigenvectors:

$$\boldsymbol{A} = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

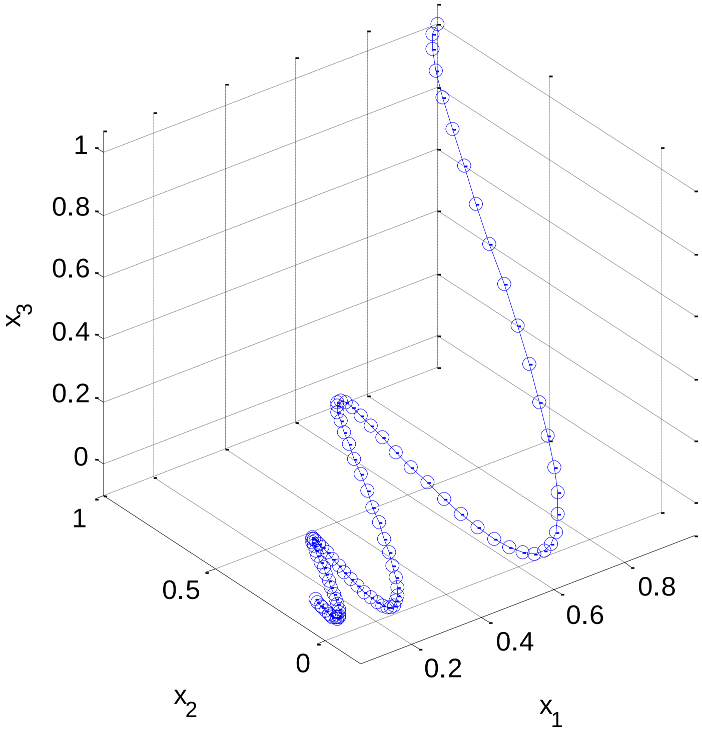
```
[v,d]=eig([2 -12;1 -5])
```

Matrix Exponential: `expm()`

- A typical linear time-invariant system is usually formulated as

$$\boldsymbol{y} = \frac{d\boldsymbol{x}(t)}{dt} = \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$$

```
A = [0 -6 -1; 6 2 -16; -5 20 -10];
x0 = [1 1 1]';  X = [];
for t = 0:.01:1
    X = [X expm(t*A)*x0];
end
plot3(X(1,:),X(2,:),X(3,:),'-o');
xlabel('x_1');  ylabel('x_2');
zlabel('x_3');  grid on;
axis tight square;
```



End of Class